Weakly, Semi Compatible Mappings and Common Fixed Points in Fuzzy Metric Spaces

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Singh and Chauhan [13] and Cho[1] proved fixed point theorems in fuzzy metric space for four self maps using the concept of compatibility where two mappings needed to be continuous. The purpose of this paper is to obtain common fixed point theorem in fuzzy metric space for six self maps using the concept of semi-compatibility and weak compatibility and only one map is needed to be continuous, which generalizes the result of Singh and Chauhan[13] and Cho[1].

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1. INTRODUCTION.

Jungck[6] proposed the concept of compatibility. The concept of compatibility in fuzzy metric space was introduced by Mishra et al.[11]. Later on, Jungck[7] generalized the concept of compatibility by introducing the concept of weak compatibility. Cho et al.[3] introduced the concept of semi-compatible maps in d-topological space. Singh and Jain[14] defined the concept of semi-compatible maps in fuzzy metric space. In this paper, we deal with the fuzzy metric space defined by Kramosil and Michalek [9] and modified by George and Veeramani [4].

Definition 1.1. [15] Let X be any set. A fuzzy set A in X is a function with domain in X and values in [0, 1].

Definition 1.2. [12] A binary operation * : [0, 1] × [0, 1] → [0, 1] is called a continuous t-norm if it satisfies the following conditions:

(i) * is associative and commutative,
(ii) * is continuous,
(iii) a * 1 = a, for all a ∈ [0, 1],
(iv) a * b ≤ c * d, whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0, 1].

Examples of t-norms are

a * b = min {a, b} (minimum t-norm),
a * b = ab (product t-norm).

Definition 1.3. [4] The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X × (0, ∞) satisfying the following conditions:

(FM-1) M(x, y, t) > 0,
(FM-2) M(x, y, t) = 1 if and only if x = y,
(FM-3) M(x, y, t) = M(y, x, t),
(FM-4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),
(FM-5) M(x, y, .) : (0, ∞) → [0, 1] is left continuous,
for all x, y, z ∈ X and t, s > 0.

Definition 1.4. [5] A sequence {x_n} in a fuzzy metric space (X, M, *) is said to be convergent to a point x ∈ X if \( \lim_{n \to \infty} M(x_n, x, t) = 1 \) for all t > 0. Further, the sequence \{x_n\} is said to be Cauchy if \( \lim_{n \to \infty} M(x_{n^+p}, x_{n+p}, t) = 1 \), for all t > 0 and p > 0. The space (X, M, *) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 1.5. [5] Let (X, M, *) be a fuzzy metric space. Then M(x, y, .) is non-decreasing.

Definition 1.6. [10] Let (X, M, *) be a fuzzy metric space. Then M is a continuous function on X × (0, ∞).

Throughout this paper (X, M, *) will denote the fuzzy metric space with the following condition:

(FM-6) \( \lim_{t \to \infty} M(x, y, t) = 1 \) for all x, y ∈ X and t > 0.

Lemma 1.7. [11] If there exists k ∈ (0, 1) such that M(x, y, kt) ≥ M(x, y, t) for all x, y ∈ X and t > 0, then x = y.

Lemma 1.8. [8] The only t-norm * satisfying \( r * r ≥ r \) for all r ∈ [0, 1] is the minimum t-norm, that is a*b = min(a, b) for all a, b ∈ (0, 1).
Lemma 1.9. [2] Let \( \{y_n\} \) be a sequence in a fuzzy metric space \((X, M, \ast)\) with condition (FM-6). If there exists a number \( k \in (0, 1) \), such that
\[
M(y_{n+2}, y_{n+1}, k^t) \geq M(y_{n+1}, y_n, t)
\]
for all \( t > 0 \)
Then \( \{y_n\} \) is a Cauchy sequence in \( X \).

Definition 1.10. [11] Let \( A \) and \( B \) be self mappings on a fuzzy metric space \((X, M, \ast)\). The pair \((A, B)\) is said to be compatible if
\[
\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1
\]
for all \( t > 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x \), for some \( x \in X \).

Definition 1.11. [14] Let \( A \) and \( B \) be self mappings on a fuzzy metric space \((X, M, \ast)\). Then the pair \((A, B)\) is said to be weakly compatible if
\[
Ax = Bx \implies ABx = BAx
\]
It is known that a pair \((A, B)\) of compatible maps is weakly compatible but converse is not true in general.

Definition 1.12. [14] A pair \((A, B)\) of self maps of a fuzzy metric space \((X, M, \ast)\) is said to be semi-compatible if
\[
\lim_{n \to \infty} M(ABx_n, Bx, t) = 1
\]
for all \( t > 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x \).

