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Two-Sided β-Expectation Tolerance Interval for Lifetime Distribution of K-Unit Parallel System

R. R. Kumbhar
Department of Statistics
P.D.V.P. Mahavidyalaya, Tasgaon.
Dist.: Sangali. India.416312

D.T. Shirke
Department of Statistics
Shivaji University, Kolhapur.
India. 416004.

***Abstract*- Kumbhar and Shirke [12] provided one-sided β-expectation and β-content γ-level tolerance limits for the lifetime distribution of k-unit parallel system, when lifetimes of the units are independent and exponentially distributed with the same mean. This article provides the Two-Sided β-Expectation Tolerance Interval for lifetime distribution of k-unit parallel system. The Expected coverage of the proposed Tolerance Interval is obtained by using Taylor series expansion about mean. The numerical values of the expected coverage for different values of k when values of sample size n and β are specified are also computed for a fixed sample size.**

Keywords- Parallel system; Exponential distribution; β-Expectation Tolerance Interval; Expected Covrage.

1. INTRODUCTION

One of the important issues in reliability theory and life testing is to set tolerance limits based on observed lifetimes. These lifetimes may be of components or of the system composed of various components. Let X be lifetime of the unit having a distribution function F(x;θ), $\theta \in \Theta \in \Re$. Let L(X) and U(X) be two functions of observations such that L(X) < U(X). Then (L, U) is called a β-content γ-level tolerance interval (TI), if

$$P\{\int_L^U f(x;\theta)dx \geq \beta\} = \gamma \text{ , for every } \theta \in \Theta \qquad (1.1)$$

Where, $f(x;\theta)$ is probability density function of X.
If L and U are determined so that

$$E\{\int_L^U f(x;\theta)dx\} = \beta \text{ , for every } \theta \in \Theta \qquad (1.2)$$

then (L, U) is called a β-expectation TI. The quantity $\int_L^U f(x;\theta)dx$ is called the sample coverage and L and U are called lower and upper tolerance limits, respectively.

Guttman ([1]; [2]), Goodman and Madansky [3], are few among others, who have provided tolerance limits for exponential distribution. Guenther et al. [8] have obtained one sided TI, while Engelhardt and Bain [9] have obtained TI on reliability for two parameter exponential distribution. Patel [11] has given a good review of some known results on TIs, which takes an account of TI for various discrete and continuous distributions. Now as far as lifetime distribution of the system is concerned, Hanson and Koopmans [4] have provided nonparametric tolerance limits for the class of distributions with increasing hazard rates. Barlow and Proschan [5] have obtained tolerance limits based on order statistics for the class of distributions based on failure rates. However, it appears that not much is reported on the parametric tolerance limits for the lifetime of system, when components have exponential lifetimes. In this paper, we consider the problem of setting tolerance limits for the lifetime distribution of k-unit parallel system, when the lifetimes of units are independently and identically distributed (i.i.d.) exponential random variables. When lifetimes of units are independent but have different means, the model to be considered will be mulitiparameter and hence problem of tolerance intervals requires different treatment. Therefore, we consider here the case of i.i.d. exponential variates only. Kumbhar and Shirke [12] provided one-sided β-expectation and β-content γ-level tolerance limits for the lifetime distribution of k-unit parallel system, when lifetimes of the units are independent and exponentially distributed with the same mean. They have also studied the problem of sample size determination forgiven β and γ by using the Faulkenberry and weeks [6] criterion. Pradhan [13] has discussed the problem of point and interval estimation for life time distribution of k-unit parallel system based on progressively type-II censored data.

This article provides Two-sided β-Expectation Tolerance Interval for Lifetime distribution of k-Unit Parallel System; when lifetimes of the units are independent and exponentially distributed with the same mean. The numerical values of expected coverage are also calculated.

2. TWO-SIDED Β-EXPECTATION TOLERANCE INTERVAL TOLERANCE INTERVAL

Let $Y_{i\,,}\ i = 1, 2, \dots, k$ be the lifetime of the ith unit of a k-unit parallel system. Assume that $Y_i^{'}s, i = 1, 2, \dots, k$ are i.i.d. exponential with mean θ. Then $X = Max\{Y_{i\,,}\ i = 1, 2, \dots, k\ \}$ is the lifetime of the system. The distribution function and probability density function of X is given by

$$F(x;\theta) = (1-e^{-x/\theta})^k;\ x \geq 0,\ \theta \geq 0 \qquad (2.1)$$

and

$$f(x;\theta) = \left(\frac{k}{\theta}\right) e^{-\frac{x}{\theta}} \left(1-e^{-\frac{x}{\theta}}\right)^{k-1}, \quad x \geq 0,\ \theta \geq 0,$$

$$= 0 \quad \text{, otherwise.} \qquad (2.2)$$

respectively.

