International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 2 Issue 9, September - 2013

Theory of System of Linear Differential Equations on Time Scales

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Abstract

This paper presents the criterion to construct fundamental matrices for the-system of linear differential equations with constant coefficients on time scales. We develop the procedure to compute fundamental matrices for vector differential equations on time scales.

Key words: Time scale, dynamical equation, fundamental matrix, eigenvalues, eigenvectors.

AMS Subject Classification: 34B99, 39A99

1 Introduction

The study of solutions of linear differential equations on time scales gained momentum because of unified approach nature for differential and difference systems. The theory of linear differential equations provides a broad mathematical basis for an understanding of continuous time dynamic processes. There are many results on continuous time dynamical systems which are needed in discrete time context. In recent past a new theory is emerged to unify the results not only on continuous and discrete time dynamical systems but also on discrete time dynamical system for any jump. The theory was ,first introduced by B. Aulbach et al [2]. By a time scale we mean a nonempty closed subset of $\mathbb{R}$. For the time scale calculus and notation for delta differentiation, as well as concepts for dynamic equations on time scales, we refer to the introductory book on time scales by M. Bohner et al [3]. It provides a new direction of research in dynamical process with time scales.

In this paper, for the development of theory, we construct the fundamental matrices for the system of linear differential equations on time scales. If all the eigenvalues of the coefficient matrix are real and distinct, then we can construct a solution of the system without any difficulty. But if some of the eigenvalues of the coefficient matrix are repeated we take care, since in general n^{th} delta derivative of a polynomial cannot be evaluate without assumptions.

Three conditions that we assume throughout are as follows:

(A) Every point t in T is neither simultaneously left dense and right scattered nor simultaneously left scattered and right dense.

(B) The jump is uniform at all scattered points of T. Finally,

(C) The eigenvalues of A are regressive on T.

This paper is organized as follows. In Section 2, we briefly describe some salient features of time scales, functions defined on time scales and operations with these functions. In Section 3, we construct the fundamental matrices for t he system of linear differential equations on time scale for real and distinct eigenvalues, and as an application, we also give some examples to demonstrate our results. In Section 4,first we introduce algebraic concepts for the main result and, then we invoke our assumptions, along with direct sum of solution spaces, we prove that a lemma to compute the mth delta derivative of t^n, to obtain fundamental matrices of system of linear differential equations for general case.

2 Preliminaries

We denote the time scale by the symbol T. By an interval we mean the intersection of the real interval with a given time scale. The jump operators in-troduced on a time scale T may be connected or disconnected. To overcome this topological difficulty the concept of jump operators is introduced in the following way. The operators σ and ρ from T to T, defined by $\sigma(t) = inf\ \{s \in T : s > t\}$ and $\rho(t) = sup\{s \in T : s < t\}$ are called jump operators. If σ is bounded above and ρ is bounded below then we define $\sigma(max\ T) = max\ T\ and\ \rho(min\ T) = min\ T$. These operators allow us to classify the points of time scale T. A point $t \in T$ is said to be right-dense if σ(t) = t, left-dense if ρ(t) = t, right-scattered if σ(t) > t, left-scattered if ρ(t) < t, isolated if ρ(t) < t < σ(t) and dense if ρ(t) = t = σ(t).The set T^k which is derived from the time scale T is defined as follows

$$T^k = \begin{cases} T \setminus (\rho(\sup T), \sup T) & if \sup T < \infty \\ T & if \sup T = \infty \end{cases}$$

Finally, if $f : T \to \mathbb{R}$ is a function, then we define the function $f^{\sigma} : T \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in T$

Definition 2.1 *Let* T *be a time scale,* $\mathbb{R}$ *be a real line, and* $f : T \to \mathbb{R}$. *We say That* f *is delta differentiable at a point* $s \in T^k$, *if there exists an* $a \in \mathbb{R}$ *such that for any* $\varepsilon > 0$ *there exists a neighborhood U of s such that,*

$$\left| f(\sigma(s)) - f(t) - (\sigma(s) - t)a \right| \le \varepsilon \left| \sigma(s) - t \right| \ \forall\ t \in U,$$

or more specifically, f is delta differentiable at s if the limit

$$\lim_{t \to \sigma(s)} \frac{f(t) - f(\sigma(s))}{t - \sigma(s)}$$

exists, and is denoted by $f^{\Delta}(s)$.