It follows that if \((A, B)\) is semi-compatible and \( Ax = Bx \) then \( ABx = BAx \) that means every semi-compatible pair of self maps is weak compatible but the converse is not true in general.

Cho[1] generalized the result of Singh and Chauhan[13] as follows:

Theorem 1.13. [1] Let \((X, M, \ast)\) be a complete fuzzy metric space and let \( A, B, S \) and \( T \) be mappings from \( X \) into itself such that the following conditions are satisfied:

(i) \( AX \subset TX, BX \subset SX \)

(ii) \( S \) and \( T \) are continuous,

(iii) the pairs \([A,S]\) and \([B,T]\) are compatible,
(iv) there exists \( q \in (0, 1) \) such that for every \( x, y \in X \) and \( t > 0 \),
\[
M(Ax, By, qt) \geq M(Sx, Ty, t) \ast M(Ax, Sx, t) \ast M(By, Tt, t) \ast M(Ax, Ty, t)
\]
Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

2. MAIN RESULT.

Our result generalizes the results of Singh and Chauhan [13] and Cho[1] as we are using the concept of semi-compatibility and weak compatibility which are lighter conditions than that of compatibility, also only one map is needed to be continuous. We are proving the result for six self maps using another functional inequality.

Theorem 2.1. Let \((X, M, \ast)\) be a complete fuzzy metric space with \( r \ast r \geq r \) for all \( r \in [0,1] \) and let \( A, B, S, T, P \) and \( Q \) be mappings from \( X \) into itself such that the following conditions are satisfied:

(2.1.1) \( A(X) \subset ST(X), B(X) \subset PQ(X) \)

(2.1.2) either \( A \) or \( PQ \) is continuous;

(2.1.3) \((A, PQ)\) is semi-compatible and \((B, ST)\) is weakly compatible;

(2.1.4) \( PQ = QP, ST = TS, AQ = QA \) and \( BT = TB \);

(2.1.5) there exists \( q \in (0, 1) \) such that for every \( x, y \in X \) and \( t > 0 \),
\[
M(Ax, By, qt) \geq M(Ax, STy, t) \ast M(Ax, PQx, t) \ast M(By, STy, t) \ast M(PQx, STy, t) \ast M(PQx, By, 2t).
\]
Then \( A, B, S, T, P \) and \( Q \) have a unique common fixed point in \( X \).

Proof. Let \( x_0 \) be an arbitrary point in \( X \). As \( A(X) \subset ST(X) \) and \( B(X) \subset PQ(X) \), then there exists \( x_1, x_2 \in X \) such that
\[
Ax_0 = STx_1 = y_0 \quad \text{and} \quad Bx_1 = PQx_2 = y_1.
\]
We can construct sequences \( \{y_n\} \) and \( \{x_n\} \) in \( X \) such that \( y_{2n} = STx_{2n+1} = Ax_{2n} \) and \( y_{2n+1} = Bx_{2n+1} = PQx_{2n+2} \) for \( n = 0, 1, 2, \ldots \).

Now, we first show that \( \{y_n\} \) is a Cauchy sequence in \( X \).
From (2.1.5), we have
\[
M(y_{2n}, y_{2n+2}, qt) = M(Ax_{2n}, Bx_{2n+1}, qt)
\]
\[
\geq M(Ax_{2n}, STx_{2n+1}, t) \ast M(Ax_{2n}, PQx_{2n}, t) \ast M(Bx_{2n+1}, STx_{2n+1}, t) \ast M(PQx_{2n}, STx_{2n+1}, t)
\]
\[
= M(y_{2n}, y_{2n}, t) \ast M(y_{2n}, y_{2n-1}, t) \ast M(y_{2n+1}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t).
\]
Using definition 1.2 and definition 1.3, we get
\[
M(y_{2n}, y_{2n+2}, qt) \geq M(y_{2n-1}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t)
\]
Thus we have
\[
M(y_{2n}, y_{2n+2}, qt) \geq M(y_{2n-1}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n+2}, t)
\]
Putting (ii) in (i), we get
\[ M(y_{2n^2}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n+1}, t) \ast M(y_{2n-1}, y_{2n}, t/q) \ast M(y_{2n+1}, y_{2n-1}, t/q) \]

Using lemma 1.5 and lemma 1.8, we get
\[ M(y_{2n^2}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n}, t/q) \]

Proceeding in the similar manner, we get
\[ M(y_{2n^2}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n+1}, t) \ast M(y_{2n+1}, y_{2n}, t/q) \]

