International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 04 , April - 2026

The Uniqueness of Differential-Difference Polynomials of Meromorphic Functions Sharing A Small Function

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Abstract - In this article, we establish the uniqueness outcomes of a meromorphic function in which a small function is shared through Differential Difference polynomials. We use second order difference operator $\triangle^2 f(z) = f(z+2c) - 2f(z+c) + f(z)$ and Utilizing a non-constant differential polynomial $L(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f^{(1)} + a_0 f$ we aim to establish the uniqueness of two transcendental meromorphic functions with zero order. To achieve this, we will apply Leibnitz's theorem to substantiate the obtained result.

Keywords: Entire function, Meromorphic function, Uniqueness, Shift operator, Difference polynomial, Differential polynomial.

Subject Code: MSC 30D35

1 INTRODUCTION

The Nevanlinna theory of value distribution focuses on the distribution density of points where a meromorphic function attains a specific value in the complex plane. For standard definitions and concepts in Nevanlinna theory, one can refer to works by W. K. Hayman [9] and C. C. Yang [15]. Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane $\mathbb{C}$ that are not constant. For $\beta \in \mathbb{C} \cup \{\infty\}$, we say that $f(z)$ and $g(z)$ are $\beta\ CM$ if $f-\beta$ and $g-\beta$ have the same zero's and f and g share $\beta\ IM$ if multiplicity is not counted.

The amalgamation of complex derivatives and complex differences results in a complex differential-difference polynomial. Nevanlinna extended classical results on the distribution of a-points from entire functions to meromorphic functions. A differential-difference polynomial comprises terms involving the function f(z), its shifts, and the derivatives of these shifts.

Numerous research articles have delved into the study of entire and meromorphic functions, particularly focusing on scenarios where differential polynomials exhibit sharing of specific values, small functions, or fixed points. Mathematicians worldwide have contributed to this body of work, as evidenced by publications such as [2, 4, 12, 13].

Contemporary research efforts have witnessed a significant number of scholars exploring and attaining uniqueness results pertaining to difference polynomials. In 2006, R. G. Halburd and R. J. Korhonen [6] introduced a version of Nevanlinna theory grounded in difference operators, accompanied by the formulation of the difference logarithmic derivative lemma [7], thus advancing the development of the difference analogue of Nevanlinna theory. Chiang and Feng [3] have independently played a crucial role in contemplating the difference analogues of this theory. Consequently, mathematicians worldwide have directed their focus towards scrutinizing the distribution of zeros in various types of difference polynomials.

In 2007, I. Laine and C.C Yang [11] gave the result that "If n≥2, then expression $f^n(z)f(z+\eta)$ for a finite-order transcendental entire function f(z) and a non-zero complex constant η takes on any non-zero value $a \in \mathbb{C}$ infinitely often."

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ISSN: 2278-0181
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In 2010, J. Zhang [16] explored the zeros of difference polynomials and derived the result that "for every integer n≥2, the expression $f^n(z)(f(z)-1)f(z+\eta)-\alpha(z)$ has infinitely many zeros if $f(z)$ is a transcendental entire function with finite order, $\alpha(z)(\neq 0)$ is a small function with respect to $f(z)$, and η is a non-zero complex constant."

In the same paper, the author also established the following result regarding uniqueness.
"Let $\alpha(z)$ be the small function with regard to both $f(z)$ and $g(z)$ whose are transcendental entire functions with finite order. Assuming c to be a non-zero complex constant with $n \geq 7, f(z) \equiv g(z)$ if $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM."

Recently, the uniqueness results on difference polynomial of entire functions of the form $f^n(z)(f(z)-1)^{(\kappa)}f(z+c)$ and $g^n(z)(g(z)-1)^{(\kappa)}g(z+c)$ as well as q-shift difference polynomial of meromorphic functions of the form $f^n(z)(f(qz+c)^{(\kappa)}-1$ and $g^n(z)g(qz+c)^{(\kappa)}-1$ have been investigated by R.S Dyavanal [4] and Q. Zhao et.al [18] respectively.

