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Weak Vector Saddle Point Theorem under Vector ρ,η - Convexity

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Abstract:

In this paper we derive sufficient optimality condition and weak vector saddle point theorem and also duality results for non smooth multiobjective fractional programming problem have been proved.

Key words:

weak vector saddle point, non smooth multiobjective fractional programming, vector ρ-η convexity-invexity for locally Lipschitz theorem.

Introduction:

Xu described saddle point optimality criteria and established duality theorems in terms of generalized Lagrangian functions. Jeya Kumar defined ρ- invexity for non-smooth scalar-valued functions, studied duality theorem for non-smooth optimization problems and gave relationships between Saddle Points & optimality. But no serious attempt is made in utilizing the recent developed concept like Saddle Point Theorem under v-ρ-η-convexity. Hence in this paper an attempt is made to fill the gap by developing vector valued functions under v-ρ-η-convexity which is generalization of the concept of V – convexity and (β, η) convexity and establish sufficient optimality condition and weak vector saddle point theorems and also duality results for non-smooth multiobjective fractional programming problems are obtained.

Definition:

The following are the definitions of Vector, v- ρ -η-convexity -invexity for locally Lipschitz functions:

Definition:- Let $\frac{f_i}{g_i}: R^n \to R$ and $h_j : R^n \to R$ be locally Lipschitz functions for i = 1, 2, …p, and j = 1, 2, ….m, respectively

(i) $\frac{f_i}{g_i}, i = 1, 2, \ldots\ldots p$ is V- ρ - η-convex with respect to functions η and

$\theta : R^n \times R^n \to R^n$ if there exists $\alpha_i : R^n \times R^n \to R_+ \setminus \{ 0 \}$ and $\rho_i \in R$, i = 1, 2, …p such that for any x, u $\in R^n$ and any $\xi_i \in \partial \frac{f_i}{g_i}(u)$,

$$\alpha_i (x, u) \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(u)}{g_i(u)} \right) \geq \xi_i \, \eta(x, u) + \rho_i \, \| \, \theta(x, u) \, \|^2 .$$

(ii) h_j, j = 1, 2,…m is V- ρ - η-convex with respect to functions η and

$\theta : R^n \times R^n \to R^n$ if there exist $\beta_j : R^n \times R^n \to R_+ \setminus \{ 0 \}$ and $\sigma_j \in R$, j = 1, 2, ….m

Such that for any x, u $\in R^n$ and any $\phi_j \in dhj(u)$.

$$\beta_j (x, u) [h_j(x) - h_j(u)] \geq \phi_j \, \eta (x, u) + \sigma_j \, \| \, \theta(x, u) \, \|^2$$

Let u $\in$ x is said to be a weak minimum of (FP) if there exists no x$\in$X such that $\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}$, i = 1, 2, …. P

Formulation

Primal problem:

Consider the following non-smooth multi objective fractional programming problems.

(FP) : $\underset{x \in X}{Min} \; \underset{1 \leq i \leq p}{Max} \left[\frac{f_i(x)}{g_i(x)} \right]$,

subject to $h_j(x) \leq 0$, j = 1, 2, …m,

where $\frac{f_i}{g_i}: R^n \to R, i = 1, 2, \ldots p$ and $h_j : R^n \to R$, i= 1, 2….p and $h_j : R^n \to R$, j = 1, 2, …m are locally lipschitz function.

