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# Weakly, Semi Compatible Mappings and Common Fixed Points in Fuzzy Metric Spaces

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***Abstract.*** **Zadeh[15] proposed a mathematical way by defining the notion of fuzzy set. Fuzzy metric space was defined by several researchers to use this concept in Analysis and Topology. Jungck[6] proposed the concept of compatibility. The concept of compatibility in fuzzy metric space was introduced by Mishra et al.[11]. Later on, Jungck[7] generalized the concept of compatibility by introducing the concept of weak compatibility. Cho et al.[3] introduced the concept of semi-compatible maps in d-topological space. Singh and Jain[14] defined the concept of semi-compatible maps in fuzzy metric space.**

**Singh and Chauhan [13] and Cho[1] proved fixed point theorems in fuzzy metric space for four self maps using the concept of compatibility where two mappings needed to be continuous. The purpose of this paper is to obtain common fixed point theorem in fuzzy metric space for six self maps using the concept of semi-compatibility and weak compatibility and only one map is needed to be continuous, which generalizes the result of Singh and Chauhan[13] and Cho[1].**

**AMS (2000) Subject Classification. 54H25, 47H10.**

***Keywords. Common fixed point, fuzzy metric space, semi-compatible mappings, weakly compatible mappings***

## 1. INTRODUCTION

Jungck[6] proposed the concept of compatibility.The concept of compatibility in fuzzy metric space was introduced by Mishra et al.[11]. Later on, Jungck[7] generalized the concept of compatibility by introducing the concept of weak compatibility. Cho et al.[3] introduced the concept of semi-compatible maps in d-topological space. Singh and Jain[14] defined the concept of semi-compatible maps in fuzzy metric space. In this paper, we deal with the fuzzy metric space defined by Kramosil and Michalek [9] and modified by George and Veeramani [4].

Definition 1.1. [15] Let X be any set. A fuzzy set A in X is a function with domain in X and values in [0, 1].

Definition 1.2. [12] A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,

(ii) $*$ is continuous,

(iii) $a*1 = a$, for all $a \in [0, 1]$,

(iv) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Examples of t-norms are

$a * b = \min\{a, b\}$ (minimum t-norm),

$a * b = ab$ (product t-norm).

Definition 1.3. [4] The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions :

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \rightarrow [0, 1]$ is left continuous,

for all $x, y, z \in X$ and $t, s > 0$.

Definition 1.4. [5] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. Further, the sequence $\{x_n\}$ is said to be Cauchy if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$, for all $t > 0$ and $p > 0$. The space (X, M, *) is said to be complete if every Cauchy sequence in X is convergent in X.

Lemma 1.5. [5] *Let (X, M, *) be a fuzzy metric space. Then M(x, y, .) is non-decreasing for all* $x, y \in X$.

Lemma 1.6. [10] *Let (X, M, *) be a fuzzy metric space. Then M is a continuous function on* $X^2 \times (0, \infty)$.

Throughout this paper (X, M, *) will denote the fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Lemma 1.7.[11] *If there exists* $k \in (0,1)$ *such that* $M(x, y, kt) \geq M(x, y, t)$ *for all* $x, y \in X$ *and* $t > 0$, *then* $x = y$.

Lemma 1.8.[8] *The only t-norm * satisfying* $r * r \geq r$ *for all* $r \in [0,1]$ *is the minimum t-norm, that is* $a*b = \min\{a, b\}$ *for all* $a, b \in (0, 1)$.

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Lemma 1.9.[2] *Let $\{y_n\}$ be a sequence in a fuzzy metric space ( X, M, *) with condition (FM-6). If there exists a number $k \in (0, 1)$, such that*

$$M(y_{n+2}, y_{n+1}, k\,t) \geq M(y_{n+1}, y_n, t) \quad \text{for all } t > 0$$

*Then $\{y_n\}$ is a Cauchy sequence in X.*

Definition 1.10. [11] Let A and B be self mappings on a fuzzy metric space (X, M, *). The pair (A, B) is said to compatible if $\lim_{n\to\infty}$ $(ABx_n, BAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, for some $x \in$ X.