If f is delta differentiable for every $t \in T^k$ we say that $f : T^k \to \mathbb{R}$ is delta differentiable on T. If *f* and *g* are two delta differentiable functions at *s* then f*g* is delta differentiable at *s* and

$$(fg)^{\Delta}(s) = f(s)g^{\Delta}(s) + f^{\Delta}(s)g^{\sigma}(s) = f^{\Delta}(s)g(s) + f^{\sigma}(s)g^{\Delta}(s)$$

Definition 2.2 *A function* $g : T^k \to \mathbb{R}$ *.is rd- continuous if it is continuous in every right-dense point* $t \in T^k$ *and if* $\lim_{s \to t^-} g(s)$ *exists for each left-dense* $t \in T^k$.

We say that a function $p: T^k \to \mathbb{R}$ is regressive provided $1+\mu(t)p(t) \neq 0$ for all $t \in T^k$. For $s \in T$, we define the graininess function $\mu: T \to [0,\infty)$ by $\mu(s) = \sigma(s) - s$.

Definition 2.3 *For h > 0 we define the Hilger complex number*
$\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq \frac{-1}{h} \right\}$ *For h = 0 : Let* $\mathbb{C}_0 = \mathbb{C}$.

Definition 2.4 *For h > 0, we define the cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by $\xi_h(z) = \frac{1}{h} Log(1+zh)$ *,where Log is the principal logarithm function. For h = 0, we define* $\xi_o(z) = z$ *for all* $z \in \mathbb{C}$.

Definition 2.5 *If p is regressive, then we define the exponential function by* $e_p(t,s) = \exp(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau)$ *for* $s,t \in T$,where $\xi_{\mu(\tau)}$ *is the cylinder transformation.*

Definition 2.6 *Let* $p: T^k \to \mathbb{R}$ *be regressive and rd-continuous, then a mapping* $e_p : T \to \mathbb{R}$ *is said to be a solution of the linear homogenous dynamic equation* $y^\Delta = p(t)y$*, if* $e_p^\Delta(t,t_0) = p(t)e_p(t,t_0)$ $\forall t \in T^k$ *, and a fixed* $t_0 \in T^k$.

Definition 2.7 *Any set of n linearly independent solutions of* $y^\Delta = Ay$ *is a fundamental set of solutions of the equation. The matrix with these particular solutions as columns is a fundamental matrix for the given equation.*

Definition 2.8 *Let* $y_1, y_2, ...y_n$ *be a fundamental set of solutions of equation* $y^\Delta = Ay$ *and let* $Y = (y_1, y_2, ..., y_n)$ *be the corresponding fundamental matrix. For any constant n-vector c, Yc is a solution of* $y^\Delta = Ay$.

3 Real Distinct Eigenvalues

In this section, we consider a system of differential equations

$$y^\Delta = Ay \qquad (1)$$

on a time scale T^k, where A is $n \times n$ constant matrix ,and y is $n \times 1$ vector, assume that the eigenvalues of A are regressive on T^k. By using a non-singular transformation,

$$y = Sx \qquad (2)$$

where S is $n \times n$ non-singular constant matrix and x is $n \times 1$ vector, the equation (1) can be transformed into

$$x^\Delta = Dx \text{ where } D = S^{-1}AS \qquad (3)$$

D will take different forms depending on the eigenvalues of *A*. This case is treated to provide an introduction for the general case.

Theorem 3.1 *Assume that the equation* (1) *satisfies the above assumptions of A and, if we assume the eigenvalues* $\lambda_1, \lambda_2, ..., \lambda_n$ *of the matrix A are real and distinct, then the fundamental matrix Y for* (1) *is of the form*

$Y(t)= \quad [s_1, s_2, ..., s_n]E(t)$

where s_j *is an* $n\times 1$ *eigenvector of* A *corresponding to eigenvalue* $\lambda_j, E(t) = \delta_{ij} e_{\lambda_j}(t,t_0), \quad i, j = 1,2,.....,n$ *and a fixed* $t_0 \in T^k$.

