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Transfer Function of Dynamical Multi-agent Consensus Systems with Less Than Five Agents

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Abstract— **This paper deals with transfer functions of agents in a linear multi-agent consensus system where every agent in the system has the same dynamics described by a common transfer function. Depending on network topology, the transfer function of an agent in a consensus system varies in a highly complicated way and analytic methods for specifying the orders of transfer functions are yet unavailable except for some special cases. This paper presents a table of transfer functions of agents in a consensus system composed of less than five agents under all possible network topologies.**

Keywords— Consensus System, Transfer Function, Laplacian Graph Angle

I. INTRODUCTION

Consensus multi-agent systems composed of a number of identical dynamical agents have attracted much attention in various fields of science and engineering.

A typical approach for changing overall behaviors of a consensus system is to directly control only a small number of agents and to let remaining uncontrolled agents follow a certain predefined consensus protocol. The leader-follower approach [1,2,3], the single agent control (SAC) [4,5,6] and the pinning control [7,8] share the same idea. Among those approaches, only the SAC is a frequency domain approach where a transfer function description of agents is critically important.

One key issue of the SAC is how to choose a controlled agent in a given consensus system and a network topology. In favor of an easy controller synthesis, it is desirable to select a controlled agent whose transfer function order is the smallest among all agents in a given consensus system. Unfortunately, however, there is no analytic methods which can characterize an agent with the smallest transfer function order, except a special case where a consensus system has a *hub agent* which has direct connections with all other agents [9].

Motivated by this difficulty, from numerical computations, this paper gives a complete description of transfer functions of consensus systems with less than five agents under all possible (connected) network topology.

An obvious contribution of this work is to serve an easy reference for transfer function representation of small consensus systems under the SAC scheme. Additionally our results provide useful inspiration on interesting, but theoretically unsubstantiated yet, correlations between transfer function order and agent location in a given network. For instances, our numerical results suggests that agents located at more symmetric positions in a given network topology, have smaller transfer function orders.

II. PRELIMINARY

This section gives a short summary on the transfer function representation of a consensus system in [9].

In the SAC, every agent in a consensus system composed of $n \in N$ identical agents is assumed to have a common linear time-invariant dynamics described by a SISO (single-input single-output) transfer function

$$g_a(s) = \frac{y_i(s)}{u_i(s)} = \frac{b(s)}{a(s)} \tag{1}$$

where $y_i(s)$ and $u_i(s)$ denote the Laplace transforms of output and input of agent labelled $i \in [1, n]$. The polynomials $b(s)$ and $a(s)$ are *coprime* and $a(s)$ is a *monic* polynomial. Moreover, we assume that every agent $i \in [1, n]$ follows a linear consensus protocol given by

$$u_i = \sum_{k \in N_i} (y_k - y_i) \tag{2}$$

where $N_i \subset [1, n]$ denotes the set of neighbor agents of the agent $i \in [1, n]$. In contrast, only one agent, call it $j \in [1, n]$, is to be actively controlled by an exogenous controller and thus has an external input u_j^{ext}, i.e.,

$$u_j = \sum_{k \in N_j} (y_k - y_j) + u_j^{ext} \tag{3}$$

The neighbor sets $\{N_i ; i \in [1, n]\}$ of a given network topology can be completely described with a *Laplacian matrix* of a mathematical graph corresponding to the network topology;

$$L = (l_{ij}) = \begin{bmatrix} d_1 & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{1n} & \cdots & d_n \end{bmatrix}, d_i = -\sum_{j=1}^{n} l_{ij} \geq 0 \tag{4}$$

$$l_{ij} = \begin{cases} -1 & \text{if } i \text{ is directly connected to } j \\ 0 & otherwise \end{cases} \tag{5}$$

Let $\mu_1 > \cdots > \mu_m$ $(m \leq n)$ be the distinct eigenvalues of the Laplacian matrix L. Then from a spectral factorization $L = \mu_1 P_1 + \cdots + \mu_m P_m$ where P_k $(k = 1, \quad \ldots, m)$ is the orthogonal projection onto μ_k- eigenspace $\mathcal{E}(\mu_{1k})$, the transfer function $g_j(s)$ between the external input u_j^{ext} and output y_j can be shown to be

$$g_j(s) = \frac{y_j(s)}{u_j^{ext}(s)} = \sum_{k=-1}^{m} \frac{b(s)}{a(s) + \mu_k b(s)} \alpha_{jk}^2 \tag{6}$$

$$\alpha_{jk}^2 =< e_j, P_k e_j > \tag{7}$$

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Vol. 5 Issue 03, March-2016

denotes the cosine of an angle between the eigenspace $\mathcal{E}(\mu_{1k})$, and the standard orthonormal basis $\{e_i; i = 1, \dots, n\}$ of $\mathbb{R}^n$. We call α_{jk} the *Laplacian graph angle* or *graph angle* in short.