Definition 1.11. [14] Let A and B be self mappings on a fuzzy metric space (X, M, *). Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, Ax = Bx implies ABx = BAx.

It is known that a pair (A, B) of compatible maps is weakly compatible but converse is not true in general.

Definition 1.12. [14] A pair (A, B) of self maps of a fuzzy metric space (X, M, *) is said to be semi-compatible if $\lim_{n\to\infty} M(ABx_n, Bx, t) = 1$ for all $t > 0$, Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$.

It follows that if (A, B) is semi-compatible and Ax = Bx then ABx = BAx that means every semi-compatible pair of self maps is weak compatible but the converse is not true in general.

Cho[1] generalized the result of Singh and Chauhan[13] as follows:

Theorem 1.13. [1] *Let (X, M, *) be a complete fuzzy metric space and let A, B, S and T be mappings from X into itself such that the following conditions are satisfied :*

*(i) $AX \subset TX$, $BX \subset SX$,*

*(ii) S and T are continuous,*

*(iii) the pairs [A,S] and [B,T] are compatible,*

*(iv) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$ ,*

$$M(Ax, By, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t)$$

*Then A, B, S and T have a unique common fixed point in X.*

## 2. MAIN RESULT.

Our result generalizes the results of Singh and Chauhan [13] and Cho[1] as we are using the concept of semi-compatibility and weak compatibility which are lighter conditions than that of compatibility, also only one map is needed to be continuous. We are proving the result for six self maps using another functional inequality.

Theorem 2.1. *Let (X, M, *) be a complete fuzzy metric space with $r * r \geq r$ for all $r$ □ [0,1] and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied :*

*(2.1.1) $A(X) \subset ST(X)$, $B(X) \subset PQ(X)$*

*(2.1.2) either A or PQ is continuous;*

*(2.1.3) (A, PQ) is semi-compatible and (B, ST) is weakly compatible;*

*(2.1.4) PQ= QP, ST=TS, AQ = QA and BT = TB;*

*(2.1.5) there exists $q \in (0, 1)$ such that for every x, y in X and $t > 0$,*

$$M(Ax, By, qt) \geq M(Ax, STy, t) * M(Ax, PQx, t) * M(By, STy, t) * M(PQx, STy, t) * M(PQx, By, 2t).$$

*Then A, B, S, T, P and Q have a unique common fixed point in X.*

Proof. Let $x_0$ be an arbitary point in X. As $A(X) \subset ST(X)$ and $B(X) \subset PQ(X)$, then there exists $x_1, x_2 \in X$ such that $Ax_0 = STx_1 = y_0$ and $Bx_1 = PQx_2 = y_1$. We can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = STx_{2n+1} = Ax_{2n}$ and

$$y_{2n+1} = Bx_{2n+1} = PQx_{2n+2} \quad \text{for } n = 0, 1, 2, \ldots$$

Now, we first show that $\{y_n\}$ is a Cauchy sequence in X.

From (2.1.5), we have

$$M(y_{2n}, y_{2n+1}, qt) = M(Ax_{2n}, Bx_{2n+1}, qt)$$

$$\geq M(Ax_{2n}, STx_{2n+1}, t) * M(Ax_{2n}, PQx_{2n}, t) * M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQx_{2n}, STx_{2n+1}, t) * M(PQx_{2n}, Bx_{2n+1}, 2t).$$

$$= M(y_{2n}, y_{2n}, t) * M(y_{2n}, y_{2n-1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n+1}, 2t)$$

Using definition 1.2 and definition 1.3, we get

$$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t) \quad \text{(i)}$$

Thus we have

$$M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, t/q) * M(y_{2n+1}, y_{2n}, t/q) \quad \text{(ii)}$$

Putting (ii) in (i), we get

$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n}, t/q) * M(y_{2n+1}, y_{2n}, t/q)$