Proof: The canonical form of A is a diagonal matrix given by $D = (\delta_{ij}\lambda_j)$. Let the matrix S be

$S = [s_1, s_2, ..., s_n]$, where the j^{th} column is the vector s_j. It follows that *AS* = *SD*, and, since S is non-singular, that $D = S^{-1}AS$. If

$x^{\Delta} = Dx$ (4) is written in scalar form, and each scalar equation, has a relation it is that

$$x_j = e_{\lambda_j}(t,t_0)d_j, \qquad j = 1, 2, ..., n,$$

where d_j is real constant and a fixed $t_0 \in T^k$. The matrix $E = (\delta_{ij} e_{\lambda_j}(t,t_0))$ is a fundamental matrix for the equation (4). It follows that a fundamental matrix Y for the equation (1) is $Y = SE$.

Example

An example illustrates the above result. Find a fundamental matrix for the following equation.

$$y^{\Delta} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} y .$$

The eigenvalue of the coefficient matrix A are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, the corresponding eigenvectors are $s_1 = [1, \quad 0, \quad 0]^T$, $s_2 = [1, \quad 1, \quad 0]^T$, $s_3 = [1, \quad 1, \quad 1]^T$, where T is transpose. Hence, a fundamental set of solutions is given by

$$y_1 = e_1(t,t_0)s_1^T, \; y_2 = e_2(t,t_0)s_2^T, \; y_3 = e_3(t,t_0)s_3^T$$

The matrix

$$Y = SV = \begin{pmatrix} e_1(t,t_0) & e_2(t,t_0) & e_3(t,t_0) \\ 0 & e_2(t,t_0) & e_2(t,t_0) \\ 0 & 0 & e_3(t,t_0) \end{pmatrix}$$

is a fundamental matrix for the equation

(1) If $T = \square$,

$$\text{then} \quad Y = \begin{pmatrix} e^{(t-t_0)} & e^{2(t-t_0)} & e^{3(t-t_0)} \\ 0 & e^{2(t-t_0)} & e^{3(t-t_0)} \\ 0 & 0 & e^{3(t-t_0)} \end{pmatrix}$$

(2) If $T = \square$, then $Y = \begin{pmatrix} 2^{(t-t_0)} & 3^{(t-t_0)} & 4^{(t-t_0)} \\ 0 & 3^{(t-t_0)} & 4^{(t-t_0)} \\ 0 & 0 & 4^{(t-t_0)} \end{pmatrix}$

(3) If $T = h\square$, h>0,

$$\text{then} \quad Y = \begin{pmatrix} (1+h)^{\frac{(t-t_0)}{h}} & (1+2h)^{\frac{(t-t_0)}{h}} & (1+3h)^{\frac{(t-t_0)}{h}} \\ 0 & (1+2h)^{\frac{(t-t_0)}{h}} & (1+3h)^{\frac{(t-t_0)}{h}} \\ 0 & 0 & (1+3h)^{\frac{(t-t_0)}{h}} \end{pmatrix}$$

4 General case

In this section, we state and prove the main results of this paper. We need the following algebraic concepts and theorems.

The direct sum of r- vector spaces can be used advantageously in this section. Given $Y_1, Y_2, \ldots\ldots\ldots, Y_r$ as r- finite dimensional vector spaces, their direct sum $Y_1 \oplus Y_2 \oplus \ldots\ldots \oplus Y_r$ is the set of all ordered r^{th} tuples $(a_1, a_2, \ldots\ldots\ldots, a_r)$ where $a_i \in Y_i$, $i = 1, 2, \ldots\ldots, r$. It may be established, if addition and scalar multiplication are appropriately defined, that this set is a vector space and that its dimension is the sum of the dimension of $Y_1, Y_2, \ldots\ldots, Y_r$. It is of significance that a subspace of the direct sum consisting of all ordered rth tuples of the form $(a_1, 0, \ldots\ldots\ldots, 0)$ is the isomorphism to Y_1, the subspace containing all r$^{\text{th}}$ tuples of the form $(0, a_2, \ldots\ldots\ldots, 0)$ is the isomorphism to Y_2 , similarly, $(0, 0, \ldots\ldots, a_i, \ldots, 0, 0)$ is the isomorphism to Y_i, i=1, 2, ...,r.

