Weak Vector Saddle Point Theorem under
Vector ρ,η - Convexity

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Abstract:
In this paper we derive sufficient optimality condition and weak vector saddle point theorem and also duality results for non smooth multiobjective fractional programming problem have been proved.

Key words:
weak vector saddle point, non smooth multiobjective fractional programming, vector ρ-η convexity-invexity for locally Lipschitz theorem.

Introduction:
Xu described saddle point optimality criteria and established duality theorems in terms of generalized Lagrangian functions. Jeya Kumar defined ρ - invexity for non-smooth scalar-valued functions, studied duality theorem for non-smooth optimization problems and gave relationships between Saddle Points & optimality. But no serious attempt is made in utilizing the recent developed concept like Saddle Point Theorem under ν-ρ-η-convexity. Hence in this paper an attempt is made to fill the gap by developing vector valued functions under ν-ρ-η-convexity which is generalization of the concept of V – convexity and (β, η) convexity and establish sufficient optimality condition and weak vector saddle point theorems and also duality results for non-smooth multiobjective fractional programming problems are obtained.
Definition: The following are the definitions of Vector, \(\nu\)-\(\rho\)-\(\eta\)-convexity -invexity for locally Lipschitz functions:

**Definition:** Let \(f_i : R^n \to R\) and \(h_j : R^n \to R\) be locally Lipschitz functions for \(i = 1, 2, \ldots p\), and \(j = 1, 2, \ldots m\), respectively

(i) \(\frac{f_i}{g_i} : R^n \to R\) is \(V\)-\(\rho\)-\(\eta\)-convex with respect to functions \(\eta\) and \(\phi\) if there exists \(\alpha_i : R^n \times R^n \to R, i = 1, 2, \ldots p\) such that for any \(x, u \in R^n\) and any \(\xi_i \in \partial \frac{f_i}{g_i}(u)\),

\[
\alpha_i(x, u) \left( \frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)} \right) \geq \xi_i \eta(x, u) + \rho_i \| \theta(x, u) \|^2.
\]

(ii) \(h_j, j = 1, 2, \ldots m\) is \(V\)-\(\rho\)-\(\eta\)-convex with respect to functions \(\eta\) and \(\phi\) if there exist \(\beta_j : R^n \times R^n \to R, j = 1, 2, \ldots m\) such that for any \(x, u \in R^n\) and any \(\sigma_j \in dh_j(u)\),

\[
\beta_j(x, u) [h_j(x) - h_j(u)] \geq \phi_j \eta(x, u) + \sigma_j \| \theta(x, u) \|^2.
\]

Let \(u \in x\) is said to be a weak minimum of (FP) if there exists no \(x \in X\) such that \(\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}, i = 1, 2, \ldots P\)

**Formulation**

**Primal problem:**

Consider the following non-smooth multi objective fractional programming problems.

\[\text{(FP)} : \quad \min_{x \in X} \max_{1 \leq i \leq p} \left[ \frac{f_i(x)}{g_i(x)} \right],\]

subject to \(h_j(x) \leq 0, j = 1, 2, \ldots m,\)

where \(f_i : R^n \to R, i = 1, 2, \ldots p\) and \(h_j : R^n \to R, j = 1, 2, \ldots p\) and \(h_j : R^n \to R, j = 1, 2, \ldots m\) are locally lipschitz function.
Dual Problem: -

For the problem (FP), consider the dual problem (FD):

\[(FD) \quad \max \frac{f_i(u)}{g_i(u)}\]

subject to \(0 \in \sum_{i=1}^{p} \tau_i \partial \left[ \frac{f_i(u)}{g_i(u)} \right] + \sum_{j=1}^{m} \lambda_j \partial [h_j(u)]\)

\[
\lambda_j h_j(u) \geq 0, j = 1, 2, ..., m \\
\tau_i \geq 0, i = 1, 2, ...p \\
\lambda_j \geq 0, j = 1, 2, ...m,
\]

where \(e = (1, 1, ...1)^t \in \mathbb{R}^p\)

Sufficiency and Duality Theorems:

In this section we show that the generalized karush-kuhn-tucker conditions are sufficient for a weak minimum of (FP)

Theorem: - Let \((u, \tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m\) satisfy the generalized karush-kuhn-Tucker conditions as follows.

\[
0 \in \sum_{i=1}^{p} \tau_i \partial \left[ \frac{f_i(u)}{g_i(u)} \right] + \sum_{j=1}^{m} \lambda_j \partial [h_j(u)]
\]

\[
h_j(u) \leq 0, \lambda_j h_j(u) = 0 , j = 1, 2, ...m, \\
\tau_i \geq 0, i = 1, 2, ..., p \\
\tau^t e > 0 \\
\lambda_j \geq 0 , j = 1, 2, ....m
\]

