

The Quadratic Equation and its Numerical Roots

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Abstract— In this paper an alternative procedure is proposed to obtain the proof of the general formula to solve quadratic equations, which can be taught to high school and university students. Likewise, the solution through power series is discussed, which can be used when there are values much less than 1 in the equation. In the same way, an analysis is presented on the significant digits at the roots of the quadratic equation.

Keywords— Quadratic equation; Exact roots; School mathematical speech; Significant digits.

I. INTRODUCTION

In current times, pedagogy plays a very important role in the teaching-learning processes at each of the educational levels since different learning strategies are used, such as the flipped classroom [1], project-based learning, meaningful learning [2], grid technique [3], etc.,

In the case of textbooks, there is a trend that is continuously repeated when content is updated when presenting a new edition, because when a new edition is published as in [4-18], favorable changes are made in the pedagogy with which the thematic concepts are presented, more demonstrative examples are included, more proposed exercises, the use of more than one ink in their printing, the use of schemes and illustrations, etc. In other cases, the new editions incorporate the use of mathematical software such as Maple [19,20], Matlab [21,22], GeoGebra [23], Excel [24], among others. However, there are contents that remain intact, preserving the school discourse.

In mathematics teaching, school mathematical discourse (dME) is all the language that is introduced into a class. It is a system of reason that produces symbolic violence due to the imposition of arguments, meanings and procedures [25,26]. It is everything that remains unchanged despite the innovations in relation to school mathematics, since this innovation does not modify the concepts that are being taught, but the way in which it is being taught [27].

Many textbooks, such as those used in mathematics, are designed under the dME. It is frequent to blame mainly the teachers for the failure in the learning processes of mathematics, ignoring that the books are influenced by the same school mathematical discourse [27].

The school speech is reflected in the textbooks used in the different educational and professional areas. For example, in control engineering texts [10] the way in which the transfer function is determined does not change, comparing an updated edition against a previous one. Similarly, it occurs when another work is reviewed within the same discipline [11].

Something similar happens in other areas of knowledge. For example, in the field of administration, important concepts are taught, including various ratios and relationships that allow knowing the solvency, liquidity and productivity of a business, known as financial ratios, such as the Net Present Value, Internal rate of return, Balance point, among others [28]. Project evaluation should not consider the aforementioned coefficients in isolation, but should be analyzed comprehensively and together in order to provide validity, timeliness and objectivity to the information that supports and bases decision-making in an investment project. In the science of management, the dME is present when the procedures to calculate these coefficients are shown, since many books in this branch of knowledge approach these concepts practically the same. However, despite the fact that many books are conceived by the dME, in various works in the different fields of study, it has sought to overcome the dME. For example, [29] an analysis was carried out for a bias circuit for two rectifier diodes in series, where an algebraic solution was proposed in terms of the parameters of the circuit components. In this way, it is possible to calculate the bias point of the circuit without falling into the dME that occurs in the classroom when teaching analog electronics, on the subject of rectifier diodes. In other words, it is customary to teach diode bias using the graphical method, where the diode curve is plotted against the load line of the circuit. Obtaining the polarization point for this circuit can also be obtained using numerical algorithms such as Newton-Raphson's [30,31].

In [32] an approximation was presented for the error function and another for the normal cumulative function, which have many advantages in their implementation, such as reducing computation times, but also seek to change an alternative in the teaching the use of the normal distribution to calculate a probability, and thus avoid the dME.

In this work, an alternative method is proposed to obtain the proof of the general formula to solve second degree algebraic equations and that can be taught to high school and engineering students. This article presents the solution through power series, presented in [33] which is not detailed, since this text assumes that the reader has a good background in mathematics, incurring the dME. In this way, it is sought to teach this solution method to students who are in the last semesters of high school and to university students who do not have a rigorous training in the area of mathematics. Also in this work an analysis of significant digits in the roots of the quadratic equation is presented when the general formula is used or when it is applied in a rationalized way. The aforementioned concepts are not taught in conventional algebra courses, nor are they mentioned in algebra textbooks as they are influenced by the dME.

The purpose of this paper is aimed at awakening the interest in mathematics to students and teachers in the area of mathematics, and is organized as follows. Section 2 presents the importance of the quadratic equation. In section 3 the alternative deduction for the general formula for solving quadratic equations is presented. Section 4 presents the solution using power series. In section 5 two case studies are presented; the first is about the computational implications when the general formula is rationalized; The second case study presents a comparison of the solution obtained through the general formula against the solution through the power series. In section 6 the conclusions of this work are presented.