The properties of a direct sum in this case evolve from the properties of matrix multiplication. Let $A_1, A_2, \ldots, A_r$ be r square matrices of orders $n_1, n_2, \ldots, n_r$, , respectively, and let the vector space Y_i be the solution space of

$$y^{\Delta} = A_i y, \quad i = 1, 2, \ldots, r. \qquad (5)$$

If Y_i is a fundamental matrix for the equation (5), then $y_i \in Y_i$ if and only if

$$y_i = Y_i c_i \qquad (6)$$

for some vector c_i in $V_{n_i}(R)$. We may represent an element in the direct sum of

$Y_1, Y_2, ..., Y_r$ by
$[y_1, y_2, ..., y_r]^T$

(7)

It may be observed here that an ordered r[th] tuple is an ordered r[th] tuple whether it be written in horizontal or vertical form. The vertical form is preferred here because the solutions of vector equations are usually written as column vectors. Because of its vertical form, the ordered r[th] tuple (7) may be thought of as a partitioned column vector of dimension $n_1 + n_2 + ... + n_r$. Hence, we write

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{pmatrix} = \begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & Y_r \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix}$$

where $c_1, c_2 ..., c_r$ are the vector appearing in formula (6). It is clear from this relation that

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{pmatrix} \in Y_1 \oplus Y_2 \oplus ... \oplus Y_r \quad \text{if and only if} \quad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{pmatrix} \in V_{n_1 + n_2 + ... + n_r}(\square)$$

This establishes the fact that $Y_1 \oplus Y_2 \oplus ... \oplus Y_r$ is a vector space of dimension $n_1 + n_2 + ... + n_r$ .It is equally clear that elements of the form $[y_1, 0, 0, ..., 0]^T$, $[0, y_2, 0, ..., 0]^T$,..., and $[0, , 0, , 0 , ..., y_r]^T$ are, respectively, subspaces of the direct sum. The first of these subspaces is isomorphic to Y_1, the second to Y_2, ... , and finally r[th] subspace to Y_r .

Our understanding of the formation of a direct sum and its properties can now be applied to establish the following theorem. The notation that was introduced above is used in theorem.

Theorem 4.1 *If* $\{A_i : i = 1, 2, ..., r\}$ *is a set of constant square matrices, then the solution space of*

$$y^{\Delta} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_r \end{pmatrix} y \qquad (8)$$

is the direct sum of the solution spaces of the equations in the set

$\{y^{\Delta} = A_i y \ : i = 1, 2, \ldots, r\}$. *Moreover, a fundamental matrix for* (8) *is*

$$\begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & Y_r \end{pmatrix}$$

where Y_i *is a fundamental matrix for* $y^{\Delta} = A_i y, i = 1, 2, \ldots, r$.

Lemma 4.2 *Let* $n \in \mathbb{N}$ *, define a function* $f : T \to \mathbb{R}$ *by* $f(t) = t^n$ *, if we assume that the conditions (A) and (B) are satisfied, then*

$$f^{\Delta^m}(t) = \frac{n!}{(n-m)!}\left(\sum_{r=0}^{n-m}\left[t^{n-m-r}\sum_{n_1,n_2,\ldots,n_r \in \mathbb{N}\cup\{0\}}^{n_1+n_2+\ldots+n_m=r}\left[\prod_{i=1}^{m}(\sigma^i(t))^{n_i}\right]\right]\right) \quad (9)$$

$\forall\ t \in T^{k^m}$ *holds for all* $m \le n \in N$, *where* $\sum_{n_1,n_2,\ldots\ldots,n_m \in \mathbb{N}\cup\{0\}}^{n_1+n_2+\ldots\ldots+n_m=r}$ *is the set of all distinct combinations of* $\{n_1, n_2, \ldots, n_m\}$ *such that the sum is equal to given r.*