Suppose n, k-unit parallel systems are put on test and *X1, X2, . . . , Xn*, be the observed lifetimes of these systems. Based on these observations the problem is to find two-sided β -expectation TI for the distribution function F(.; θ). Let $X_{(1-\beta)/2}$ and $X_{(1+\beta)/2}$ be the lower ((1- β)/2)th and ((1+ β)/2)th percentiles of the cdf (2.1).

Define

$$L_1(\underline{X}) = -\theta \ln\left(1-((1-\beta)/2)^{1/k}\right)$$

and

$$U_1(\underline{X}) = -\theta \ln\left(1-((1+\beta)/2)^{1/k}\right).$$

Then

$$E\left[F\left(U_1(\underline{X})\right) - F\left(L_1(\underline{X})\right)\right] = \beta.$$

Thus if θ is known then $\left(L_1(\underline{X}), U_1(\underline{X})\right)$ is a two-sided β-Expectation Tolerance Interval. Since θ is unknown we replace it by its MLE say $\hat{\theta}$. Therefore, we propose a two-sided β-Expectation Tolerance Interval for F(. ; θ) as

$$I(\underline{X}) = \left(-\hat{\theta}\ln\left[1-((1-\beta)/2)^{1/k}\right], -\hat{\theta}\ln\left[1-((1+\beta)/2)^{1/k}\right]\right). \qquad (2.3)$$

MLE and Its Asymptotic Variance

Kumbhar and Shirke (2004) have obtained the MLE of θ in (2.2) by using Newton-Raphson Method. They have also obtained an Asymptotic Variance of of the MLE; $\hat{\theta}$ as

$$\sigma^2(\theta) = \begin{cases} \dfrac{\theta^2}{n}; & k=1 \\ \dfrac{\theta^2}{1.808n}; & k=2 \\ \dfrac{\theta^2}{(2C_k-1)n}; & k=3 \end{cases} \qquad (2.4)$$

where

$$C_k = k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(j+1)^2} - k(k-1)\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^j}{(j+2)^2} + k(k-1)\sum_{j=0}^{k-3}\binom{k-3}{j}\frac{(-1)^j}{(j+2)^3}.$$

and $\binom{a}{b} = \dfrac{a!}{(a-b)!b!}$

3. EXPECTED COVERAGE:

The expected coverage of the Two-sided β-Expectation Tolerance Interval is obtained in the following Therom.

Theorem 3.1 **:** The expected coverage of the TI (2.3) is given by

$$E\left[F\left(X_{((1+\beta)/2)};\theta\right) - F\left(X_{((1-\beta)/2)};\theta\right)\right] \cong \beta + \frac{Var(\hat{\theta})}{2\theta^2}\left\{\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left[A_k^{k-i}\left(\ln A_k^{k-i}\right)^2 - B_k^{k-i}\left(\ln B_k^{k-i}\right)^2\right]\right\}, \qquad (3.1)$$

where $A_k = \left[1-((1+\beta)/2)^{1/k}\right]$ and $B_k = \left[1-((1-\beta)/2)^{1/k}\right]$.

Proof: The expected coverage of the interval I ($\underline{X}$) is

$$E\left[F\left(X_{(1+\beta)/2};\theta\right) - F\left(X_{((1-\beta)/2)};\theta\right)\right]$$

$$= E\left[\left(1-\exp\left\{\frac{\hat{\theta}}{\theta}\ln\left[1-((1+\beta)/2)^{1/k}\right]\right\}\right)^k - \left(1-\exp\left\{\frac{\hat{\theta}}{\theta}\ln\left[1-((1-\beta)/2)^{1/k}\right]\right\}\right)^k\right].$$

Expanding terms inside expectation using binomial expansion and substituting

$$A_k = \left[1-\left((1+\beta)/2\right)^{1/k}\right] \quad \text{and} \quad B_k = \left[1-\left((1-\beta)/2\right)^{1/k}\right]$$

we get

$$E\left[F\left(X_{\left((1+\beta)/2\right)};\theta\right)-F\left(X_{\left((1-\beta)/2\right)};\theta\right)\right]$$

$$= E\left[\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left(\exp\left\{\hat{\theta}(k-i)\ln\left[A_k\right]/\theta\right\}-\exp\left\{\hat{\theta}(k-i)\ln\left[B_k\right]/\theta\right\}\right)\right] \tag{3.2}$$