Suppose the *isolated* agent transfer function $g_a(s)$ in (1) has a transfer function order $O(g_a) \in \mathbb{N}$. Then, the following fact holds [9];

Lemma 1: *The transfer function $g_j(s)$ of agent j of a consensus system in (6) has an order $|\phi_j| \times O(g_a) \in \mathbb{N}$ where*

$$\phi_j := \{k \in [1, m];\ \alpha_{jk}^2 \neq 0\} \subset \{1, \dots, m\} \qquad (8)$$

and $|\phi_j|$ denotes the number of elements in ϕ_j.

From this result, the problem of how to choose an agent with a smallest transfer function order boils down to the problem of how to find $j \in [1, n]$ with the smallest $|\phi_j|$. Unfortunately however no analytic methods exists within the author's knowledge that can characterize the smallest $|\phi_j|$ in general.

Example 1 Suppose five identical agents in network topology of Fig. 1share the same dynamics given by:

$$g_a(s) = \frac{y_i(s)}{u_i(s)} = \frac{1}{s^2 + s + 1} \qquad (9)$$

whose transfer function order is $O(g_a) = 2$.

Note that $u_i^{ext} = 0$ implicitly holds for all $i \neq j$ in Fig. 1 when $j \in [1,5]$ is a single agent to be controlled under the SAC scheme. Numerical computations give five distinct eigenvalues of L in the first line of Table 1. Other lines give agent labels $j \in [1,5]$ in the left and corresponding graph angles α_{jk} in the right. The transfer function of an agent $j \in [1,5]$ is given $g_j(s)$ in (6) with $\{\alpha_{jk}\}$ in Table 1 and $b(s) = 1, a(s) = s^2 + s + 1$. In addition, from Lemma 1, the order of $g_j(s)$ is given by $2\,|\phi_j|$ where the number 2 is the order of the isolated transfer function (9). By counting the number of non-zero angles in Table 1, one can find the quantity $\{\phi_j\}$ shown in Fig. 2 inside of circles denoting agents. For examples, the transfer function order of agent 3 is given as $2\,|\phi_3| = 2 \times 3 = 6$ whereas that of agent 1 is $2\,|\phi_3| = 2 \times 5 = 10$.

TABLE I. EIGENVALUES AND GRAPH ANGLES

	4.3028	3.6180	1.3820	0.6972	0.0000
1,2	0.6768	0.5117	0.1954	0.2049	0.4472
3	**0.0000**	0.6325	0.6325	**0.0000**	0.4472
4,5	0.2049	0.1954	0.5117	0.6768	0.4472

III. TRANSFER FUNCTIONS

There are 1, 2, 6, 21 different network topologies, equivalently connected undirected graphs, for multi-agent systems composed of 2, 3, 4 and 5 agents (vertices), respectively.

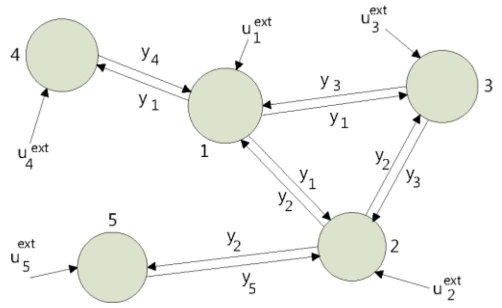

Figure 1 A Consensus System

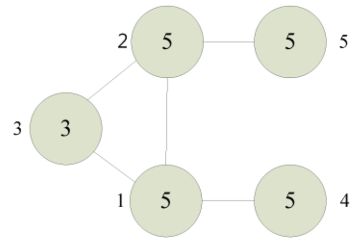

Figure 2 Transfer Function Orders

For all 30 network topologies, the Laplacian eigenvalues, graph angles and network topologies are given in Table 2. Our ordering and representations of graphs follow those in [10] (Appendix B).