Dual Problem: -

For the problem (FP), consider the dual problem (FD) :

(FD) max $\left[\frac{f_i\,(u)}{g_i\,(u)}\right]$

$$\text{subject to } O \in \sum_{i=1}^{p}\tau_i\,\partial\left[\frac{f_i\,(u)}{g_i\,(u)}\right]+\sum_{j=1}^{m}\lambda_j\;\partial\left[hj\,(u)\right]$$

$$\lambda_j\; h_j\,(u) \geq 0,\; j = 1, 2, \ldots, m$$

$$\tau_i \geq 0,\; i = 1, 2, \ldots p$$

$$\lambda_j \geq 0,\; j = 1, 2, \ldots m,$$

where $e = (1, 1, \ldots 1)^t \in R^p$

Sufficiency and Duality Theorems:

In this section we show that the generalized karush-kuhn-tucker conditions are sufficient for a weak minimum of (FP)

Theorem: - Let $(u, \tau, \lambda) \in R^n \times R^p \times R^m$ satisfy the generalized karush-kuhn-Tucker conditions as follows.

$$O \in \sum_{i=1}^{p}\tau_i\,\partial\left[\frac{f_i\,(u)}{g_i\,(u)}\right]+\sum_{j=1}^{m}\lambda_j\;\partial\left[hj\,(u)\right]$$

$$h_j\,(u) \leq 0,\; \lambda_j\; h_j\,(u) = 0\;,\; j = 1, 2, \ldots m,$$

$$\tau_i \geq 0,\; i = 1, 2, \ldots, p$$

$$\tau^t\, e > 0$$

$$\lambda_j \geq 0\;,\; j = 1, 2, \ldots .m$$

If $\frac{f_i}{g_i}$ is V- ρ - η-convex and hj is v-σ-convex with respect to the same functions η and θ and

$\sum_{i=1}^{p}\tau_i\; p_i\; + \sum_{j=1}^{m}\lambda_j\sigma_j \geq 0$, then u is weak minimum of (FP).

Proof :- Since $0 \in \sum_{i=1}^{p}\tau_i\,\partial\left[\frac{f_i\,(u)}{g_i\,(u)}\right]+\sum_{j=1}^{m}\lambda_j\;\partial\left[hj\,(u)\right]$, there exist

$\xi_i \in \partial \left[\frac{f_i\,(u)}{g_i\,(u)} \right]$ *and* $\phi j \in \partial hj\,(u)$ such that

$$\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \phi_j = 0 \qquad (4.1)$$

Suppose that u is not a weak minimum of (FP). Then there exists x ∈ X such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}, \text{ i= 1, 2, ...p,}$$

since

α_i (x, u) > 0 we have

$$\alpha_i\,(x,u)\,\frac{f_i\,(x)}{g_i\,(x)} < \alpha_i\,(x,u)\,\frac{f_i\,(u)}{g_i\,(u)}, i = 1, 2, \ldots .p$$

by the V- ρ - η-convex of $\frac{f_i}{g_i}$, for all i ,

$$\xi_i \eta\,(x,u) + \rho_i \parallel \theta\,(x,u) \parallel^2 < 0 \text{ for each } \xi_i \in \partial \left[\frac{f_i\,(u)}{g_i\,(u)} \right]$$

Hence, we have

$$\sum_{i=1}^{p} \tau_i \xi_i\, \eta\,(x,u) + \sum_{i=1}^{p} \tau_i \rho_i \parallel \partial(x,u) \parallel^2 < 0$$

Since $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j\, \sigma_j \geq 0$ it follows from (1)

$$\sum_{j=1}^{m} \lambda_j \phi_j\, \eta(x,u) + \sum_{j=1}^{m} \lambda_j\, \sigma_j \parallel \theta\,(x,u) \parallel^2 > 0$$

Then, by the v - σ - con vexity of h_j, we have

$$\sum_{j=1}^{m} \beta_j\,(x,u)\left[\lambda_j\, h_j\,(x) - \lambda_j\, h_j\,(u) > 0\right]$$

since λ_j h_j (u) = 0, j = 1, 2, …, m, we have $\sum_{j=1}^{m} \beta_j (x,u)\, \lambda_j\, h_j(x) > 0$ which contradicts the conditions β_j (x, u) > 0, $\lambda_j \geq 0$ and h_j (x) $\leq$ 0.

Thus u is week minimum of (FP).

