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The Plane Viscous Incompressible MHD Flows

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***Abstract* — The fundamental coefficients for a plane surface with a curvilinear coordinate system is used to study the two dimensional plane Magnetohydrodynamic flows of viscous incompressible fluids with mutually orthogonal magnetic and velocity fields. Geometry of streamlines are studied for some specific flows by expressing the governing equations in terms of the fundamental coefficients.**

***Keywords* — Magnetohydrodynamic flows; streamlines; fundamental coefficients.**

I. INTRODUCTION

O. P. Chandna and M. R. Garg [1] investigated the geometries of steady plane magnetohydrodynamic flows of a steady viscous incompressible fluid when the streamlines and magnetic lines form an isometric net and when magnetic force vanishes. They [2] also investigated these flows with mutually orthogonal magnetic and velocity fields using Hodograph transformation method. Kingston and Talbot [3] classified the corresponding flows of an inviscid incompressible fluid. Nath and Chandna [4] determined the flow geometries for these type of flows when the streamlines are straight lines or involutes of a curve. M. H. Martin [5] introduced the curvilinear coordinates φ, ψ in the plane of flow in which the coordinate lines ψ = constant are the stream lines of the flow and the coordinate lines φ = constant are left arbitrary.

In the present work, we obtain a system of partial differential equations for the fundamental coefficients E, F, G for a plane surface, as functions of φ, ψ, to study steady plane Magnetohydrodynamic flows of a viscous incompressible fluid, having infinite electrical conductivity, when the magnetic fields and velocity fields are mutually orthogonal. This approach is illustrated by considering two examples in which the curves ψ = constant and φ = constant form an orthogonal curvilinear coordinate system.

II. BASIC EQUATIONS

The steady plane Magnetohydrodynamic flow of a viscous incompressible fluid of infinite electrical conductivity is governed by the following system of equations

$$div\,\vec{v}=0 \tag{1}$$

$$\rho\left[\left(\vec{v}.grad\right)\vec{v}\right]+grad\;p=\mu\,curl\,\vec{H}\times\vec{H}+\eta\nabla^2\vec{v} \tag{2}$$

$$curl\left(\vec{v}\times\vec{H}\right)=\vec{0} \tag{3}$$

$$div\,\vec{H}=0 \tag{4}$$

where $\vec{v}=\left(v_1,v_2\right)$ denotes the velocity vector with $v=\left|\vec{v}\right|=\sqrt{v_1^{\,2}+v_2^{\,2}}$, $\vec{H}=\left(H_1,H_2\right)$ the magnetic field vector, ρ the density, p the pressure, η the coefficient of viscosity, μ the magnetic permeability of flow.

Also the vorticity function $\xi(x, y)$, current density function $\delta(x, y)$ and the energy function $h(x, y)$ are given by

$$\xi=\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y} \tag{5}$$

$$\delta=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y} \tag{6}$$

$$h=\frac{1}{2}\rho v^2+p \text{ or } h=\frac{1}{2}\rho\left(v_1^{\,2}+v_2^{\,2}\right)+p \tag{7}$$

Now (1) becomes

$$\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}=0 \tag{8}$$

Using (5) and (7), (2) becomes

$$\eta\frac{\partial\xi}{\partial y}-\rho\xi v_2+\mu\delta H_2=-\frac{\partial h}{\partial x} \tag{9}$$

$$\eta\frac{\partial\xi}{\partial x}-\rho\xi v_1+\mu\delta H_1=\frac{\partial h}{\partial y} \tag{10}$$

From (3) we get

$$v_1H_2-v_2H_1=k \tag{11}$$

where k is an arbitrary non – zero constant.

Finally (4) gives

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0. \tag{12}$$

We shall determine the seven unknown functions v_1, v_2, H_1, H_2, ρ, δ and h.

III. CONCEPTS OF DIFFERENTIAL GEOMETRY

Let $x=x\left(\phi,\psi\right)$ and $y=y\left(\phi,\psi\right)$ define a system of curvilinear coordinates in (x, y) plane.