For each network topology in Table 2, circles in the right side denote agents and numbers beside circles are agent labels. The multiplication factor $|\phi_j|$ of the transfer function order of agent $j \in [1, n]$ is written as a number inside of a circle, as we did in Fig. 2. In the left side of table, the first line shows the Laplacian eigenvalues with subscripts denoting multiplicities. Other lines give agent labels in the left and their graph angles in the right. The zero angles are emphasized with **bold** fonts.

The number of connections that an agent has is called as a *degree* in graph theory. The possible correlation between degree and multiplication factor $|\phi_j|$ of an agent is very complicated and no theoretical results are available yet except for hub agents.

A rather unexpected result in Table 2 is that a larger degree is not significantly beneficial for a smaller multiplication factor. One can easily find many cases where agents with smaller degrees have smaller multiplication factors including the index 25, for an example.

Note that our previous example with the Laplacian matrix (10) corresponds to the index 24 of Table 2.

IV. CONCLUSION

With no analytic methods available, we numerically characterized transfer functions of a controlled agent in general liner consensus systems composed of less than five agent under all possible network topologies. From mathematical viewpoint, our result can be seen as a specification of Laplacian graph angles of connected graphs with less than five vertices.

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TABLE II. EIGENVALUES AND GRAPH ANGLES

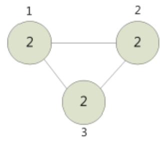
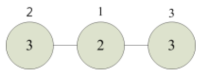
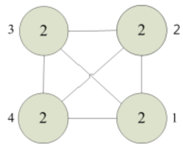
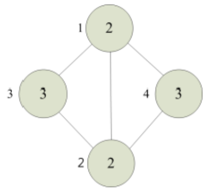
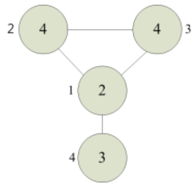
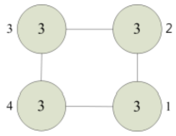
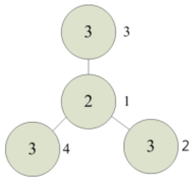
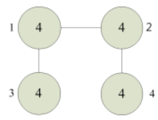
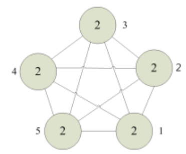
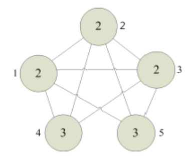
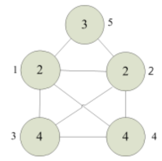
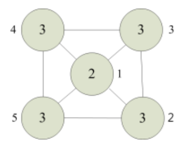
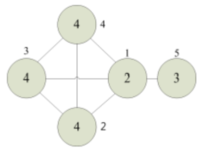
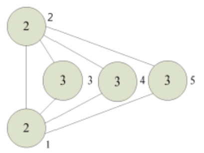
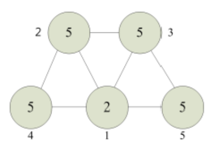
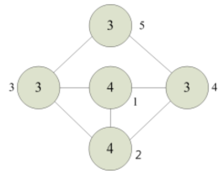
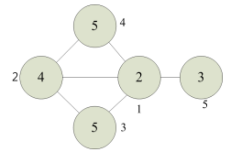
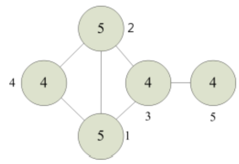
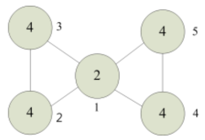
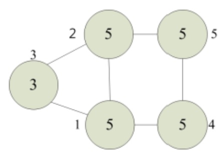
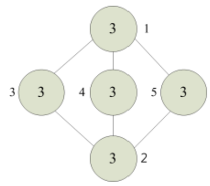
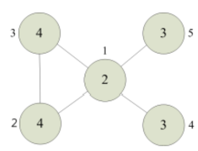
Index	n	Laplacian Eigenvalue / Agent number, Laplacian Angles					Graph Topology
1	2		2.0000^{1}	0.0000^{1}			
		1,2	0.7071	0.7071			
2	3		3.0000^{2}	0.0000^{1}			
		1,2,3	0.8165	0.5774			
3	3		3.0000^{1}	1.0000^{1}	0.0000^{1}		
		1	0.8165	**0.0000**	0.5774		
		2,3	0.4082	0.7071	0.5774		
4	4		4.0000^{3}	0.0000^{1}			
		1,2,3,4	0.8660	0.5000			
5	4		4.0000^{2}	2.0000^{1}	0.0000^{1}		
		1,2	0.8660	**0.0000**	0.5000		
		3,4	0.5000	0.7071	0.5000		
6	4		4.0000^{1}	3.0000^{1}	1.0000^{1}	0.0000^{1}	
		1	0.8660	**0.0000**	**0.0000**	0.5000	
		2,3	0.2887	0.7071	0.4082	0.5000	
		4	0.2887	**0.0000**	0.8165	0.5000	
7	4		4.0000^{1}	2.0000^{2}	0.0000^{1}		
		1,2,3,4	0.5000	0.7071	0.5000		
8	4		4.0000^{1}	1.0000^{2}	0.0000^{1}		
		1	0.8660	**0.0000**	0.5000		
		2,3,4	0.2887	0.8165	0.5000		