Letting \( m \to \infty \) and using (FM-6), we get
\[ M(y_{2n^2}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n+1}, t) \quad \forall t > 0. \]

In general,
\[ M(y_{n^2}, y_{n+1}, qt) \geq M(y_{n-1}, y_{n+1}, t) \quad \forall t > 0. \]

Therefore
\[ M(y_{n^2}, y_{n+1}, t) \geq M(y_{n-1}, y_{n+1}, t/q) \geq M(y_{n-2}, y_{n+1}, t/q^2) \geq \ldots \geq M(y_0, y_1, t/q^n) \]

Using (FM-6), we get
\[ \lim_{n \to \infty} M(y_{n^2}, y_{n+1}, t) = 1 \quad \forall t > 0. \]

Now for any positive integer \( p \),
\[ M(y_{n^2}, y_{n+p+1}, t) \geq M(y_{n^2}, y_{n+1}, t/p) \ast M(y_{n+1}, y_{n+2}, t/p) \ast \ldots \ast M(y_{n+p-1}, y_{n+p}, t/p). \]

Therefore
\[ \lim_{n \to \infty} M(y_{n^2}, y_{n+p+1}, t) = 1 \ast 1 \ast 1 \ast \ldots \ast 1 = 1. \]

Thus, \( \{y_n\} \) is a Cauchy sequence in \( X \). By completeness of \( (X, M, \ast) \), \( \{y_n\} \) converges to some point \( z \) in \( X \). Consequently, the subsequences \( \{A x_{2n^2}\}, \{B x_{2n+1}\}, \{S T x_{2n+1}\} \) and \( \{P Q x_{2n+2}\} \) of sequence \( \{y_n\} \) also converges to \( z \) in \( X \).

Case I. Suppose \( A \) is continuous, we have \( APQ x_{2n} \to Az \)

The semi-compatibility of the pair \( (A, PQ) \) gives that \( A(PQ) x_{2n} \to PQz \).

We know that the limit in a fuzzy metric space is unique, we get \( Az = PQz \)

Step 1. Putting \( x = z \) and \( y = x_{2n+1} \) in (2.1.5), we have
\[ M(Az, B x_{2n+1}, qt) \geq M(Az, S T x_{2n+1}, t) \ast M(Az, PQ z, t) \ast M(B x_{2n+1}, S T x_{2n+1}, t) \ast M(PQ z, S T x_{2n+1}, t) \ast M(PQ z, B x_{2n+1}, 2t). \]

Letting \( n \to \infty \) and using above results, we get
\[ M(Az, z, qt) \geq M(Az, z, t) \ast M(Az, Az, t) \]

\[ \ast M(z, t) \ast M(Az, z, t) \ast M(Az, 2t) \]

Now by Lemma 1.7, we get \( Az = z \). Hence \( Az = z = PQz \).

Step 2. Putting \( x = z \) and \( y = x_{2n+1} \) in (2.1.5), we have
\[ M(AQ z, B x_{2n+1}, qt) \geq M(AQ z, S T x_{2n+1}, t) \ast M(AQ z, P Q z, t) \ast M(B x_{2n+1}, S T x_{2n+1}, t) \ast M(P Q z, S T x_{2n+1}, t) \ast M(P Q z, B x_{2n+1}, 2t). \]

As \( AQ = QA \) and \( PQ = QP \), we have \( A(Q z) = Q(Az) = Qz \) and \( PQ(Q z) = Q(PQ z) = Qz \).

Letting \( n \to \infty \) and using above results, we get
\[ M(Q z, z, qt) \geq M(Q z, z, t). \]

Now by Lemma 1.7, we get \( Qz = z \)

Now \( PQ z = z \) implies that \( Pz = Qz \). Therefore \( Az = Pz = Qz = z \)

Step 3. Since \( A(X) \subset ST(X) \), there exists \( u \in X \) such that \( z = Az = ST u \). Putting \( x = x_{2n} \) and \( y = u \) in (2.1.5) then letting \( n \to \infty \) and using above results, we get
\[ M(z, Bu, qt) \geq M(Bu, z, t) \]

Using Lemma 1.7, we get \( z = Bu = ST u \). Which implies that \( u \) is a coincidence point of \( (B, ST) \). The weak compatibility of the pair \( (B, ST) \) gives that \( STBu = BSTu \) implies \( STz = Bz \).