To improve the above result, the author N.V Thin[14] in 2017, gave a new conception(idea) for the unicity theorems by considering q-difference polynomial of meromorphic function. In this paper, we prove the result.

Inspired by the extensive research in this field, we have demonstrated the following results by extending the findings of N.V. Thin and Renukadevi S. Dyavanal, among others, focusing on the differential-difference polynomial.

Theorem 1.1: *Consider $f(z)$ and $g(z)$ as two transcendental meromorphic (or entire) functions of zero order, such that $\triangle^2 f(z) \neq 0$ and $\triangle^2 g(z) \neq 0$ where c is non-zero complex constant. k, n, m are positive integers. $\alpha(z)$ be a small function and let $L(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f^{(1)} + a_0 f$ be a non-constant differential polynomial with constant coefficients a_0, a_1, a_{n-1}, $a_n(\neq 0)$ and m be the distinct zeros of $L(f)$. If $n > 2km + 2m + 2k + 14$. and $[L(f) \triangle^2 f(z)]^{(k)}$ and $[L(g) \triangle^2 g(z)]^{(k)}$ share $\alpha(z)$, ∞ CM.* Then, one of the two *cases* listed below is true.

1. $f \equiv tg$ for a constant t with $ht^d = 1$ when $d = LCM\{\lambda_j; j = 0,1,(n-1)\}$ and

$$\lambda_j = \begin{cases} j, & if\ a_j \neq 0 \\ n, & if\ a_j = 0 \end{cases}$$

2. f and g satisfy the algebraic equations $R(f,g) = 0$ when

$$R(w_1,w_2) = P(w_1)\big(w_1(z+2c) - 2w_1(z+c) + w_1(z)\big) - P(w_2)\big(w_2(z+2c) - 2w_2(z+c) + w_2(z)\big)$$

Theorem 1.2: *Let $f(z)$ and $g(z)$ be transcendental meromorphic $(resp\text{-}entire)$ functions of zero order, such that $\triangle^2 f(z) \neq 0$ and $\triangle^2 g(z) \neq 0$ where c is non-zero complex constant. k, n, $m \in Z^+$. $\alpha(z)$ be a small function and let $L(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f^{(1)} + a_0$f be a non-constant differential polynomial with constant coefficients a_0, a_1, a_{n-1}, $a_n(\neq 0)$ and m be the distinct zeros of $L(f)$. If $n > 5mk + 7m + 8k + 38$ and $[L(f) \triangle^2 f(z)]^{(k)}$ and $[L(g) \triangle^2 g(z)]^{(k)}$ share$\alpha(z)$ IM.* Then, one of the three *cases* listed below is true.

1. $[L(f) \triangle^2 f(z)]^{(k)}[L(g) \triangle^2 g(z)]^{(k)} = \alpha^2$

2. $f \equiv tg$ for a constant t with $ht^d = 1$ when $d = LCM\{\lambda_j; j = 0,1,(n-1)\}$ and

$$\lambda_j = \begin{cases} j, & if\ a_j \neq 0 \\ n, & if\ a_j = 0 \end{cases}$$

3. f and g satisfy the algebraic equations $R(f,g) = 0$ when

$$R(w_1,w_2) = P(w_1)\big(w_1(z+2c) - 2w_1(z+c) + w_1(z)\big) - P(w_2)\big(w_2(z+2c) - 2w_2(z+c) + w_2(z)\big)$$

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ISSN: 2278-0181
Vol. 15 Issue 04 , April - 2026

2 LEMMA SECTION

Lemma 2.1 [*3*] Assuming $f(z)$ is a transcendental meromorphic function of finite order, the Nevanlinna characteristic function equality holds:

$$T(r, f(z+c)) = T(r,f) + S(r,f)$$

.

Lemma 2.2 [*3*] *By using Lemma 2.1 and properties of* $T(r,f)$

$$T(r, \triangle_c f(z)) = T(r, f(z+c) - f(z))$$

$$\leq T(r, f(z+c)) + T(r,f) + (r,f)$$

$$\leq 2T(r,f) + S(r,f)$$

.