TABLE II. EIGENVALUES AND GRAPH ANGLES (CONTINUED)

Index	n	Laplacian Eigenvalue Agent number, Laplacian Angles	Graph Topology
9	4	3.4142^{1} 2.0000^{1} 0.5858^{1} 0.0000^{1} 1,2 0.6533 0.5000 0.2706 0.5000 3,4 0.2706 0.5000 0.6533 0.5000	
10	5	5.0000^{4} 0.0000^{1} 1,2,3,4,5 0.8944 0.4472	
11		5.0000^{1} 3.0000^{1} 0.0000^{1} 1,2,3 0.8944 **0.0000** 0.4472 4,5 0.5477 0.7071 0.4472	
12		5.0000^{2} 4.0000^{1} 2.0000^{1} 0.0000^{1} 1,2 0.8944 **0.0000** **0.0000** 0.4472 3,4 0.3651 0.7071 0.4082 0.4472 5 0.3651 **0.0000** 0.8165 0.4472	
13		5.0000^{2} 3.0000^{2} 0.0000^{1} 1 0.8944 **0.0000** 0.4472 2,3,4,5 0.5477 0.7071 0.4472	
14		5.0000^{1} 4.0000^{2} 1.0000^{1} 0.0000^{1} 1 0.8944 **0.0000** **0.0000** 0.4472 2,3,4 0.2236 0.8165 0.2887 0.4472 5 0.2236 **0.0000** 0.8660 0.4472	
15		5.0000^{2} 2.0000^{2} 0.0000^{1} 1,2 0.8944 **0.0000** 0.4472 3,4,5 0.3651 0.8165 0.4472	
16		5.0000^{1} 4.4142^{1} 3.0000^{1} 1.5858^{1} 0.0000^{1} 1 0.8944 **0.0000** **0.0000** **0.0000** 0.4472 2,3 0.2236 0.6533 0.5000 0.2706 0.4472 4,5 0.2236 0.2706 0.5000 0.6533 0.4472	

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 5 Issue 03, March-2016

TABLE II. EIGENVALUES AND GRAPH ANGLES (CONTINUED)

Index	n	Laplacian Eigenvalue / Agent number, Laplacian Angles						Graph Topology
17			5.0000^{1}	4.0000^{1}	3.0000^{1}	2.0000^{1}	0.0000^{1}	
		1,2	0.3651	0.7071	**0.0000**	0.4082	0.4472	
		3,4	0.5477	**0.0000**	0.7071	**0.0000**	0.4472	
		5	0.3651	**0.0000**	**0.0000**	0.8165	0.4472	
18			5.0000^{1}	4.0000^{1}	2.0000^{1}	1.0000^{1}	0.0000^{1}	
		1	0.8944	**0.0000**	**0.0000**	**0.0000**	0.4472	
		2	0.2236	0.8165	**0.0000**	0.2887	0.4472	
		3,4	0.2236	0.4082	0.7071	0.2887	0.4472	
		5	0.2236	**0.0000**	**0.0000**	0.8660	0.4472	
19			4.4812^{1}	4.0000^{1}	2.6889^{1}	0.8299^{1}	0.0000^{1}	
		1,2	0.4193	0.7071	0.2422	0.2560	0.4472	
		3	0.7024	**0.0000**	0.5362	0.1380	0.4472	
		4	0.3380	**0.0000**	0.7031	0.4375	0.4472	
		5	0.2018	**0.0000**	0.3175	0.8115	0.4472	
20	5		5.0000^{1}	3.0000^{2}	1.0000^{1}	0.0000^{1}		
		1	0.8944	**0.0000**	**0.0000**	0.4472		
		2,3,4,5	0.2236	0.7071	0.5000	0.4472		
21			4.6180^{1}	3.6180^{1}	2.3820^{1}	1.3820^{1}	0.0000^{1}	
		1,2	0.6015	0.5117	0.3717	0.1954	0.4472	
		3	**0.0000**	0.6325	**0.0000**	0.6325	0.4472	
		4,5	0.3717	0.1954	0.6015	0.5117	0.4472	
22			5.0000^{1}	3.0000^{2}	2.0000^{1}	0.0000^{1}		
		1,2	0.5477	0.7071	**0.0000**	0.4472		
		3,4,5	0.3651	**0.0000**	0.8165	0.4472		
23			5.0000^{1}	3.0000^{1}	1.0000^{2}	0.00000^{1}		
		1	0.8944	**0.0000**	**0.0000**	0.4472		
		2,3	0.2236	0.7071	0.5000	0.4472		
		4,5	0.2236	**0.0000**	0.8660	0.4472		