II. THE IMPORTANCE OF THE QUADRATIC EQUATION

The quadratic equation
 $az^2 + bz + c = 0,$

(1)

represents a parabola and its solution is of high importance in physics and engineering. In physics it is used to solve mathematical procedures that lead to its solution [34]. In the area of ordinary differential equations its usefulness is of high importance since to give a solution to linear equations it is necessary to solve the characteristic equation, which in many cases presents a second degree algebraic equation [6,7]. In engineering areas such as electronics, field effect transistor (FET) polarization analyzes require solving Shockley's second degree equation [35,36].

In electric circuit theory, the solution of the quadratic equation is also important, in some analyzes as the frequency response using Bode diagrams [10], it is required to analyze the damping criterion in the response of an analog filter [37,38].

It is important to remember that the study of the quadratic equation begins in the introductory courses of algebra in secondary education in Mexico and deepens in the mathematics courses of the high school, however different authors have made many proposals for the study of the second degree equation in order to make the teaching of the second degree equation and its geometric behavior meaningful. For example, in [39] a didactic design was proposed to re-signify the parabola and the graphic models related to situations of a person's displacement.

In traditional textbooks used for teaching algebra [4], solution procedures are presented using classical algebraic methods to solve the quadratic equation. In addition, notable

products such as the binomial squared and difference of squares are studied when it presents the characteristics of being a perfect square trinomial, or when the form is incomplete, $az^2 + c = 0$. However, the general formula to solve these equations is very useful when we find coefficients that do not allow us to find their solution immediately.

In [4] the proof is presented by adding terms and completing squares to obtain the general formula to solve quadratic equations, given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{2}$$

III. OBTAINING IN AN ALTERNATIVE WAY THE GENERAL FORMULA

We begin by making $z = x + yi$, and substituting in equation (1), where z is a complex number with real part given by x , and imaginary part given by y , simplifying we obtain

$$ax^2 + 2iaxy - ay^2 + bx + iby + c = 0. \tag{3}$$

From (3) we get the real part and the imaginary part. Solving for the imaginary part we have

$$x = -\frac{b}{2a}. \tag{4}$$

Substituting (4) in the real part we obtain

$$-\frac{b^2}{4a} - ay^2 + c = 0. \tag{5}$$

From (5) we can see that it is easy to solve for y , solving

$$y = \pm \sqrt{\frac{4ca - b^2}{2a}}. \tag{6}$$

Substituting (4) and (6) into $z = x + yi$, we have

$$z = -\frac{b}{2a} \pm \sqrt{\frac{4ac - b^2}{2a}}i. \tag{7}$$

Furthermore, taking into account that $i = \sqrt{-1}$, equation (7) can be simplified, obtaining then the general formula given by (2). However, when $b^2 \gg 4ac$, the numerical difference in the numerator can be very small. In these cases, it is useful to make use of double precision variables [40] when formula (2) is implemented in a programming language. To minimize the effects of cancellation due to subtraction, equation (7) can be rewritten as

$$z = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}} \tag{8}$$

IV. SOLUTION BY POWER SERIES

The solution through power series is possible when there are perturbative parameters of ϵ , being a constant whose restriction is $\epsilon \ll 1$. Consider the quadratic equation given by

$$x^2 + k_1\epsilon x - k_2 = 0, \tag{9}$$

where k_1 is a real integer constant and k_2 a real constant. To solve this equation, perturbative methods used [33] are used. Now consider

$$x \approx x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots, \tag{10}$$

using the first three terms of the series and substituting in (9) we have

$$\epsilon^4 x_2^2 + \epsilon^3(k_1 x_2 + 2x_1 x_2) + \epsilon^2(k_1 x_1 + 2x_0 x_2 + x_1^2) + \epsilon(k_1 x_0 + 2x_0 x_1 + x_0^2) = k_2, \tag{11}$$

equating the coefficients of ϵ^n , we obtain, the system given by

$$\begin{aligned} \epsilon^0: & \quad x_0^2 = k_2, \\ \epsilon^1: & \quad k_1 x_0 + 2x_0 x_1 = 0, \\ \epsilon^2: & \quad k_1 x_1 + 2x_0 x_2 + x_1^2 = 0. \end{aligned} \tag{12}$$