Lemma 2.3 [*8*] Given $f(z)$ as a transcendental meromorphic function of finite order and c as a non-zero complex constant, the relation

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f)$$

holds.

Lemma 2.4. [*3*] Considering $f(z)$ as a meromorphic function of finite order and c as a non-zero complex constant, the expression

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f)$$

holds true.

Lemma 2.5 [*15*] *If* F *and* G *are non-constant meromorphic functions sharing a common value, then one of the following three cases must occur:*

1.

$$\max\{T(r,F), T(r,G)\} \leq N_2\left(r, \frac{1}{F}\right) + N_2(r,F) + N_2\left(r, \frac{1}{G}\right) + N_2(r,G) + S(r,F) + S(r,G)$$

2.

$$F \equiv G$$

3.

$$FG \equiv 1$$

Lemma 2.6 [*12*] *Assume* f *is a non-constant meromorphic function, and consider two positive integers* p *and* k*. Then,*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f)$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f)$$

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ISSN: 2278-0181
Vol. 15 Issue 04 , April - 2026

Lemma 2.7 [17] Consider two non-constant meromorphic functions, f and g, *and let* $a(z)(a \not\equiv 0, \infty)$ *be a small function of both* f *and* g. *If* f *and* g *share* $a(z) - IM$, then one of the following three cases must be true:

(i)

$$T(r,f) \leq N_2(r,f) + N_2\left(r,\frac{1}{f}\right) + N_2(r,g) + N_2\left(r,\frac{1}{g}\right) + 2\left[\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f)\right]$$

$$+\left[\overline{N}\left(r,\frac{1}{g}\right) + \overline{N}(r,g)\right] + S(r,f) + S(r,g)$$

and similar equality for $T(r,g)$

(ii) $f \equiv g$

(iii) $fg \equiv a^2$.

Lemma 2.8 *Let* $f(z)$ *be a transcendental meromorphic function of zero order and let* $F = L(f) \triangle^2 \left(f(z)\right)$, *where* n *is positive integer, then*

$$(n-4)T(r,f) + S(r,f) \leq T(r,f)$$

Proof. Derived from the first fundamental theorem, we acquire:

$$(n+2)T(r,f) = T\big(r, f(z)L(f)\big) + S(r,f)$$

$$\leq T\left(r, f(z)\frac{F}{\triangle^2\left(f(z)\right)}\right) + S(r,f)$$

$$\leq T(r,F) + T\left(r,\frac{\triangle^2(f)}{f(z)}\right) + S(r,f)$$

$$\leq T(r,F) + T\left(r,\frac{f(z+2c)}{f(z)}\right) + 2T\left(r,\frac{f(z+c)}{f(z)}\right) + S(r,f)$$

$$\leq T(r,F) + m\left(r,\frac{f(z+2c)}{f(z)}\right) + N\left(r,\frac{f(z+2c)}{f(z)}\right) + 2m\left(r,\frac{f(z+c)}{f(z)}\right)$$

$$+2N\left(r,\frac{f(z+c)}{f(z)}\right) + S(r,f)$$

$$(n+2)T(r,f) \leq T(r,F) + 6T(r,f) + S(r,f)$$

$$(n-4)T(r,f) - S(r,f) \leq T(r,F)$$

.

Lemma 2.9 *If* $f(z)$ *be a transcendental entire function of zero order and* $L(f) = a_n f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0 f$. *Let* $F = L(f) \triangle^2 (f)$, *where* n *is positive integer. Then*

$$(n-1)T(r,f) + S(r,f) \leq T(r,F)$$

Proof.