Proof: We will show the equation (9) by induction. First, if $m = 1$, then

$$f^{\Delta}(t) = \left(\sum_{r=0}^{n-1}\left[t^{n-1-r}\sum_{n_1 \in \mathbb{N}\cup\{0\}}^{n_1=r}\left[\prod_{i=1}^{1}(\sigma^i(t))^{n_i}\right]\right]\right)$$

i.e $$f^{\Delta}(t) = t^{n-1} + t^{n-2}\sigma(t) + t^{n-3}(\sigma(t))^2 + t^{n-4}(\sigma(t))^3 + \ldots + t(\sigma(t))^{n-2} + (\sigma(t))^{n-1}.$$

Therefore the equation (9) is true for $m = 1$. Next, we assume that equation (9) is true for $m = s \in N$, then, by using the properties of delta derivatives, define $n_0 = 0$, we btain

$$
\begin{aligned}
f^{\Delta^{s+1}}(t) &= \frac{n!}{(n-s)!}\left(\sum_{r=0}^{n-s}\left[t^{n-s-r}\sum_{n_1,n_2,\dots,n_s\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\dots+n_s=r}\left[\prod_{i=1}^{s}(\sigma^{i}(t))^{n_i}\right]\right]\right)^{\Delta} \\
&= \frac{n!}{(n-s)!}\left(t^{n-s}+\sum_{r=1}^{n-s}\left[t^{n-s-r}\sum_{n_1,n_2,\dots,n_s\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\dots+n_s=r}\left[\prod_{i=1}^{s}(\sigma^{i}(t))^{n_i}\right]\right]\right)^{\Delta} \\
&= \frac{n!}{(n-s)!}\left(\left(t^{n-s}\right)^{\Delta}+\sum_{r=1}^{n-s}\left[t^{n-s-r}\sum_{n_1,n_2,\dots,n_s\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\dots+n_s=r}\left[\prod_{i=1}^{s}(\sigma^{i}(t))^{n_i}\right]\right]\right)^{\Delta} \\
&= \frac{n!}{(n-s)!}\left(\sum_{r=0}^{n-s-1}\left[t^{n-s-1-r}\sum_{n_1\in\mathbb{N}\cup\{0\}}^{n_1=r}\left[\prod_{i=1}^{1}(\sigma^{i}(t))^{n_i}\right]\right]\right) \\
&\quad + \frac{n!}{(n-s)!}\left(\sum_{r=1}^{n-s}\left[t^{n-s-r}\sum_{n_1,n_2,\dots,n_s\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\dots+n_s=r}\left[\prod_{i=1}^{s}(\sigma^{i}(t))^{n_i}\right]\right]\right)^{\Delta} \\
&\quad + \frac{n!}{(n-s)!}\left(\sum_{r=1}^{n-s}\left[(t^{n-s-r})^{\Delta}\sum_{n_1,n_2,\dots,n_s\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\dots+n_s=r}\left[\prod_{i=1}^{s}(\sigma^{i+1}(t))^{n_i}\right]\right]\right) \\
&= \frac{n!}{(n-s)!}\left(\sum_{r=0}^{n-s-1}\left[t^{n-s-1-r}\sum_{n_1\in\mathbb{N}\cup\{0\}}^{n_1=r}\left[\prod_{i=1}^{1}(\sigma^{i}(t))^{n_i}\right]\right]\right) \\
&\quad + \frac{n!}{(n-s)!}\left(\sum_{r=1}^{n-s}\left[t^{n-s-r}\sum_{n_1,n_2,\dots,n_s\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\dots+n_s=r}\sum_{l=0}^{s-1}\left[\prod_{i=0}^{l}(\sigma^{i+1}(t))^{n_i}\right]\right]\right)\times \\
&\quad \left(\sum_{r_1=0}^{n_{l+1}-1}\left[(\sigma^{l+2}(t)^{r_1})\right]\left[(\sigma^{l+1}(t)^{n_{l+1}-r_1-1})\right]\right)\times\left[\prod_{i=l+2}^{s}(\sigma^{i}(t))^{n_i}\right] \\
&\quad + \frac{n!}{(n-s)!}\left(\sum_{r=1}^{n-s}\left[\sum_{r_1=0}^{n-s-r-1}t^{n-s-r-1-r_1}\sum_{n_1\in\mathbb{N}\cup\{0\}}^{n_1=r_1}\left[\prod_{i=1}^{1}(\sigma^{i}(t))^{n_i}\right]\right]\right)\times \\
&\quad \left[\sum_{n_1,n_2,\dots,n_s\in\mathbb{N}\cup\{0\}}^{n_1+n_2+\dots+n_s=r}\left[\prod_{i=1}^{s}(\sigma^{i+1}(t))^{n_i}\right]\right]
\end{aligned}
$$