We expand the exponential terms in (3.2) using Taylor series expansion about θ

$$\exp\left(\hat{\theta}(k-i)\ln(A_k)/\theta\right) = \exp\left(\ln A_k^{k-i}\right) + \left[\frac{d}{d\hat{\theta}}\exp\left(\hat{\theta}\ln(A_k^{k-i})/\theta\right)\right]_{\hat{\theta}=\theta}(\hat{\theta}-\theta)$$

$$+\left[\frac{d^2}{d\hat{\theta}^2}\exp\left(\hat{\theta}\ln(A_k^{k-i})/\theta\right)\right]_{\hat{\theta}=\theta}\frac{(\hat{\theta}-\theta)^2}{2!}+R_1,$$

where $$\left[\frac{d}{d\hat{\theta}}\exp\left(\hat{\theta}\ln(A_k^{k-i})/\theta\right)\right]_{\hat{\theta}=\theta} = A_k^{k-i}\,\frac{\ln A_k^{k-i}}{\theta},$$

and $$\left[\frac{d^2}{d\hat{\theta}^2}\exp\left(\hat{\theta}\ln(A_k^{k-i})/\theta\right)\right]_{\hat{\theta}=\theta} = \left(A_k^{k-i}\right)^2\frac{\left(\ln A_k^{k-i}\right)^2}{\theta^2}.$$

Therefore ,

$$\exp\left(\hat{\theta}(k-i)\ln(A_k)/\theta\right) = A_k^{k-i} + A_k^{k-i}\ln(A_k^{k-i})\frac{(\hat{\theta}-\theta)}{\theta} + A_k^{k-i}\left(\ln(A_k^{k-i})\right)^2\frac{(\hat{\theta}-\theta)^2}{2\theta^2}+R_1. \tag{3.3}$$

Similarly we get

$$\exp\left(\hat{\theta}(k-i)\ln(B_k)/\theta\right) = B_k^{k-i} + B_k^{k-i}\ln(B_k^{k-i})\frac{(\hat{\theta}-\theta)}{\theta} + B_k^{k-i}\left(\ln(B_k^{k-i})\right)^2\frac{(\hat{\theta}-\theta)^2}{2\theta^2}+R_2. \tag{3.4}$$

where R_1, R_2 are the remainders with higher order terms in the expansion. Substituting the expressions for exponential terms in (3.2) from (3.3) and (3.4) and ignoring the higher order terms in the expressions after simplification we get

$$E\left[F\left(X_{(1+\beta)/2};\theta\right)-F\left(X_{(1-\beta)/2};\theta\right)\right]=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left\{A_k^{k-i}-B_k^{k-i}\right\}$$
$$+\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left\{A_k^{k-i}(\ln A_k^{k-i})-B_k^{k-i}(\ln B_k^{k-i})\right\}\frac{E(\hat{\theta}-\theta)}{\theta}$$
$$+\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\left\{A_k^{k-i}(\ln A_k^{k-i})^2-B_k^{k-i}(\ln B_k^{k-i})^2\right\}\frac{E(\hat{\theta}-\theta)^2}{2\theta^2}.$$

The first term in in the above expression reduces to β. Using Consistency property of $\hat{\theta}$, after simplification we get (3.1).

4. NUMERICAL EXPECTED COVERAGE

Using the values of A_k, B_k and an Asymptotic Variance of MLE; $\hat{\theta}$, for different values of the k we have obtained the expected coverage of the proposed Two-sided Tolerance Interval in (2.3).

For k=2,

$$A_k = \left[1-\left((1+\beta)/2\right)^{1/2}\right], \quad B_k = \left[1-\left((1-\beta)/2\right)^{1/2}\right]$$

and $\sigma^2(\theta)=0.5531\theta^2$.

Therefore,

Expected Coverage = β

$$+\frac{0.27655}{n}\left[A_k^2\left(\ln(A_k^2)\right)^2-B_k^2\left(\ln(B_k^2)\right)^2\right]$$
$$-\frac{0.5531}{n}\left[A_k\left(\ln(A_k)\right)^2-B_k\left(\ln(B_k)\right)^2\right].$$

Similarly expressions for expected coverage are obtained for k = 3, 5 etc. The numerical values of the expected coverage for k=1, 2, 3 and 5 when β =.90 and .99 are tabulated for different values of n in Table 4.1.

Table 4.1: Numerical expected coverage

k	Sample Size=10				Sample Size=20			
	β				β			
	.90	.95	.975	.99	.90	.95	.975	.99
2	.8805	.9361	.9657	.9848	.8903	.9430	.9703	.9874
3	.8815	.9372	.9666	.9855	.8908	.9436	.9708	.9877
5	.8649	.9381	.9674	.9820	.8825	.9440	.9884	.9742
	Sample Size=50				Sample Size=100			
2	.8961	.9472	.9731	.9890	.8981	.9486	.9741	.9895
3	.8963	.9487	.9733	.9881	.8992	.9487	.9742	.9895
5	.8930	.9476	.9735	.9884	.8965	.9488	.9742	.9892

We observe from Table 4.1 that as n increases I (X) attain desired coverage. As such there is no significant effect of increase in k; the number of units in the paralle system.

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