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$$(n+2)T(r,f) = T\big(r, f(z)L(f)\big) + S(r,f)$$

$$\leq T(r,F) + T\left(r, \frac{\Delta^2(f)}{f(z)}\right) + S(r,f)$$

$$\leq T(r,F) + T\left(r, \frac{f(z+2c) - 2f(z+c) + f(z)}{f(z)}\right) + S(r,f)$$

$$\leq T(r,F) + T\left(r, \frac{f(z+2c)}{f(z)}\right) + 2T\left(r, \frac{f(z+c)}{f(z)}\right) + S(r,f)$$

$$\leq T(r,F) + m\left(r, \frac{f(z+2c)}{f(z)}\right) + N\left(r, \frac{f(z+2c)}{f(z)}\right) + 2m\left(r, \frac{f(z+c)}{f(z)}\right)$$

$$+2N\left(r, \frac{f(z+c)}{f(z)}\right) + S(r,f)$$

$$(n+2)T(r,f) \leq T(r,F) + 3T(r,f) + S(r,f)$$

$$(n-1)T(r,f) - S(r,f) \leq T(r,F)$$

.

3 MAIN RESULTS

Proof of Theorem 1.1:

Proof. Let

$$F = L(f)\,\Delta^2(f)$$

$$F^{(k)} = [L(f)\,\Delta^2(f)]^{(k)}$$

and

$$G = L(g)\,\Delta^2(g)$$

$$G^{(k)} = [L(g)\,\Delta^2(g)]^{(k)}.$$

Since, $F^{(k)}$ and $G^{(k)}$ share $\alpha(z)$, ∞, *CM*, then there exists a non-zero constant γ such that

$$(([L(f)\ \Delta\ \text{^}2\ (f)]\text{^}((k)\ )])/\alpha(z)\ - 1)/(([L(g)\ \Delta\ \text{^}2\ (g)]\text{^}((k)\ )])/\alpha(z)\ - 1) = \gamma$$

(3.1)

we get,

$$[L(f)\,\Delta^2(f)]^{(k)} - \alpha(z)(1-\gamma) = \gamma[L(g)\,\Delta^2(g)]^{(k)}$$

.

Now, we will prove that $\gamma = 1$

On the contrary, if $\gamma \neq 1$. UsingSecondFundamental Theorem and Lemma2.6

$$T\big(r, F^{(k)}\big) \leq \overline{N}\big(r, F^{(k)}\big) + \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F(k)_{-\alpha(z)(1-\gamma)}}\right) + S(r,f)$$

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$$T(r,F^{(k)}) \leq \overline{N}(r,F) + T(r,F^{(k)}) - T(r,F) + N_{k+1}\left(r,\frac{1}{F}\right) + k\overline{N}(r,F) + N_{k+1}\left(r,\frac{1}{F}\right)$$
$$+S(r,f) + S(r,g)$$

.

This implies,

$$T(r,L(f)\triangle^2(f)) \leq \overline{N}(r,L(f)\triangle^2(f)) + N_{k+1}\left(r,\frac{1}{L(f)\triangle^2(f)}\right) + k\overline{N}(r,L(g)\triangle^2(g))$$
$$+N_{k+1}\left(r,\frac{1}{L(g)\triangle^2(g)}\right) + S(r,f) + S(r,g)$$
$$T(r,L(f)\triangle^2(f)) \leq \overline{N}(r,f) + \overline{N}(r,\triangle^2(f)) + N_{k+1}\left(r,\frac{1}{L(f)}\right) + N\left(r,\frac{1}{\triangle^2(f)}\right) + k\overline{N}(r,g)$$
$$+k\overline{N}(r,\triangle^2(g)) + N_{k+1}\left(r,\frac{1}{L(g)}\right) + N\left(r,\frac{1}{\triangle^2(g)}\right) + S(r,f) + S(r,g)$$

Combining this with Lemma 2.7, we get

$$(n-4)T(r,f) \leq 2T(r,f) + 2kT(r,g) + (k+1)(m)T(r,f) + (k+1)(m)T(r,g)$$
$$+4T(r,f) + 4T(r,g) + S(r,f) + S(r,g)$$

,

$$(n-4)T(r,f) \leq (6+km+m)T(r,f) + (4+2k+km+m)T(r,g) + S(r,f) + S(r,g) \tag{3.2}$$

$$(n-4)T(r,g) \leq (6+km+m)T(r,g) + (4+2k+km+m)T(r,f) + S(r,f) + S(r,g) \tag{3.3}$$

Combining (3.2) and (3.3) we get

$$(n-4)[T(r,f)+T(r,g)] \leq (10+2km+2k+2m)(T(r,f)+T(r,g)) + S(r,f) + S(r,g)$$
$$(n-2km-2k-2m-14)[T(r,f)+T(r,g)] \leq S(r,f) + S(r,g)$$

which is contradiction to $n > 2km + 2m + 2k + 14$.