If the Jacobian J is given by

$$J=\frac{\partial(x,y)}{\partial(\psi,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\psi}\\ \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\psi}\end{vmatrix}=\frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\psi}-\frac{\partial x}{\partial\psi}\frac{\partial y}{\partial\phi},$$

then we can obtain $\phi=\phi(x,y)$ and $\psi=\psi(x,y)$ such that

$$\frac{\partial x}{\partial\phi}=J\frac{\partial\psi}{\partial y},\quad \frac{\partial y}{\partial\phi}=-J\frac{\partial\psi}{\partial x},\quad \frac{\partial x}{\partial\psi}=-J\frac{\partial\phi}{\partial y},\quad \frac{\partial y}{\partial\psi}=J\frac{\partial\phi}{\partial x}$$

(13)

provided $0<|J|<\infty$

Then the first fundamental form for the xy plane is

$$ds^2=Ed\phi^2+2Fd\phi d\psi+Gd\psi^2 \quad \text{where}$$

$$E=\left(\frac{\partial x}{\partial\phi}\right)^2+\left(\frac{\partial y}{\partial\phi}\right)^2,$$

$$F=\frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\psi}+\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\phi},$$

$$G=\left(\frac{\partial x}{\partial\psi}\right)^2+\left(\frac{\partial y}{\partial\psi}\right)^2$$

The incompressibility constraint equation (8) implies the existence of a stream function ψ (x, y) and the solenoidal equation (12) implies the existence of a magnetic flux function φ (x, y) such that

$$\frac{\partial\psi}{\partial x}=-v_2,\ \frac{\partial\psi}{\partial y}=v_1,\ \frac{\partial\phi}{\partial x}=H_2,\ \frac{\partial\phi}{\partial y}=-H_1$$

(15)

We assume that the curves $\psi=$ constant and the curves $\phi=$ constant form the curvilinear coordinate system.

Then using (15) we get from (11),

$$\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}=\frac{\partial(\phi,\psi)}{\partial(x,y)}=\frac{1}{J}=k\neq 0$$

$$\text{i.e. } J=\frac{1}{k}\neq 0 \qquad (16)$$

And

$$W^2=EG-F^2=J^2=\frac{1}{k^2}\Rightarrow J=\pm W,\ EG-F^2=\frac{1}{k^2}$$

$\therefore$ By inverse theorem of differential calculus, if we know x and y as functions of ϕ and ψ, then we can find ϕ and ψ as functions of x and y.

IV. CONTINUITY EQUATION AND VORTICITY

Martin [5] has shown that the equation of continuity implies the fluid flows along the streamlines towards the higher or lower parameter values of ϕ according as J is positive or negative. He has also proved that

$$WV=\sqrt{E}\ \text{and}\ v_1+iv_2=\sqrt{\frac{E}{J}}e^{i\alpha} \qquad (17)$$

where is α is the angle between the tangent to the co-ordinate line and $\psi=$ constant, directed in the sense of increasing φ and the new form of the vorticity is

$$\xi=\frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right)-\frac{\partial}{\partial\psi}\left(\frac{E}{W}\right)\right] \qquad (18)$$

Nath and Chandna [4] have shown that the magnetic field acts along the magnetic lines towards the higher or lower parameter values of ψ according as J is positive or negative. Also they have shown that

$$WH=\sqrt{G}\ \text{AND}\ H_1+iH_2=\sqrt{\frac{G}{J}}e^{i\beta} \qquad (19)$$

where is β is the angle between the tangent to the co-ordinate (14) line and $\varphi=$ constant, directed in the sense of increasing ψ with x – axis. The new form of the current density function is

$$\delta=\frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right)-\frac{\partial}{\partial\psi}\left(\frac{F}{W}\right)\right] \qquad (20)$$

Using (14), (15) and (16), we get from (9)

$$\frac{\partial h}{\partial\psi}=\frac{1}{2}\frac{\eta}{J}\left(G\frac{\partial\xi}{\partial\phi}-F\frac{\partial\xi}{\partial\psi}\right)-\rho\xi \qquad (21)$$

Using (14), (15) and (16), we get from (10)

$$\frac{\partial h}{\partial\phi}=\frac{1}{2}\frac{\eta}{J}\left(F\frac{\partial\xi}{\partial\phi}-E\frac{\partial\xi}{\partial\psi}\right)-\rho\delta \qquad (22)$$

Differentiating (21) w. r. t. φ and (22) w.r.t. ψ, and then using the integrability condition $\frac{\partial^2 h}{\partial\phi\partial\psi}=\frac{\partial^2 h}{\partial\psi\partial\phi}$ we obtain

$$\frac{\eta}{2}\left[\frac{\partial}{\partial\phi}\left\{\frac{1}{J}\left(G\frac{\partial\xi}{\partial\phi}-F\frac{\partial\xi}{\partial\psi}\right)\right\}-\frac{\partial}{\partial\psi}\left\{\frac{1}{J}\left(F\frac{\partial\xi}{\partial\phi}-E\frac{\partial\xi}{\partial\psi}\right)\right\}\right]$$

$$+\mu\frac{\partial\delta}{\partial\psi}-\rho\frac{\partial\xi}{\partial\phi}=0 \qquad (23)$$