Hence the proof.

Weak Duality Theorem:

Let x be a feasible for (FP) and (u, τ, λ) a feasible for (FD), assume that $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \rho_j \geq 0$. If $\frac{f_i}{g_i}$ is V-ρ- η-convex and h_j is v - σ -convex with respect to same functions η and θ, then

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(u)}{g_i(u)}$$

From feasibility conditions and β_j (x, u) > 0, we have

β_j(x,u) $\lambda_j h_j(x) \leq \beta_j$ (x, u) λ_j h_j (u). Then, by th v - σ - convexity of h_j, we have

$$\lambda_j\, \phi_j\, \eta\, (x, u) + \lambda_j\, \sigma_j\, \| \theta(x, u) \|^2 \leq 0,$$

for each $\phi_j \in \partial h_j(u)$. Hence we have

$\sum_{j=1}^{m} \lambda_j\, \phi_j\, \eta\,(x,u) + \sum_{j=1}^{m} \lambda_j\, \sigma_j\, \|\, \theta\,(x,u)\, \|^2$ $\leq$ for each $\phi_j \in \partial h_j(u)$.

Since O$\in \sum_{j=1}^{p} \tau_j\, \partial \left[\frac{f_i\,(u)}{g_i\,(u)}\right] + \sum_{j=1}^{m} \lambda_j\, \partial hj\,(u),$

There exists $\xi_i \in \partial \left[\frac{f_i\,(u)}{g_i\,(u)}\right]$ and $\phi_j \in \partial h_j(u)$ such that

$$\sum_{j=1}^{p} \tau_i\, \xi_i + \sum_{j=1}^{m} \lambda_j\, \phi_j\, \eta\,(x,u) = 0$$

Hence, from the assumption $\sum_{i=1}^{p} \tau_i\, \rho_i + \sum_{j=1}^{m} \lambda_j\, \rho_j \geq 0$

We have,

$$\sum_{j=1}^{p} \lambda_i \xi_i \ \eta(x,u) + \sum_{i=1}^{p} \tau_i \ \rho_i \| \ \theta(x,u) \ \|^2 \geq 0$$

from the V- ρ - η-convex of $\frac{fi}{gi}$, we have

$$\sum_{i=1}^{p} \alpha_i \ (x,u) \left[\tau_i \frac{f_i(x)}{g_i(x)} - \tau_i \frac{f_i(u)}{g_i(u)} \right] \geq 0$$

Since α_i (x, u) > 0, $\tau_i \geq 0$, $\tau^+ e = 1$ we have

$$\frac{f_i(x)}{g_i(x)} \ \not\leq \ \frac{f_i(u)}{g_i(u)}$$

Strong Duality Theorem:

Let $\bar{x}$ be a weak minimum of (FP) at which constraint qualification is satisfied then there exists $\bar{\tau} \in R^p$ and $\bar{\lambda} \in R^m \Rightarrow (\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for (FD).

If $\frac{f_i}{g_i}$ is V- ρ - η-convex and hj is v - σ - convex with respect to same function η and θ, then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a weak maximum of (FD)

Proof :-

Since $\bar{x}$ is weak minimum of (FP) and a constraint qualification is satisfied $\bar{x}$, from the generalized Karush-Kuhn-Tucker theorem there exist

$\tau_i \in R^p$ and $\lambda_j \in R^m$ such that

$$O \in \sum_{i=1}^{p} \tau_i \ \partial \left[\frac{f_i \ (x)}{g_i \ (\bar{x})} \right] + \sum_{j=1}^{m} \lambda_j \ \partial h_j \ (\bar{x})$$

$\lambda_j \ h_j \ (\bar{x}) = 0$, j = 1, 2, …. m

$\tau_i \geq 0$, i = 1, 2, … p

$\tau^+ e > 0$

$\lambda_j \geq 0$, j = 1, 2, ….m

Since $\tau_i \geq 0$, i = 1, 2, ….p and $\tau^+ e > 0$,

we can consider that $\bar{\tau}$ i and $\bar{\tau}$ j as

$$\bar{\tau}_i = \frac{\tau_i}{\sum_{i=1}^{p} \tau_i}, \bar{\lambda}_j = \frac{\lambda_j}{\sum_{i=1}^{p} \tau_j}$$

Then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is feasible for (FD).