Thus, we get $\gamma = 1$, Hence from (3.1) we have

$$[L(f)\triangle^2(f)]^{(k)} = [L(g)\triangle^2(g)]^{(k)}.$$

We get

$$L(f)\triangle^2(f) = L(g)\triangle^2(g) + r(z) \tag{3.4}$$

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ISSN: 2278-0181
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where $r(z)$ is a polynomial. Suppose $r(z) \neq 0$, then we get

$$\frac{L(f)\,\Delta^2(f)}{r(z)} = \frac{L(g)\,\Delta^2(g)}{r(z)} + 1$$

By utilizing Lemma 2.7 and Nevanlinna's Second Fundamental Theorem, we obtain:

$$(n-4)T(r,f) \le T\left(r, \frac{L(f)\,\Delta^2(f)}{r(z)}\right) + S(r,f)$$

$$\le \overline{N}\left(r, \frac{L(f)\,\Delta^2(f)}{r(z)}\right) + \overline{N}\left(r, \frac{r(z)}{L(f)\,\Delta^2(f)}\right) + \overline{N}\left(r, \frac{r(z)}{L(g)\,\Delta^2(g)}\right) + S(r,f)$$

$$\le 2T(r,f) + \overline{N}\left(r, \frac{1}{L(f)}\right) + \overline{N}\left(r, \frac{1}{\Delta^2(f)}\right) + \overline{N}\left(r, \frac{1}{L(g)}\right) + \overline{N}\left(r, \frac{1}{\Delta^2(g)}\right) + S(r,f)$$

$$(n-4)T(r,f) \le (m+4)[T(r,f) + T(r,g)] + 2T(r,f) + S(r,f) \qquad (3.5)$$

Similarly, we have

$$(n-4)T(r,g) \le (m+4)[T(r,f) + T(r,g)] + 2T(r,g) + S(r,g) \qquad (3.6)$$

Combining (3.5) and (3.6) we get

$$(n-4)[T(r,f) + T(r,g)] \le 2(m+4)[T(r,f) + T(r,g)] + 2[T(r,f) + T(r,g)] + S(r,f) + S(r,g)$$

$$(n-4-2m-10)[T(r,f) + T(r,g)] \le S(r,f) + S(r,g)$$

.

Which contradicts to $n > 2km + 2m + 2k + 14 > 2m + 14$. Therefore, $r(z) = 0$.

Therefore, deriving from equation (3.4), we obtain:

$$L(f)\,\Delta^2(f) \equiv L(g)\,\Delta^2(g) \qquad (3.7)$$

That is

$$\left(a_n f^{(n)} + a_{n-1} f^{(n-1)} + \ +a_0 f\right)\Delta^2(f) = \left(a_n g^{(n)} + a_{n-1} g^{(n-1)} + \ +a_0 g\right)\Delta^2(g)$$

which implies

$$\left(a_n f^{(n)} + a_{n-1} f^{(n-1)} + \ +a_0 f\right)\left(f(z+2c) - 2f(z+c) + f(z)\right)$$
$$= \left(a_n g^{(n)} + a_{n-1} g^{(n-1)} + \ +a_0 g\right)\left(g(z+2c) - g(z+c) + g(z)\right)$$

Substitute $f = gh$ or $h = \frac{f}{g}$, and we examine the following scenarios:

Case 1: If $h(z)$ is a constant, then substitute $f = gh$ in (3.7), we have

$$\left[\left(a_n (gh)^{(n)} + a_{n-1}(gh)^{(n-1)} + \ +a_0(gh)\right)\right] hg(z+2c) -$$

$$2\left[a_n (gh)^{(n)} + a_{n-1}(gh)^{(n-1)} + \ +a_0(gh)\right] hg(z+c)$$

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$$+\left[a_n(gh)^{(n)}+a_{n-1}(gh)^{(n-1)}+\ +a_0(gh)\right]hg(z)$$

$$=\left[\left(a_n g^{(n)}+a_{n-1}g^{(n-1)}+\ +a_0 g\right)\right]g(z+2c)-2\left[\left(a_n g^{(n)}+a_{n-1}g^{(n-1)}+\ +a_0 g\right)\right]g(z+c)+\left[\left(a_n g^{(n)}+a_{n-1}g^{(n-1)}+\ +a_0 g\right)g\right]g(z)$$

Applying Leibnitz theorem,

$$[g(z+2c)-g(z+c)+g(z)]\{a\_n\,[h(g+h)\text{^}n-g\text{^}n\,]+a\_(n-1)\,g\text{^}(n-1)\,[h(g+h)\text{^}(n-1)-g\text{^}(n-1)\,]+a\_(n-2)\,[h(g+h)\text{^}(n-2)-g\text{^}(n-2)\,]+\cdots+a\_0\,g(h\text{^}2-1)\}=0$$
,

$$[g(z+2c)-g(z+c)+g(z)]\{a\_n\,g\text{^}n\,[h(1+h/g)\text{^}n-1]+a\_(n-1)\,g\text{^}(n-1)\,[h(1+h/g)\text{^}(n-1)-1]+a\_(n-2)\,g\text{^}(n-2)\,[h(1+h/g)\text{^}(n-2)-1]+\cdots+a\_0\,g(h\text{^}2-1)\}=0$$
.

Put $1+\frac{h}{g}=t$

$$[g(z+2c)-g(z+c)+g(z)]\{a_n g^n(ht^n-1)a_{n-1}g^{n-1}(ht^{n-1}-1)+a_{n-2}g^{n-2}(ht^{n-2}-1)+\ \dots+a_0 g(h^2-1)\}=0$$

which implies $ht^d=1$ where $d=\{\lambda_j : j=0,1,\dots,(n-1)\}$ and $\lambda_j=\begin{cases} j, & \text{if } a_j\neq 0\\ n, & \text{if } a_j=0\end{cases}$

Thus, $f=hg$, where$h$ is constantwith $ht^d=1$, where $d=LCM\{\lambda_j : j=0,1,(n-1)\}$ and

$$\lambda_j=\begin{cases} j, & \text{if } a_j\neq 0\\ n, & \text{if } a_j=0\end{cases}$$

Case 2: Assume that, $h(z)$ is notaconstant, In such a case, f and g fulfill the algebraic equation $R(f,g)=0$, $R(w_1,w_2)=P(w_1)[w_1(z+2c)-2w_1(z+c)+w_1(z)]-P(w_2)[w_2(z+2c)-2w_2(z+c)+w_2(z)]$.

Note: In the scenario where $f(z)$ and $g(z)$ are transcendental entire functions, it follows that $N(r,F)=0$ and $N(r,G)=0$.

By performing calculations analogous to the scenario of meromorphic functions, we straightforwardly deduce the conclusion of Theorem 1.1 when $n>2km+2m+2k+11$.

Proof of Theorem 1.2:
Proof. Let

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ISSN: 2278-0181
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$$F(z) = L(f) \triangle^2 (f)$$

and

$$G(z) = L(g) \triangle^2 (g)$$

,

we see that $F^{(k)}$ and $G^{(k)}$ share $\alpha(z)\ IM$. If the condition stated in equation 1 of Lemma 2.5 is satisfied, then, through the application of Lemma 2.6, we derive:

$$T\left(r, \left(L(f) \triangle^2 (f)\right)^{(k)}\right) \leq N_2\left(r, \frac{1}{\left(L(f) \triangle^2 (f)\right)^{(k)}}\right) + N_2\left(r, \left(L(f) \triangle^2 (f)\right)^{(k)}\right) + N_2\left(r, \frac{1}{\left(L(g) \triangle^2 (g)\right)^{(k)}}\right)$$