Now we collect the terms t^{n-s-1}, t^{n-s-2}, ..., from the above expression, we have

$$= \frac{n!}{(n-s-1)!} t^{n-s-1} + \frac{n!}{(n-s-1)!} t^{n-s-2} \sum_{n_1, n_2, \ldots, n_{s+1} \in \mathbb{N} \cup \{0\}}^{n_1+n_2+\ldots+n_{s+1}=1} \left[\prod_{i=1}^{s+1} (\sigma^i(t))^{n_i} \right] + \ldots..$$

$$\ldots\ldots + \frac{n!}{(n-s-1)!} \sum_{n_1, n_2, \ldots, n_{s+1} \in \mathbb{N} \cup \{0\}}^{n_1+n_2+\ldots+n_{s+1}=n-s-1} \left[\prod_{i=1}^{s+1} (\sigma^i(t))^{n_i} \right]$$

$$= \frac{n!}{(n-s-1)!} \left(\sum_{r=0}^{n-s-1} \left[t^{n-s-1-r} \sum_{n_1, n_2, \ldots, n_{s+1} \in \mathbb{N} \cup \{0\}}^{n_1+n_2+\ldots+n_{s+1}=r} \left[\prod_{i=1}^{s+1} (\sigma^i(t))^{n_i} \right] \right] \right)$$

So that equation (9) holds for $m = s+1$. By the principle of mathematical induction, (9) holds for all $m \le n \in N$.

NOTE: For $l > s$, $\prod_{i=l}^{s} = 1$ and $\sum_{i=1}^{s} = 0$

We assume that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of A are real and distinct, with multiplicity $n_1, n_2, \ldots, n_r$ respectively, for each eigenvalue λ_i there exists only one linearly independent eigenvector and regressive on T^k, such that $n_1 + n_2 + \ldots + n_r = n$ As we discussed in the Section 3, by using a non-singular transformation (2), (1) can be transform into (3). Thus, that D has r block matrices.

i.e $$D = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D_r \end{pmatrix} \qquad (10)$$

where Di is a square sub matrix of order n_i, $i = 1, 2, \ldots, r$, and is given by $D_i = \lambda_i I + J$ and J is defined by

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n_i \times n_i}$$

Suppose for each eigenvalue λ_i , if there exists $m_i \le n_i$ linearly independent eigen-vectors and remaining generalized eigenvectors are computed for last linearly inde-pendent eigenvector, then Di has the following form

$$\begin{pmatrix} \lambda_i I_{m_i-1 \times m_i-1} & \square \\ \square & \lambda_i I_{n_i-m_i+1 \times n_i-m_i+1} + J_{n_i-m_i+1 \times n_i-m_i+1} \end{pmatrix}_{n_i \times n_i} \qquad (11)$$

by using this technique, we can establish a fundamental matrix for equation (1), for each eigenvalue λ_i there exists only one linearly independent eigenvector, as stated in the following theorem.