If \(\frac{f_i}{g_i}\) is \(V-\rho - \eta\)-convex and \(h_j\) is \(v-\sigma\)-convex with respect to the same functions \(\eta\) and \(\sigma\) and

\[
\sum_{i=1}^{p} \tau_i p_i + \sum_{j=1}^{m} \lambda_j \sigma_j \geq 0, \text{ then } u \text{ is weak minimum of } (FP).
\]

Proof: - Since \(0 \in \sum_{i=1}^{p} \tau_i \partial \left[ \frac{f_i(u)}{g_i(u)} \right] + \sum_{j=1}^{m} \lambda_j \partial [h_j(u)]\), there exist
\[ \xi_i \in \partial \left[ \frac{f_i(u)}{g_i(u)} \right] \text{ and } \phi_j \in \partial h_j(u) \text{ such that} \]

\[ \sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \phi_j = 0 \quad (4.1) \]

Suppose that \( u \) is not a weak minimum of (FP). Then there exists \( x \in X \) such that

\[ \frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}, \ i = 1, 2, ..., p, \]

since

\[ \alpha_i(x, u) > 0 \text{ we have} \]

\[ \alpha_i(x, u) \frac{f_i(x)}{g_i(x)} < \alpha_i(x, u) \frac{f_i(u)}{g_i(u)}, \ i = 1, 2, ..., p, \]

by the \( V - \rho - \eta \)-convex of \( \frac{f_i}{g_i} \), for all \( i \),

\[ \xi_i \eta(x, u) + \rho_i \left\| \theta(x, u) \right\|^2 < 0 \text{ for each } \xi_i \in \partial \left[ \frac{f_i(u)}{g_i(u)} \right] \]

Hence, we have

\[ \sum_{i=1}^{p} \tau_i \xi_i \eta(x, u) + \sum_{i=1}^{p} \tau_i \rho_i \left\| \theta(x, u) \right\|^2 < 0 \]

Since \( \sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \sigma_j \geq 0 \) it follows from (1)

\[ \sum_{j=1}^{m} \lambda_j \phi_j \eta(x, u) + \sum_{j=1}^{m} \lambda_j \sigma_j \left\| \theta(x, u) \right\|^2 > 0 \]

Then, by the \( v - \sigma \)-convexity of \( h_\mu \), we have

\[ \sum_{j=1}^{m} \beta_j(x, u) \left[ \lambda_j h_j(x) - \lambda_j h_j(u) > 0 \right] \]
since $\lambda_j h_j (u) = 0, \ j = 1, 2, \ldots, m$, we have $\sum_{j=1}^{m} \beta_j (x,u) \lambda_j h_j (x) > 0$ which contradicts the conditions $\beta_j (x, u) > 0, \lambda_j \geq 0$ and $h_j (x) \leq 0$.

Thus $u$ is weak minimum of (FP).

Hence the proof.

**Weak Duality Theorem:**

Let $x$ be a feasible for (FP) and $(u, \tau, \lambda)$ a feasible for (FD), assume that

$$\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho_j \geq 0.$$ If $f_i / g_i$ is $\nu$- $\rho$ - $\eta$-convex and $h_j$ is $\nu$ - $\sigma$ -convex with respect to same functions $\eta$ and $\theta$, then

$$\frac{f_i (x)}{g_i (x)} < \frac{f_i (u)}{g_i (u)}$$

From feasibility conditions and $\beta_j (x, u) > 0$, we have

$$\beta_j (x,u) \lambda_j h_j (x) \leq \beta_j (x,u) \lambda_j h_j (u).$$ Then, by the $\nu$ - $\sigma$ -convexity of $h_j$, we have

$$\lambda_j \phi_j \eta (x, u) + \lambda_j \sigma_j \| \theta (x, u) \|^2 \leq 0,$$

for each $\phi_j \in \partial h_j (u)$. Hence we have

$$\sum_{j=1}^{m} \lambda_j \phi_j \eta (x, u) + \sum_{j=1}^{m} \lambda_j \sigma_j \| \theta (x, u) \|^2 \leq 0$$

for each $\phi_j \in \partial h_j (u)$.