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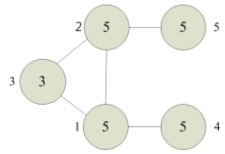
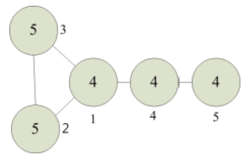
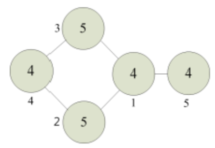
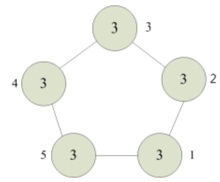
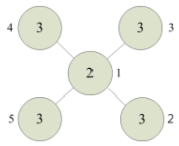
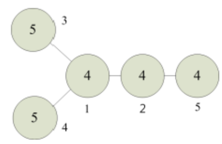
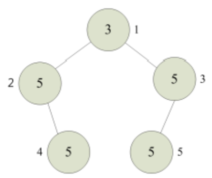
24			4.3028^{1}	3.6180^{1}	1.3820^{1}	0.6972^{1}	0.0000^{1}	
		1,2	0.6768	0.5117	0.1954	0.2049	0.4472	
		3	**0.0000**	0.6325	0.6325	**0.0000**	0.4472	
		4,5	0.2049	0.1954	0.5117	0.6768	0.4472	
25			4.1701^{1}	3.0000^{1}	2.3111^{1}	0.5188^{1}	0.0000^{1}	
		1	0.8115	**0.0000**	0.3175	0.2018	0.4472	
		2,3	0.2560	0.7071	0.2422	0.4193	0.4472	
		4	0.4375	**0.0000**	0.7031	0.3380	0.4472	
		5	0.1380	**0.0000**	0.5362	0.7024	0.4472	
26			4.4812^{1}	2.6889^{1}	2.0000^{1}	0.8299^{1}	0.0000^{1}	
		1	0.7024	0.5362	**0.0000**	0.1380	0.4472	
		2,3	0.4193	0.2422	0.7071	0.2560	0.4472	
		4	0.3380	0.7301	**0.0000**	0.4375	0.4472	
		5	0.2018	0.3175	**0.0000**	0.8115	0.4472	
27	5		3.6180^{2}	1.3820^{2}	0.0000^{1}			
		1,2,3,4,5	0.6325	0.6325	0.4472			
28			5.0000^{2}	3.0000^{2}	0.0000^{1}			
		1	0.8944	**0.0000**	0.4472			
		2,3,4,5	0.5477	0.7071	0.4472			
29			4.1701^{1}	2.3111^{1}	1.0000^{1}	0.5188^{1}	0.0000^{1}	
		1	0.8115	0.3175	**0.0000**	0.2018	0.4472	
		2	0.4375	0.7031	**0.0000**	0.3380	0.4472	
		3,4	0.2560	0.2422	0.7071	0.4193	0.4472	
		5	0.1380	0.5362	**0.0000**	0.7024	0.4472	
30			3.6180^{1}	2.6180^{1}	1.3820^{1}	0.3820^{1}	0.0000^{1}	
		1	0.6325	**0.0000**	0.6325	**0.0000**	0.4472	
		2,3	0.5117	0.6015	0.1954	0.3717	0.4472	
		4,5	0.1954	0.3717	0.5117	0.6015	0.4472	

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