Solving the system of equations given by (12), and substituting the solutions of x_0 , x_1 y x_2 in (10), the solutions in power series are given by

$$\begin{aligned} x_{s1} &= \sqrt{k_2} - \frac{k_1\epsilon}{2} + \frac{\sqrt{k_2}k_1^2\epsilon^2}{8}, \\ x_{s2} &= -\sqrt{k_2} - \frac{k_1\epsilon}{2} - \frac{\sqrt{k_2}k_1^2\epsilon^2}{8}. \end{aligned} \tag{13}$$

V. STUDY CASES

In the first case study, we analyze the behavior of the significant digits for the when solving the quadratic equation using (2) against (8). The second case study discusses the comparison of the solution calculated by the general formula against the solution obtained with perturbation in power series. For the development of the case studies presented in this work, the mathematical software Maple 15 and its fsolve command have been used, which is an inter-build function that has a Newton-Raphson (NR) algorithm and that we will use as an exact solution to validate the results presented in this article.

A. Significant digits using the general formula and its rationalization

We will consider the quadratic equation given by

$$x^2 + 5x - c = 0,$$

with solutions $x_1 = -5$ and $x_2 = 0$ when $c = 0$.

To see the behavior of the significant digits of the roots of this equation, it is considered to give different values to c , such that $c \ll 1$. Then, to determine the roots of (14) the formulas of (2) and (8) will be used. In addition, we will consider that the exact solutions are those that have been determined with the numerical differentiation NR. In [30, 31] algorithms and their implementation are presented using a programming language such as C [40] or Fortran [41]. Table 1 presents the comparison of using equations (2) and (8) against NR for $1E - 5 \leq c \leq 1$.

TABLE I. COMPARISON OF SIGNIFICANT DIGITS BETWEEN EQUATIONS (2) AND (8) AGAINST NR TABLE STYLES

c values	Roots with (2)	Roots with (8)	Roots using NR.
	x_1 x_2	x_1 x_2	x_1 x_2
1	-5.192582405 0.1925824035	-5.192582406 0.1925824035	-5.192582404 0.1925824036
1E-1	-5.019920635 0.01992063350	-5.019920676 0.01992063367	-5.019920634 0.01992063367
1E-2	-5.001999200 0.00199920064	-5.001999550 0.00199920064	-5.001999201 0.01999200639
1E-3	-5.000199990 0.00019999200	-5.000200008 0.0001999920	-5.000199992 0.00019999201
1E-4	-5.000020000 0.00002000000	-5.000000000 0.00001999999	-5.000020000 0.00001999992
1E-5	-5.000002000 0.00000200000	-5.000000000 0.00000199999	-5.000002000 0.00000199999

The formulas for solving the quadratic equation (8) and (2) are algebraically the same expression. In other words, to obtain eq. (8), it has proceeded to rationalize the numerator in (2), however, carrying out this process has a very important implication in numerical calculations, that the traditional books, following the dME, do not warn the student. In Table 1 we can observe the behavior of the significant digits of both roots in equations (2) and (8) as the value of c is changed. For example, if $c = 1E - 5$, for x_1 in eq. (2) the number of significant digits remains constant, whereas this does not occur for eq. (8). In the case of x_2 in eq. (8), the significant digits remain constant, but not for eq. (2).

In order to show the accuracy of the roots of the equations under discussion, we will use the formula (15) to obtain the significant digits [42, 43], given by

$$SD = -\log_{10} \left| \frac{x - \tilde{x}}{x} \right|, \tag{15}$$

where SD is the number significant digits, x is the exact value computed with NR, \tilde{x} are the roots calculated with equations (2) and (8)

Figure 1 shows the graphs of significant digits of the data is shown in table 1. In eq. (2) the number of significant digits for x_1 remains constant, in this case 11, while for x_2 they decrease as c tends to 1, that is, 5 significant digits. On the

other hand, in the case of eq. (8) x_2 is the root that keeps the significant digits constant, while x_1 shows a reduction of these. From this figure an inverse behavior can be seen in the roots of equations (2) and (8).