Since $0 \in \sum_{j=1}^{p} \tau_j \partial \left[ \frac{f_i (u)}{g_i (u)} \right] + \sum_{j=1}^{m} \lambda_j \partial h_j (u)$,

There exists $\xi_i \in \partial \left[ \frac{f_i (u)}{g_i (u)} \right]$ and $\phi_j \in \partial h_j (u)$ such that

$$\sum_{j=1}^{m} \lambda_j \phi_j \eta (x, u) = 0$$

Hence, from the assumption $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho_j \geq 0$

We have,
\[ \sum_{j=1}^{p} \lambda_j \zeta_j \eta(x,u) + \sum_{i=1}^{p} \tau_i \rho_i \mid \theta(x,u) \mid^2 \geq 0 \]

from the V- \( \rho \) - \( \eta \)-convex of \( \frac{f_i}{g_i} \), we have

\[ \sum_{i=1}^{p} \alpha_i (x,u) \left[ \tau_i \frac{f_i(x)}{g_i(x)} - \tau_i \frac{f_i(u)}{g_i(u)} \right] \geq 0 \]

Since \( \alpha_i (x,u) > 0, \tau_i \geq 0, \tau^* e = 1 \) we have

\[ \frac{f_i(x)}{g_i(x)} \downarrow \frac{f_i(u)}{g_i(u)} \]

**Strong Duality Theorem:**

Let \( \overline{x} \) be a weak minimum of (FP) at which constraint qualification is satisfied then there exists \( \overline{\tau} \in \mathbb{R}^p \) and \( \overline{\lambda} \in \mathbb{R}^m \Rightarrow (\overline{x}, \overline{\tau}, \overline{\lambda}) \) is feasible for (FD).

If \( \frac{f_i}{g_i} \) is V- \( \rho \) - \( \eta \)-convex and \( h_j \) is v- \( \sigma \)-convex with respect to same function \( \eta \) and \( \theta \), then \( (\overline{x}, \overline{\tau}, \overline{\lambda}) \) is a weak maximum of (FD)

**Proof :-**

Since \( \overline{x} \) is weak minimum of (FP) and a constraint qualification is satisfied \( \overline{x} \), from the generalized Karush-Kuhn-Tucker theorem there exist

\( \tau_i \in \mathbb{R}^p \) and \( \lambda_j \in \mathbb{R}^m \) such that

\[ 0 \in \sum_{i=1}^{p} \tau_i \frac{\partial}{\partial x} \left( \frac{f_i(x)}{g_i(x)} \right) + \sum_{j=1}^{m} \lambda_j \frac{\partial h_j}{\partial x} (\overline{x}) \]

\( \lambda_j h_j (\overline{x}) = 0, \ j = 1, 2, \ldots, m \)

\( \tau_i \geq 0, \ i = 1, 2, \ldots p \)

\( \tau^* e > 0 \)

\( \lambda_j \geq 0, \ j = 1, 2, \ldots, m \)
Since $\tau_i \geq 0$, $i = 1, 2, ... p$ and $\tau^+ e > 0$,
we can consider that $\tau_i$ and $\tau_j$ as

$$\tau_i = \frac{\tau_i}{\lambda_j}, \quad \lambda_j = \frac{\sum_{i=1}^{p} \tau_i}{\sum_{i=1}^{p} \tau_j}$$

Then $(\bar{x}, \tau, \lambda)$ is feasible for (FD).

Since $\bar{x}$ is feasible for (FP), it follows from weak duality that $\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}$ for any feasible $u$ for (FD). Hence $(\bar{x}, \tau, \lambda)$ is a weak maximum of (FD).

**Weak Vector Saddle – Point Theorem:**

In this section, we prove Weak Vector Saddle Point theorem for the non smooth multiobjective fractional program (FP) in which functions are locally lipschitz. For the problem (FP), a point $(x, \tau, \lambda)$ is said to be a critical point if, $x$ is a feasible point for (FP), and

$$O \in \partial \left[ \sum_{i=1}^{p} \tau_i \partial \left[ \frac{f_i(x)}{g_i(x)} \right] + \sum_{j=1}^{m} \lambda_j h_j(x) \right]$$

$\lambda_j h_j(x) = 0, \lambda_j \geq 0, j = 1, 2, ... m$

$\tau_i \geq 0, \quad i = 1, 2, ... p,$

$\tau^t e = 1$

Note, that

$$\partial \left[ \sum_{i=1}^{p} \tau_i \frac{f_i(x)}{g_i(x)} + \sum_{j=1}^{m} \lambda_j h_j(x) \right] = \sum_{i=1}^{p} \tau_i \partial \left[ \frac{f_i(x)}{g_i(x)} \right] + \sum_{j=1}^{m} \lambda_j h_j(x)$$

Let $L(x, \lambda) = \frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x) e,$
Where \( x \in \mathbb{R}^m \) and \( \lambda \in \mathbb{R}^m_+ \). Then, a point \((x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m_+\) is said to be a weak vector Saddle Point if whenever we introduce \( L(x, \lambda, \mu) \) it means that \( L(x, \lambda, \mu) \) has \( p \) components like

\[
\begin{pmatrix}
\frac{f_i(x)}{g_i(x)} - \lambda_j h_j(x) + \mu h_j(x)e, & i = 1, 2, \ldots, p, & j = 1, 2, \ldots, m
\end{pmatrix}
\]

for all \( x \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R}^m_+ \).