Since $\bar{x}$ is feasible for (FP), it follows from weak duality that $\frac{f_i(\bar{x})}{g_i(\bar{x})} <$

$\frac{f_i(\bar{u})}{g_i(\bar{u})}$ |

for any feasible u for (FD). Hence $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a weak maximum of (FD).

Weak Vector Saddle – Point Theorem :

In this section, we prove Weak Vector Saddle Point theorem for the non smooth multiobjective fractional program (FP) in which functions are locally lipschitz. For the problem (FP), a point (x, τ, λ) is said to be a critical point if, x is a feasible point for (FP), and

$$O \in \partial \left[\sum_{i=1}^{p} \tau_i \, \partial \left[\frac{f_i(x)}{g_i(x)} \right] + \sum_{j=1}^{m} \lambda_j \, h_j(x) \right]$$

$$\lambda_j \, h_j(x) = 0, \lambda_j \geq 0, \text{ j = 1, 2, …. m}$$

$$\tau_i \geq 0, \text{ i = 1, 2, … p,}$$

$$\tau^t e = 1$$

Note, that

$$\partial \left[\sum_{i=1}^{p} \tau_i \frac{f_i(x)}{g_i(x)} + \sum_{j=1}^{m} \lambda_j \, h_j(x) \right] = \sum_{i=1}^{p} \tau_i \, \partial \left\{ \left[\frac{f_i(x)}{g_i(x)} \right] + \sum_{j=1}^{m} \lambda_j \, h_j(x) \right\}$$

Let $L(x, \lambda) = \frac{f_i(x)}{g_i(x)} + \lambda_j h_j(x) e$, Where x $\in$ R^m and $\lambda \in R^m{}_+$. Then, a point $(\bar{x}, \bar{\lambda}) \in R^n \times R^m{}_+$ is said to be a weak vector Saddle Point if when ever we introduce L (x, λ, μ) it means that L (x, λ, μ) has p – components like $\left(\frac{f_i\ (x)}{g_i\ (x)} - \lambda_j\ h_j\ (x)\right) + \mu^t h_j(x)e,$ i = 1, 2, ...p, j = 1, 2,...m

$$L\ (\bar{x}, \lambda) \nless L(\bar{x}, \bar{\lambda}) \nless L(x, \bar{\lambda})$$

for all x $\in$ R^n and $\lambda \in R^m{}_+$

Theorem Saddle Point Condition:- Let $(\bar{x}, \bar{\tau}, \bar{\lambda})$ be a critical point of (FP) assume that $\frac{f_i(\bar{x})}{g_i(\bar{x})} + \bar{\lambda}^\tau\ h_j(\bar{x})\ e$ is V- ρ - η-convex with respect to function η and θ and $\sum_{i=1}^{p} \bar{\tau_i}\ \rho_i \geq 0$. Then $(\bar{x}, \bar{\lambda})$ is a weak vector Saddle Pont of (FP).