Thus we conclude that when streamlines, ψ = constant and the magnetic lines φ = constant of steady plane viscous Magnetohydrodynamic flows are taken as curvilinear co-ordinates system (φ, ψ) in the physical plane then the set of seven of differential equations (5), (6), (8), (9), (10), (13), (14) for v_1, v_2, H_1, H_2, ξ, δ and h as functions of x and y may be replaced by the system of six equations (16), (17), (18), (19), (20) and (23) in five dependent variables E, F, G, ξ and δ. If the solutions to these equations are given we can find x and y as functions of φ and ψ and hence E, F, G, ξ, δ as functions of x and y, since $0<|J|<\infty$. Once we obtain E, F, G, ξ, δ as functions of x and y then v_1, v_2, H_1, H_2 as functions of x and y are given by (17) and (19). Finally, the energy and

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pressure functions h and p of x & y can be obtained from momentum equation and energy expression (3).

We now study an example in which streamlines are straight lines and not parallel but envelop a curve C. Taking the tangent lines to the curve C and their orthogonal trajectories, the involutes of C as the system of orthogonal curvilinear coordinates, we find the square of the element of arc length ds in this orthogonal curvilinear coordinate system is given by

$$ds^2 = d\alpha^2 + (\alpha - \sigma)^2 \, d\beta^2 \tag{24}$$

where $\sigma = \sigma(\beta)$ denotes the arc length of the curve C and β the angle of elevation of the tangent line to the curve C. In this coordinate system, α = constant are the involutes of the curve C and β = constant are the tangent lines to the curve C.

Investigating the flows for which

$$\varphi = \varphi(\alpha) \text{ and } \psi = \psi(\beta) \tag{25}$$

We find the values of E, F, G, J, ξ and δ as

$$E = \frac{1}{\phi'^2}, \quad F = 0, \quad G = \frac{(\alpha-\sigma)^2}{\psi'^2},$$
$$J = \frac{\alpha-\sigma}{\phi'\psi'}, \xi = -\frac{\psi''(\alpha-\sigma)+\psi'\sigma'}{(\alpha-\sigma)^3}, \tag{26}$$
$$\delta = \frac{\phi' + (\alpha-\sigma)\phi''}{(\alpha-\sigma)}$$

Then the Gauss equation

$$K = \frac{1}{W}\left[\frac{W}{E}\left(\frac{-FE_\alpha + 2EF_\alpha - EE_\beta}{2W^2}\right)_\beta - \frac{W}{E}\left(\frac{EG_\alpha - FE_\beta}{2W^2}\right)_\alpha\right] = 0 \tag{27}$$

is satisfied by the E, F and G.

Substituting all these values in (23) we get

$$\begin{aligned}&-2\mu\phi'^2\sigma'(\alpha-\sigma)^4 + \left[\eta\psi^{iv} + 4\eta\psi'' + 4\rho\psi''\psi'\right](\alpha-\sigma)^3 \\ &+\left[6\eta\psi'''\sigma' + 4\eta\psi''\sigma'' + 9\eta\psi'\sigma' + \eta\psi'\sigma''' + 6\eta\psi'^2\sigma'\right](\alpha-\sigma)^2 \\ &+\left[15\eta\psi''\sigma'^2 + 10\eta\psi'\sigma'\sigma''\right](\alpha-\sigma) + 15\eta\psi'\sigma'^3 = 0\end{aligned} \tag{28}$$

Since α and β are independent variables, for the above relation to hold identically, it must hold on the curve C, $\alpha = \sigma(\beta)$ and consequently $15\eta\psi'\sigma'^3 = 0$.

Since ψ' cannot vanish identically, $\sigma' = 0$. Therefore, the radius of curvature of C vanishes identically and hence the streamlines are concurrent lines.

Thus we conclude that when the streamlines in a two dimensional flow of a plane viscous MHD fluid are straight lines, then they must be concurrent or parallel.