$$+N_2\left(r, \left(L(g) \triangle^2 (g)\right)^{(k)}\right) + 2\left[\overline{N}\left(r, \frac{1}{\left(L(f) \triangle^2 (f)\right)^{(k)}}\right)\right] + \overline{N}\left(r, \left(L(f) \triangle^2 (f)\right)^{(k)}\right)$$

$$+\overline{N}\left(r, \frac{1}{\left(L(g) \triangle^2 (g)\right)^{(k)}}\right) + \overline{N}\left(r, \left(L(g) \triangle^2 (g)\right)^{(k)}\right) + S(r, f) + S(r, g)$$

$$T\left(r, \left(L(f) \triangle^2 (f)\right)^{(k)}\right) \leq N_2\left(r, \left(L(f) \triangle^2 (f)\right)^{(k)}\right) + T\left(r, \left(L(f) \triangle^2 (f)\right)^{(k)}\right) - T\left(r, L(f) \triangle^2 (f)\right)$$

$$+N_{k+2}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right) + N_{k+2}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right) + k\overline{N}\left(r, L(f) \triangle^2 (f)\right)$$

+N_2 (r,(L(g) △ ^2 (g)))^((k))) + 2[N_(k + 1) (r, 1/(L(f) △ ^2 (f))) + k¯N (r, L(f) △ ^2 (f))]

$$+\overline{N}\left(r, \left(L(g) \triangle^2 (g)\right)^{(k)}\right) + N_{k+1}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right) + k\overline{N}\left(r, L(g) \triangle^2 (g)\right)$$

$$+\overline{N}\left(r, \left(L(g) \triangle^2 (g)\right)^{(k)}\right) + S(r, F) + S(r, G)$$

which implies,

$$T\left(r, L(f) \triangle^2 (f)\right) \leq N_2\left(r, L(f) \triangle^2 (f)\right) + N_{k+2}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right) + N_{k+2}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right)$$

+k¯N (r, L(f) △ ^2 (f)) + N_2 (r, L(g) △ ^2 (g)) + 2[N_(k + 1) (r, 1/(L(g) △ ^2 (g)))

+k¯N (r, L(f) △ ^2 (f)) + ¯N (r, L(f) △ ^2 (f))] + N_(k + 1) (r, 1/(L(g) △ ^2 (g)))

$$+k\overline{N}\left(r, L(g) \triangle^2 (g)\right) + \overline{N}\left(r, L(g) \triangle^2 (g)\right) + S(r, f) + S(r, g)$$

Therefore,

$$T\left(r, L(f) \triangle^2 (f)\right) \leq (2k + 4)\overline{N}\left(r, L(f) \triangle^2 (f)\right) + N_{k+2}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right) + 2N_{k+1}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right)$$

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$$+(2k+3)\overline{N}\left(r, L(g) \triangle^2 (g)\right) + N_{k+2}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right) + N_{k+1}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right)$$
$$+S(r,F) + S(r,G)$$

(3.8)

similarly,

$$T\left(r, L(g) \triangle^2 (g)\right) \leq (2k+4)\overline{N}\left(r, L(g) \triangle^2 (g)\right) + N_{k+2}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right) + 2N_{k+1}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right)$$
$$+(2k+3)\overline{N}\left(r, L(f) \triangle^2 (f)\right) + N_{k+2}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right) + N_{k+1}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right)$$
$$+S(r,F) + S(r,G)$$

.
(3.9)

We have,

$$\overline{N}\left(r, L(f) \triangle^2 (f)\right) \leq \overline{N}(r,f) + \overline{N}\left(r, \triangle^2 (f)\right)$$
$$\leq 2T(r,f) + S(r,f)$$

(3.10)

$$N_{k+2}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right) \leq N_{k+2}\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{\triangle^2 (f)}\right)$$
$$\leq \left((m)(k+2)+4\right)T(r,f) + S(r,f)$$

(3.11)

and

$$N_{k+1}\left(r, \frac{1}{L(f) \triangle^2 (f)}\right) \leq N_{k+1}\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{\triangle^2 (f)}\right)$$
$$\leq \left((m)(k+1)+4\right)T(r,f) + S(r,f)$$