Proof :- Since $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a critical point for (FP), there exists

$$\xi_i \in \partial\left[\sum_{i=1}^{p} \frac{f_i\ (\bar{x})}{g_i(\bar{x})} + \sum_{j=1}^{m} \bar{\lambda}_j\ g_j\ (\bar{x})\right]$$

such that

$$\sum_{i=1}^{p} \bar{\tau}_i\ \xi_i = 0 \quad \text{since} \quad \sum_{i=1}^{p} \bar{\tau}_i\ \rho_i \geq 0$$

$$\sum_{i=1}^{p} \tau_i\ \xi_i\ \eta\ (x, \bar{x}) + \sum_{i=1}^{p} \bar{\tau}_i\ \rho_i \parallel \theta\ (x, \bar{x}) \parallel^2 \geq 0$$

Then, by the V- ρ - invexity of $\frac{f_i(\bar{x})}{g_i(\bar{x})} + \bar{\lambda}^t h_j\ (\bar{x})e,$

we have $\sum_{i=1}^{p} \alpha_i\ (x, \bar{x})\ \bar{\tau}_i \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right] + \left[\bar{\lambda}_j\ h_j\ (x) - \bar{\lambda}_j\ h_j\ (\bar{x})\ \right] \geq 0$ for any x $\in$ R^n. Since α_i (x, $\bar{x}$) > 0, $\bar{\tau}_i \geq 0$ and $\bar{\tau}^t\ e = 1$

$$\frac{f_i\ (\bar{x})}{g_i(\bar{x})} + \bar{\lambda}_j\ h_j\ (\bar{x})\ e > \frac{f_i(x)}{g_i(x)} + \bar{\lambda}_j\ h_j(x)\ e \qquad (2)$$

for any $x \in R^n$,that is L $(\bar{x},\bar{\lambda}) > L(x,\bar{\lambda})$, for any $x \in R^n$.

Now, since $\lambda_j h_j(\bar{x}) \leq 0$ for any $\lambda \in R^m_+$.

$$\bar{\lambda}_j h_j(x) - \bar{\lambda}_j h_j(\bar{x}) \geq 0, \text{for any} \lambda \in R^m_+$$

Thus, $\frac{f_i(x)}{g_i(x)} + \bar{\lambda}_j h_j(\bar{x}) e - \left(\frac{f_i(x)}{g_i(x)} + \bar{\lambda}_j h_j(\bar{x}) e \right) \in R^p_+$

and hence, L $(\bar{x},\lambda) > L(\bar{x},\bar{\lambda})$, for any $\lambda \in R^m_+$.

Therefore, $(\bar{x},\bar{\lambda})$ is a weak vector Saddle Point of (FP).

Theorem :- If there exists $\bar{\lambda} \in R^m_+$ such that $(\bar{x},\bar{\lambda})$ is a weak Vector Saddle Point, then $\bar{x}$ is a weak minimum of (FP).

Proof :- Assume that $(\bar{x},\bar{\lambda})$ is a weak Vector Saddle Point from left of 2nd Equation.

$\frac{f_i(\bar{x})}{g_i(\bar{x})} + \lambda_j h_j(\bar{x}) e \not> \frac{f_i(\bar{x})}{g_i(\bar{x})} + \bar{\lambda}_j h_j(\bar{x}) e$, for any $\lambda \in R^m_+$.

Thus $\lambda_j h_j(\bar{x}) e \not> \bar{\lambda}_j h_j(\bar{x}) e$ for any $\lambda \in R^m_+$,and hence we have

$\lambda_j h_j(\bar{x}) \leq \bar{\lambda}_j h_j(\bar{x})$, for any $\lambda \in R^m_+$ (3)

Since λ_j can be taken arbitrary large, $hj(\bar{x}) \leq 0$. Hence $\lambda_j h_j(\bar{x}) \leq 0$.

Let $\lambda_j = 0$ in (3), $\bar{\lambda}_j h_j(\bar{x}) \geq 0$. Therefore, $\lambda_j h_j(\bar{x}) = 0$. Now, from the right inequality of (2) equation and $\bar{\lambda}_j h_j(\bar{x})$ = 0, we have for any feasible x for (FP),

$$\frac{f_i(\bar{x})}{g_i(\bar{x})} \not> \frac{f_i(x)}{g_i(x)}$$

Hence $\bar{x}$ is a weak minimum for (FP).

Hence the proof.

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