Using lemma 1.5 and lemma 1.8, we get

$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t/q)$

Proceeding in the similar manner, we get

$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t/q^m)$

Letting $m \to \infty$ and using (FM-6), we get

$M(y_{2n}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) \ \forall t > 0.$

In general,

$M(y_n, y_{n+1}, qt) \geq M(y_{n-1}, y_n, t) \ \forall t > 0.$

Therefore

$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t/q) \geq M(y_{n-2}, y_{n-1}, t/q^2) \geq \ldots \geq M(y_0, y_1, t/q^n)$

Using (FM-6), we get

$\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1 \ \forall t > 0.$

Now for any positive integer p,

$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) *\ldots\ldots * M(y_{n+p-1}, y_{n+p}, t/p).$

Therefore

$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1 * 1 * 1 * \ldots * 1 = 1.$

Thus, $\{y_n\}$ is a Cauchy sequence in X. By completeness of (X, M, *), $\{y_n\}$ converges to some point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{STx_{2n+1}\}$ and $\{PQx_{2n+2}\}$ of sequence $\{y_n\}$ also converges to z in X.

Case I. Suppose A is continuous, we have $APQx_{2n} \to Az$

The semi-compatibility of the pair (A, PQ) gives that $A(PQ)x_{2n} \to PQz$.

We know that the limit in a fuzzy metric space is unique, we get Az = PQz

Step 1. Putting x = z and y = $x_{2n+1}$ in (2.1.5), we have

$M(Az, Bx_{2n+1}, qt) \geq M(Az, STx_{2n+1}, t) * M(Az, PQz, t) * M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQz, STx_{2n+1}, t) * M(PQz, Bx_{2n+1}, 2t).$

Letting $n \to \infty$ and using above results, we get

$M(Az, z, qt) \geq M(Az, z, t) * M(Az, Az, t)$

$* M(z, z, t) * M(Az, z, t) * M(Az, z, 2t)$

$M(Az, z, qt) \geq M(Az, z, t).$

Now by Lemma 1.7, we get Az = z. Hence Az = z = PQz.

Step 2. Putting x = Qz and y = $x_{2n+1}$ in (2.1.5), we have

$M(AQz, Bx_{2n+1}, qt) \geq M(AQz, STx_{2n+1}, t) * M(AQz, PQQz, t) * M(Bx_{2n+1}, STx_{2n+1}, t) * M(PQQz, STx_{2n+1}, t) * M(PQQz, Bx_{2n+1}, 2t).$

As AQ = QA and PQ = QP, We have A(Qz) = Q(Az) = Qz and PQ(Qz) = QP(Qz) = Q(PQz) = Qz

Letting $n \to \infty$ and using above results, we get

$M(Qz, z, qt) \geq M(Qz, z, t).$

Now by Lemma 1.7, we get Qz = z

Now PQz = z implies that Pz = z. Therefore Az = Pz = Qz = z

Step 3. Since $A(X) \subset ST(X)$, there exists $u \in X$ such that z = Az = STu. Putting x = $x_{2n}$ and y = u in (2.1.5) then letting $n \to \infty$ and using above results, we get

$M(z, Bu, qt) \geq M(Bu, z, t)$

Using Lemma 1.7, we get z = Bu = STu. Which implies that u is a coincidence point of (B, ST). The weak compatibility of the pair (B, ST) gives that STBu = BSTu implies STz = Bz.

Step 4. Putting x = $x_{2n}$ and y = z in (2.1.5), then letting $n \to \infty$ and using above results, we get

$M(z, Bz, qt) \geq M(z, Bz, t)$

Using Lemma 1.7 Bz = z.

Thus STz = Bz = z.

Step 5. Putting x = $x_{2n}$ and y = Tz in (2.1.5). Since BT = TB and ST = TS, we have BTz = TBz = Tz and ST(Tz) = TS(Tz) = T(STz) = Tz

Letting $n \to \infty$ and using above results, we get

$M(z, Tz, qt) \geq M(z, Tz, t) * M(z, z, t)$

$* M(Tz, Tz, t) * M(z, Tz, t) * M(z, Tz, 2t)$

$M(z,Tz, qt) \geq M(z, Tz, t).$

By using Lemma 1.7, we get Tz = z. Now STz = z implies that Sz = z.