(3.12)

Similarly,

$$\overline{N}\left(r, L(g) \triangle^2 (g)\right) \leq 2T(r,g) + S(r,g)$$

(3.13)

$$N_{k+2}\left(r, \frac{1}{L(g) \triangle^2 (g)}\right) \leq \left((m)(k+2)+4\right)T(r,g) + S(r,g)$$

(3.14)

and

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Vol. 15 Issue 04 , April - 2026

$$N_{k+1}\left(r,\frac{1}{L(g)\triangle^2(g)}\right) \leq \big((m)(k+1)+4\big)T(r,g)+S(r,g) \tag{3.15}$$

Substituting (3.10) to (3.12) in (3.8), we get

$$\begin{aligned} T\big(r,L(f)\triangle^2(f)\big) &\leq 2(2k+4)T(r,f)+\big((m)(k+2)+4\big)T(r,f)+2\big((m)(k+1)+4\big)T(r,f) \\ &+(2k+3)2T(r,g)+\big((m)(k+2)+4\big)T(r,g)+\big((m)(k+1)+4\big)T(r,g)+S(r,f) \\ &+S(r,g) \end{aligned}$$

.

Using Lemma 2.8 in L.H.S

$$(n-4)T(r,f) \leq (3mk+4m+4k+20)T(r,f)+(2mk+3m+4k+14)T(r,g)+S(r,f)+S(r,g) \tag{3.16}$$

Similarly, substituting (3.13) to (3.15) in (3.9), we get

$$\begin{aligned}(n-4)T(r,g) \leq (3mk+4m+4k+20)T(r,g)+(2mk+3m+4k+14)T(r,f)+S(r,f) \\ +S(r,g)\end{aligned} \tag{3.17}$$

Now, following is derived from equations (3.16) and (3.17),

$$\begin{aligned}(n-4)[T(r,f)+T(r,g)] &\leq (3mk+4\mathrm{m}+4k+20)[T(r,f)+T(r,g)] \\ &+(2mk+3m+4k+14)[T(r,f)+T(r,g)]+S(r,f)+S(r,g) \\ &\leq (5mk+7m+8k+34)[T(r,f)+T(r,g)]+S(r,f)+S(r,g) \\ (n-4-5mk-7m-8k-34)[T(r,f)+T(r,g)] &\leq s(r,f)+S(r,g)\end{aligned}$$

which is contradiction to $n > 5mk+7m+8k+38$.

Therefore, based on Lemma 2.7, we observe either $F^{(k)}G^{(k)} \equiv \alpha^2(z)$ or $F^{(k)} \equiv G^{(k)}$.

Case 1: Assume that $F^{(k)}G^{(k)} \equiv \alpha^2(z)$

$$[L(f)\triangle^2 f(z)]^{(k)}[L(g)\triangle^2 f(z)]^{(k)} = \alpha^2(z),$$

This corresponds to one of the conclusions drawn from Theorem 1.1.

Case 2: Now, let's examine $F^{(k)} \equiv G^{(k)}$, implying that, following a reasoning similar to Theorem 1.1, we deduce that f and g satisfy one of the following two statements.

1. $f = tg$ for a constant t with $ht^d = 1$, where $d = LCM\ \{\lambda_j : j = 0,1,\ldots,(n-1)\}$ and

$$\lambda = \begin{cases} j, & \text{if } a_j \neq 0 \\ n, & \text{if } a_j = 0 \end{cases}$$

2. f and g meeting the requirements of the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2) = L(w_1)[w_1(z+2c) - 2w_1(z+c) + w_1(z)] - L(w_2)[w_2(z+2c) - 2w_2(z+c) + w_2(z)]$. Hence the proof.

Conclusion: The uniqueness of a differential-difference polynomial sharing a small function is confirmed through the verification of Lemmas (2.8) and (2.9). To establish the uniqueness of two transcendental meromorphic functions with zero order, we applied Leibniz's theorem. The utilization of a second-order operator is instrumental in demonstrating the conditional existence of two meromorphic functions with zero order.

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