**Theorem Saddle Point Condition:** Let \((x, \tau, \lambda)\) be a critical point of (FP) assume that \( \frac{f_i(x)}{g_i(x)} + \lambda^\tau h_j(x)e \) is \( V - \rho - \eta \)-convex with respect to function \( \eta \) and \( \theta \) and \( \sum \tau_i \rho_i \geq 0 \). Then \((x, \lambda)\) is a weak vector Saddle Point of (FP).

**Proof :-** Since \((x, \tau, \lambda)\) is a critical point for (FP), there exists

\[
\xi_i \in \partial \left[ \sum_{i=1}^{p} f_i(x) + \sum_{j=1}^{m} \lambda_j g_j(x) \right]
\]

such that

\[
\sum_{i=1}^{p} \tau_i \xi_i = 0 \quad \text{since} \quad \sum_{i=1}^{p} \tau_i \rho_i \geq 0
\]

\[
\sum_{i=1}^{p} \tau_i \xi_i \eta(x, x) + \sum_{i=1}^{p} \tau_i \rho_i \theta(x, x) \geq 0
\]

Then, by the \( V - \rho - \eta \) - inexactness of \( \frac{f_i(x)}{g_i(x)} + \lambda^\tau h_j(x)e \).

we have

\[
\sum_{i=1}^{p} \alpha_i (x, x) \tau \left[ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x)}{g_i(x)} \right] + \left[ \lambda_j h_j(x) - \lambda_j h_j(x) \right] \geq 0 \quad \text{for any} \ x \in \mathbb{R}^n. \text{Since} \ \alpha_i (x, x) > 0, \ \tau_i \geq 0 \ \text{and} \ \tau^\top e = 1
\]

\[
\frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x)e > \frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x)e
\]

(2)
for any $x \in R^n$, that is $L(\bar{x}, \lambda) > L(x, \bar{\lambda})$, for any $x \in R^n$.

Now, since $\lambda_j h_j(\bar{x}) \leq 0$ for any $\lambda \in R^n_+$.

$$\bar{\lambda}_j h_j(x) - \bar{\lambda}_j h_j(\bar{x}) \geq 0, \text{for any } \lambda \in R^n_+$$

Thus, $\frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x) e \geq \left( \frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x) e \right) \in R^p$

and hence, $L(\bar{x}, \lambda) > L(\bar{x}, \bar{\lambda})$, for any $\lambda \in R^n_+$.

Therefore, $(\bar{x}, \bar{\lambda})$ is a weak vector Saddle Point of (FP).

**Theorem :-** If there exists $\bar{\lambda} \in R^n_+$ such that $(\bar{x}, \bar{\lambda})$ is a weak Vector Saddle Point, then $\bar{x}$ is a weak minimum of (FP).

**Proof :-** Assume that $(\bar{x}, \bar{\lambda})$ is a weak Vector Saddle Point from left of 2nd Equation.

$$\frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x) e \geq \frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x) e, \text{ for any } \lambda \in R^n_+.$$

Thus $\lambda_j h_j(x) e \geq \lambda_j h_j(x) e$ for any $\lambda \in R^n_+$, and hence we have

$$\lambda_j h_j(x) \leq \lambda_j h_j(x), \text{ for any } \lambda \in R^n_+, \quad (3)$$

Since $\lambda_j$ can be taken arbitrary large, $h_j(\bar{x}) \leq 0$. Hence $\lambda_j h_j(\bar{x}) \leq 0$.

Let $\lambda_j = 0$ in (3), $\lambda_j h_j(\bar{x}) \geq 0$. Therefore, $\lambda_j h_j(\bar{x}) = 0$. Now, from the right inequality of (2) equation and $\bar{\lambda}_j h_j(\bar{x}) = 0$, we have for any feasible $x$ for (FP),

$$\frac{f_i(x)}{g_i(x)} > \frac{f_i(x)}{g_i(x)}$$

Hence $\bar{x}$ is a weak minimum for (FP).

Hence the proof.
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