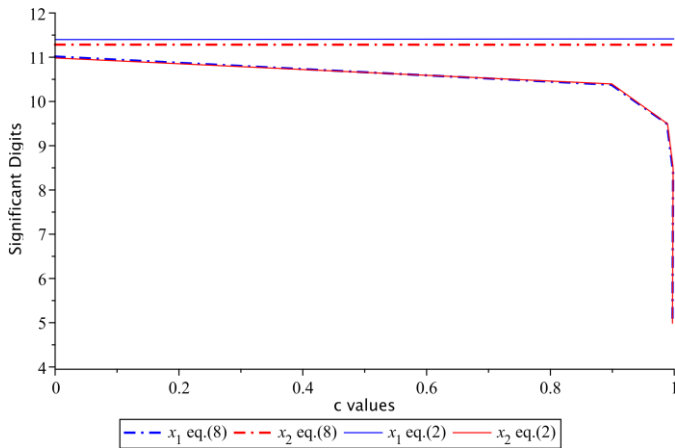


Fig. 1. Comparison of significant digits in the roots from equations (2) and (8).

From table 1 and Figure 1 it can be concluded that the advantage of using the rationalized formula (8) when we have the condition $b^2 \gg 4ac$, is that the number of significant digits can be maintained with good precision. To do this, in the denominator, both b and $\sqrt{b^2 - 4ac}$ must have the same sign. When this condition is not met, it is better to use formula (2).

B. Perturbative solution with power series.

Consider the constants $k_1 = 5$, $k_2 = 1$, and $\epsilon = 0.00$, substituting into (9) we have

$$x^2 + 0.005x - 1 = 0. \tag{14}$$

In this case, the exact solutions are calculated with the general formula and have the same significant digits as those calculated with (13), given by $x_{s1} = 0.997503125$ and $x_{s2} = -1.0025031250$.

In Figure 2 the solution for (9) has been plotted with $k_1 = 3$ and $k_2 = 2$ for $\epsilon \ll 1$. You can see that very small values of ϵ , the solution is very close to the exact value. For values of $\epsilon > 0.15$, the difference between the exact value and x_{s1} begins to be made visible.

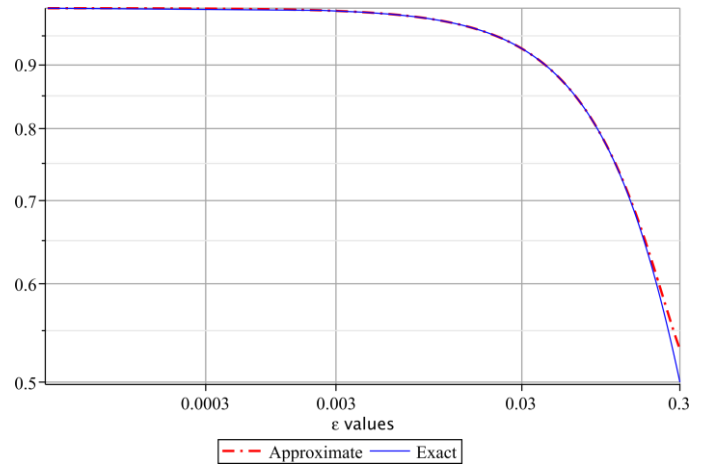


Fig. 2. Comparison between the exact positive root and x_{s1} with three terms in series expansion. For small values of ϵ the approximate solution is closer to exact solution.

VI. CONCLUSIONS

In this article an alternative deduction has been presented using a complex expression for the general formula that allows solving quadratic equations. In the same way, the perturbative solution was presented through power series.

Two case studies were also presented. In the first, the behavior of the significant digits of the roots of a quadratic equation was discussed. This behavior, following the dME in traditional algebra books is not analyzed, nor is anything commented on about it. On the other hand, in [30] the effect of rationalization is commented, however, also following the dME, the reader is presented with a detailed analysis of the advantages and disadvantages of rationalizing the general formula for solving these equations.

In the second case study, the solution was discussed through power series, which was obtained through perturbation methods when there is a small parameter such that $\epsilon \gg 1$. In the example presented, 10 significant digits were determined, which shows a high accuracy when using this method considering the first three terms of the expansion to approximate the root. Likewise, Figure 1 illustrated that the accuracy decreases if ϵ we make it grow. Figure 1 suggests that to obtain an accuracy of approximately 95%, it is necessary to use a value $\epsilon < 0.01$.

In this article we have presented some alternatives that can be used in algebra courses in teaching the quadratic equation. We believe that this work will provide an enriching experience, both for students and teachers in the teaching-learning processes, with which the school mathematical discourse that has been present in traditional books and in classrooms can be overcome from another perspective. This will allow proposing other teaching strategies for the benefit of students within the discipline of mathematics.

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