Radial Flow: The square of the element of arc length in polar coordinate system is given by

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{29}$$

Since the flows are radial, we have

$$\varphi = \varphi(r) \text{ and } \psi = \psi(\theta) \tag{30}$$

We find the values of E, F, G, J, ξ, δ and hence the values of V, H, h and p as

$$E = \frac{1}{\phi'^2}, \qquad F = 0, \qquad G = \frac{r^2}{\psi'^2},$$
$$J = \frac{r}{\phi'\psi'} = \frac{1}{k},$$
$$\xi = 0,$$
$$\delta = \frac{2k}{A} \quad \text{where} \quad \psi' = \frac{kr}{\phi'} = A \text{ (say)}$$
$$V = \frac{\sqrt{E}}{W} = \frac{k}{\phi'} = \frac{k}{kr/A} = \frac{A}{r},$$
$$H = \frac{\sqrt{G}}{W} = \frac{kr}{\psi'} = \frac{kr}{A}$$
$$h = -\frac{\rho k^2 r^2}{A^2} + f(r) + c_1$$
$$p = -\frac{\rho k^2 r^2}{A^2} - \frac{\rho A^2}{2r^2} + f(r) + c_1,$$

where c_1 is an arbitrary constant.

Next we study the example where magnetic lines are the tangent lines to the curve C and streamlines are the involutes, the orthogonal trajectories of C.

In this coordinate system, α = constant are the involutes of the curve C and the curves β = constant its tangent lines. The square of the element of arc length ds in this orthogonal curvilinear coordinate system is given by

$$ds^2 = d\alpha^2 + (\alpha - \sigma)^2 \, d\beta^2 \tag{31}$$

where $\sigma = \sigma(\beta)$ denotes the arc length. In this coordinate system, α = constant are the involutes of the curve C and the curves β = constant its tangent lines.

Investigating the flows for which

$$\varphi = \varphi(\beta) \text{ and } \psi = \psi(\alpha) \tag{32}$$

We find the values of E, F, G, J, ξ and δ as

$$E=\frac{(\alpha-\sigma)^2}{\phi'^2},\qquad F=0,\qquad G=\frac{1}{\psi'^2},$$

$$J=\frac{\alpha-\sigma}{\phi'\psi'},$$

$$\xi=-\frac{\psi'+(\alpha-\sigma)\psi''}{(\alpha-\sigma)},$$

$$\delta=\frac{(\alpha-\sigma)\phi''+\phi'\sigma'}{(\alpha-\sigma)^3} \qquad (33)$$

Also we find that the Gauss equation (27) is satisfied by E, F and G as calculated above.
Substituting all these values in (23) we get

$$\begin{aligned}&-\eta\phi'\psi^{iv}(\alpha-\sigma)^5-2\eta\phi'\psi'''(\alpha-\sigma)^4\\&+\eta\phi'\psi''(\alpha-\sigma)^3-\eta\phi'\psi'(\alpha-\sigma)^2\\&+\eta\psi'\sigma''(\alpha-\sigma)-4\mu\phi'\psi''(\alpha-\sigma)\\&+3\eta\psi'\sigma'^2-6\mu\phi'^2\sigma'-2\rho\psi'^2\sigma'=0\end{aligned} \qquad (34)$$

Moreover putting $\sigma(\eta)=0$ in (34) we get

Since α and β are independent variables, for the above relation to hold identically, it must hold on the curve C, $\alpha = \sigma(\beta)$ and consequently

$$\sigma'\left(3\eta\psi'\sigma'-6\mu\phi'^2-2\rho\psi'^2\right)=0.$$

Since ϕ' and ψ' cannot vanish identically, $\sigma'(\eta)=0$

i.e. $\sigma(\eta)=\text{constant}$ i.e. radius of curvature is same in all directions i.e. bending is same in all directions. Hence the streamlines are concentric circles.

Thus we conclude that if the streamlines in a two dimensional flow of a plane viscous MHD fluids are involutes of a curve C then C reduces to a point and the streamlines are circles concentric at this point.

$$\eta\psi^{iv}\alpha^3+2\eta\psi'''\alpha^2+(4\mu-\eta)\psi''\alpha+\eta\psi'=0 \qquad (35)$$

This is a differential equation in $\psi(\alpha)$ where α is involute in this case and hence the distance along the radius from the common centre of the circular stream line.
If q gives the velocity of flow on circular stream lines, for $\psi'=-\rho q$ the above differential equation (35) becomes

$$\eta\alpha^3q'''+2\eta\alpha^2q''+(4\mu-\eta)\alpha q'+\eta q=0$$

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