Theorem 4.3 If D is defined by relation (10) then a fundamental matrix for

$$x^{\Delta} = Dx \qquad (12)$$

is given by $X = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & X_r \end{pmatrix}$ (13)

where X_i *is a fundamental matrix for* $x^{\Delta} = D_i x, i = 1,2, ..., r.$ (14)

The matrix X_i *is given by* $X_i = e_{\lambda_i}(t,t_0)W(e_{n_i}(t)),$ (15)

Where $W(e_{n_i}(t))$ *is wronskian matrix of* $\bar{e}_{n_i} = (1,t,t^2,........,t^{n_i-1})$ (16)

on time scale $T^{k^{n_i}}$

Proof: The matrix D_i of order n_i, was defined by

$$D_i = \lambda_i I + J$$

It is clear that λ_i is the only eigenvalue of D_i and that its multiplicity is n_i. A corresponding eigenvector is d_1, and it may be noted, incidentally, that $e_{\lambda_i}(t,t_0)d_1$ is a solution of equation (14). In order to find other solutions, we note that any vector *x* can be expressed as $x = e_{\lambda_i}(t,t_0)h$. If *x*, in this form, is substitute into equation (14), we get

$$(e_{\lambda_i}(t,t_0))^{\Delta} h + e_{\lambda_i}(\sigma(t),t_0)h^{\Delta} = \lambda_i I e_{\lambda_i}(t,t_0)h + Je_{\lambda_i}(t,t_0)h$$

$$e_{\lambda_i}(\sigma(t),t_0)h^{\Delta} = Je_{\lambda_i}(t,t_0)h$$

$$h^{\Delta} = e^{1}_{\lambda_i}(\sigma(t),t_0)Je_{\lambda_i}(t,t_0)h$$

$$h^{\Delta} = Je_{\lambda_i}(t,\sigma(t))h$$

Since

$$e_{\lambda_i}(t,\sigma(t)) = \exp\{-\int_t^{\sigma(t)} \xi_{\mu}(s)\lambda_i(s)\Delta s$$

$$= [\exp\{\int_t^{\sigma(t)} \xi_{\mu}(s)\lambda_i(s)\Delta s]^{-1}$$

And since

$$exp\left\{\int_{t}^{\sigma(t)} \xi_{\mu}(s)\lambda_i(s)\Delta(s)\right\} = 1+\lambda_i\mu(t)$$

$$\left[exp\left\{\int_{t}^{\sigma(t)} \xi_{\mu}(s)\lambda_i(s)\Delta(s)\right\}\right]^{-1} = \left[1+\lambda_i\mu(t)\right]^{-1}$$

therefore $e_{\lambda_i}(t,\sigma(t)) = [1+\lambda_i\mu(t)]^{-1}$

Hence, we get $$h^{\Delta} = J[1+\lambda_i\mu(t)]^{-1}h \qquad (17)$$

Since λ_i s are regressive, $1+\lambda_i\mu(t)\neq 0$ Hence *x* is a solution of (14) if and only if *h* satisfies (17). The latter equation is the companion vector equation associated with n_i^{th} order scalar equation $\frac{u^{[n_i]}}{1+\lambda_i\mu(t)} = 0$, *i.e.* $u^{[n_i]} = 0$ the vector $e_{n_i}(t)$, defined by (16), is a fundamental vector for this equation. Hence $W\left(e_{n_i}(t)\right)$ is a fundamental matrix for equation (17). It follows that X_i, defined by (15), is a fundamental matrix for equation (14). By using theorem (4.1). We may conclude that the matrix *X* defined by (13) is a fundamental matrix for equation (12). This proves the theorem.

The main result of this section is now stated in the following theorem
.

Theorem 4.4 *A fundamental matrix for* (1) *is given by Y = SX The matrix X is defined by the relations* (13), (15) *and* (16). *The matrix S is such that* $S^{-1}AS = D$ *, where D is the Jordan canonical form of A.*

Proof: The validity of the theorem is obvious, since the result follows from the direct consequence of preliminary discussion in the Section(3) of (1), (2) and (3).
For more than one linear independent eigenvector corresponding to each eigenvalue, then the following theorem gives the fundamental matrix.