Step 4. Putting \( x = x_{2n} \) and \( y = z \) in (2.1.5), then letting \( n \to \infty \) and using above results, we get
\[ M(z, B z, qt) \geq M(z, B z, t) \]

Using Lemma 1.7 \( Bz = z. \)

Thus \( STz = Bz = z. \)

Step 5. Putting \( x = x_{2n} \) and \( y = Tz \) in (2.1.5). Since \( BT = TB \) and \( ST = TS \), we have \( BTz = TBz = Tz \) and \( ST(Tz) = T(STz) = Tz \).

Letting \( n \to \infty \) and using above results, we get
\[ M(z, Tz, qt) \geq M(z, Tz, t) \ast M(z, z, t) \ast M(z, 2t) \]

\[ \ast M(Tz, Tz, t) \ast M(Tz, Tz, t) \ast M(Tz, 2t) \]

By using Lemma 1.7, we get \( Tz = z \). Now \( STz = z \) implies that \( Sz = z. \)

Hence \( Az = Bz = Sz = Tz = Pz = Qz = z. \)

Thus, \( z \) is a common fixed point of \( A, B, S, T, P \) and \( Q. \)
Case II. Suppose PQ is continuous, we have \((PQ)Ax_{2n} \rightarrow PQz\) and \((PQ)^{2}x_{2n} \rightarrow PQz\). As the pair \((A, PQ)\) is semi-compatible, we have \(APQx_{2n} \rightarrow PQz\).

Step 6. Putting \(x = PQx_{2n}\) and \(y = x_{2n+1}\) in (2.1.5), letting \(n \rightarrow \infty\) and using above results, we get \(M(PQz, z, qt) \geq M(PQz, z, t)\).

Now by Lemma 1.7, we get \(PQz = z\).

Step 7. Putting \(x = z\) and \(y = x_{2n+1}\) in (2.1.5), letting \(n \rightarrow \infty\) and using above results, we get \(M(Az, z, qt) \geq M(Az, z, t)\).

By Lemma 1.7, we get \(Az = z\).

Using step 2, we get \(Qz = z\). Now, \(PQz = z\) implies \(Pz = z\). Applying steps 3, 4 and 5, we get \(Az = Bz = Sz = Tz = Pz = Qz = z\).

Hence, \(Az = Bz = Sz = Tz = Pz = Qz = z\).

Thus \(z\) is a common fixed point of \(A, B, S, T, P\) and \(Q\).

Uniqueness.

Let \(v\) be another common fixed point of \(A, B, S, T, P\) and \(Q\), then \(v = Av = Bv = Sv = Tv = Pv = Qv\).

Putting \(x = z\) and \(y = v\) in (2.1.5), we get,

\[ M(z, v, qt) \geq M(z, v, t). \]

Now by Lemma 1.7, we get \(z = v\).

Therefore, \(z\) is unique common fixed point of \(A, B, S, T, P\) and \(Q\).

Remark 2.2. If we take \(Q = T = I\) in theorem 2.1 then the condition (2.1.4) is satisfied trivially and we get the following result.

Corollary 2.3. Let \((X, M, \ast)\) be a complete fuzzy metric space and let \(A, B, S, T, P\) and \(Q\) be mappings from \(X\) into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and the pair \((A, PQ)\) is semi-compatible and \((B, ST)\) is semi-compatible. Then \(A, B, S, T, P\) and \(Q\) have a unique common fixed point in \(X\).

Proof. As semi-compatibility implies weak compatibility, the proof follows from theorem 2.1.

Corollary 2.4. If we take \(a \ast b = \min \{a, b\}\) where \(a, b \in [0, 1]\), then in view of remark 2.2, corollary 2.3 is a generalization of the result of Singh and Chauhan[13], as only one mapping of the first pair in (2.1.8) is needed to be continuous, also first pair of self maps is taken semi-compatible and second pair of self maps is weakly compatible in (2.1.8) which are lighter conditions than that of compatibility.

Remark 2.5. In view of remark 2.2, corollary 2.3 is also a generalization of the result of Cho[1] in the sense of another functional inequality (2.1.9), semi-compatibility for first pair and weak compatibility for second pair and continuity for only one mapping in the first pair of (2.1.8).

Corollary 2.6. Let \((X, M, \ast)\) be a complete fuzzy metric space and let \(A, B, S, T, P\) and \(Q\) be mappings from \(X\) into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and the pair \((A, PQ)\) is semi-compatible and \((B, ST)\) is semi-compatible. Then \(A, B, S, T, P\) and \(Q\) have a unique common fixed point in \(X\).

Proof. As semi-compatibility implies weak compatibility, the proof follows from theorem 2.1.