Hence Az = Bz = Sz = Tz = Pz = Qz = z.

Thus, z is a common fixed point of A, B, S, T, P and Q.

Case II. Suppose PQ is continuous, we have $(PQ)Ax_{2n} \to PQz$ and $(PQ)^2x_{2n} \to PQz$. As the pair (A, PQ) is semi-compatible, we have $APQx_{2n} \to PQz$.

Step 6. Putting $x = PQx_{2n}$ and $y = x_{2n+1}$ in (2.1.5), letting $n \to \infty$ and using above results, we get $M(PQz, z, qt) \geq M(PQz, z, t)$.

Now by Lemma 1.7, we get PQz = z.

Step 7. Putting x = z and $y = x_{2n+1}$ in (2.1.5), letting $n \to \infty$ and using above results, we get $M(Az, z, qt) \geq M(Az, z, t)$.

By Lemma 1.7, we get Az = z,

Using step 2, we get Qz = z. Now, PQz = z implies Pz = z. Therefore Az = Qz = Pz = z. Applying steps 3, 4 and 5, we get Bz = Sz = Tz = z.

Hence, Az = Bz = Sz = Tz = Pz = Qz = z

Thus z is a common fixed point of A, B, S, T, P and Q.

Uniqueness.

Let v be another common fixed point of A, B, S, T, P and Q, then

v = Av = Bv = Sv = Tv = Pv = Qv.

Putting x = z and y = v in (2.1.5), we get,

$M(z, v, qt) \geq M(z, v, t)$.

Now by Lemma 1.7, we get z = v

Therefore, z is unique common fixed point of A, B, S, T, P and Q.

Remark 2.2. If we take Q = T = I in theorem 2.1 then the condition (2.1.4) is satisfied trivially and we get the following result.

Corollary 2.3. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S and P be mappings from X into itself such that the following conditions are satisfied :*

*(2.1.6)* $A(X) \subset S(X)$, $B(X) \subset P(X)$

*(2.1.7) either A or P is continuous;*

*(2.1.8) (A, P) is semi-compatible and (B, S) is weakly compatible;*

*(2.1.9) there exists* $q \in (0, 1)$ *such that for every x, y in X and* $t > 0$ ,

$M(Ax, By, qt) \geq M(Ax, Sy, t)*M(Ax, Px, t)*M(By, Sy, t)*M(Px, Sy, t) * M(Px, By, 2t)$.

*Then A, B, S and P have a unique common fixed point in X.*

Remark 2.4. If we take a * b = min {a, b} where a, b $\in [0, 1]$, then in view of remark 2.2, corollary 2.3 is a generalization of the result of Singh and Chauhan[13], as only one mapping of the first pair in (2.1.8) is needed to be continuous, also first pair of self maps is taken semi-compatible and second pair of self maps is weakly compatible in (2.1.8) which are lighter conditions than that of compatibility.

Remark 2.5. In view of remark 2.2, corollary 2.3 is also a generalization of the result of Cho[1] in the sense of another functional inequality (2.1.9), semi-compatibility for first pair and weak compatibility for second pair and continuity for only one mapping in the first pair of (2.1.8).

Corollary 2.6. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and the pair (A, PQ) is semi-compatible and (B, ST) is semi- compatible. Then A, B, S, T, P and Q have a unique common fixed point in X.*

*Proof. As semi-compatibility implies weak compatibility, the proof follows from theorem 2.1.*

Corollary 2.7. *Let (X, M, *) be a complete fuzzy metric space and let A, B, S, T, P and Q be mappings from X into itself satisfying the conditions (2.1.1), (2.1.2), (2.1.4), (2.1.5) and the pair (A, PQ) is semi-compatible and (B, ST) is compatible.*

*Then A, B, S, T, P and Q have a unique common fixed point in X.*

*Proof. As compatibility implies weak compatibility, the proof follows from theorem 2.1.*

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