Theorem 4.5 *If D is defined by relation* (10), *then a fundamental matrix for* (12) *is given by* (13) *where* X_i *is a fundamental matrix for* (14) *and* D_i *is defined by* (11). *The matrix* X_i *is given by* $X = \begin{pmatrix} e_{\lambda_i}(t,t_0)I_{m_i-1\times m_i-1} & \square \\ \square & e_{\lambda_i}(t,t_0)W(e_{n_i-m_i+1}(t)) \end{pmatrix}$

where $W(e_{n_i-m_i+1}(t))$ *is defined by* (16) *on time scale* $T^{k^{(n_i-m_i)}}$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 2 Issue 9, September - 2013

Example 1

An example that illustrates the case of repeated eigenvalues. Find a fundamental matrix for the following equation:

$$y^{\Delta} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} y$$

The eigen value of the coefficient matrix are $\lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 3$ the corresponding eigenvectors are

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Let $S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

It may be verified that $S^{-1}AS = D$

$$D = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

The matrix D is in Jordan canonical form. A fundamental matrix Y for the given equation is

$$T = h\square, \quad h > 0 \; Y = S\begin{pmatrix} e_3(t,t_0) & te_3(t,t_0) & t^2 e_3(t,t_0) \\ 0 & e_3(t,t_0) & (t+\sigma(t))e_3(t,t_0) \\ 0 & 0 & 2e_3(t,t_0) \end{pmatrix}$$

(i) If $T = \square$ then $Y = S\begin{pmatrix} e^{3(t-t_0)} & te^{3(t-t_0)} & t^2 e^{3(t-t_0)} \\ 0 & e^{3(t-t_0)} & 2te^{3(t-t_0)} \\ 0 & 0 & 2e^{3(t-t_0)} \end{pmatrix}$

(ii) If $T = \square$ then

$$Y = S\begin{pmatrix} 4^{(t-t_0)} & t4^{(t-t_0)} & t^2 4^{(t-t_0)} \\ 0 & 4^{(t-t_0)} & (2t+1)4^{(t-t_0)} \\ 0 & 0 & (2)4^{(t-t_0)} \end{pmatrix}$$

(iii) If $T = h\square, \quad h > 0$

then $Y = S\begin{pmatrix} (1+3h)^{\frac{(t-t_0)}{h}} & t(1+3h)^{\frac{(t-t_0)}{h}} & t^2(1+3h)^{\frac{(t-t_0)}{h}} \\ 0 & (1+3h)^{\frac{(t-t_0)}{h}} & (2th+h^2)(1+3h)^{\frac{(t-t_0)}{h}} \\ 0 & 0 & (2h)(1+3h)^{\frac{(t-t_0)}{h}} \end{pmatrix}$

Example 2

Finally, an example that illustrates the case of some are repeated and some are distinct eigenvalues. Find a fundamental matrix for the following equation:

$$y^{\Delta} = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{pmatrix} y$$

The eigen value of the coefficient matrix are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 1$ the corresponding eigenvectors are

$$s_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $S = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

It may be verified that $S^{-1}AS = D$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix D is in Jordan canonical form. A fundamental matrix Y for the given equation is

$$Y = S\begin{pmatrix} e_2(t,t_0) & 0 & 0 \\ 0 & e_1(t,t_0) & te_1(t,t_0) \\ 0 & 0 & e_1(t,t_0) \end{pmatrix}$$

(i) If $T = \square$, $Y = S\begin{pmatrix} e^{2(t-t_0)} & 0 & 0 \\ 0 & e^{(t-t_0)} & te^{(t-t_0)} \\ 0 & 0 & e^{(t-t_0)} \end{pmatrix}$

(ii) If $T = \square$, then $Y = S\begin{pmatrix} 3^{(t-t_0)} & 0 & 0 \\ 0 & 2^{(t-t_0)} & t2^{(t-t_0)} \\ 0 & 0 & 2^{(t-t_0)} \end{pmatrix}$

(iii) If $T = h\square$, $h > 0$, then $Y = S\begin{pmatrix} (1+2h)^{\frac{(t-t_0)}{h}} & 0 & 0 \\ 0 & (1+h)^{\frac{(t-t_0)}{h}} & t(1+h)^{\frac{(t-t_0)}{h}} \\ 0 & 0 & (1+h)^{\frac{(t-t_0)}{h}} \end{pmatrix}$

ISSN: 2278-0181
Vol. 2 Issue 9, September - 2013

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