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THE INTEGRATED FORMULATION OF THE TAU METHOD AND ITS ERROR ESTIMATE FOR THIRD ORDER NON-OVERDETERMINED DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper is concerned with integrated formulation of the tau methods for numerical solution of initial value problems in non-overdetermined third order ordinary differential equations. The error estimate for this variant of the tau method is obtained and numerical results are provided. The numerical evidence shows that the variant is more accurate than the differential variant earlier reported.

Key Words: Tau method, Formulation, Variant, Approximant, Error estimate.

1. INTRODUCTION

Accurate approximate solution of initial value problems and boundary value problems in linear ordinary differential equations with polynomial coefficients can be obtained by the tau method introduced by Lanczos in 1938. Techniques based on this method have been reported in literature with application to more general equation including non-liner ones as well as to both deferential and integral equations. We review briefly here two of the variants of the method.

Differential form of the Tau Method

Consider the following boundary value problem in the class of m-th order ordinary differential equations:

$$Ly(x) \equiv \sum_{r=0}^{m} p_r(x) y^{(r)}(x) = f(x), a \le x \le b \qquad \text{...(1.1a)}$$

$$L^* y(x_{rk}) \equiv \sum_{r=0}^{m} a_{rk} y^{(r)}(x_{rk}) = \rho_k, k = 1(1)m \qquad ...(1.1b)$$

where $|a| < \infty, |b| < \infty, a_{rk}, x_{rk}, \rho_k, r = 0(1)m, k = 0(1)m,$ are given real numbers, and the functions $f(x)$ and

$$p_r(x) = \sum_{k=0}^{N_r} p_{r,k} x^k, r = 0(1)m \qquad ....(1.2)$$

are polynomial functions or sufficiently close polynomial approximants of given real functions .

Definition 1.1

The number of over-determination, s, of equation (1.1a) is defined as

$$s = \max\{ N_r - r : 0 \leq r \leq m\} \qquad (1.3)$$

for $N_r \geq r$ and $0 \leq r \leq m$.

Definition 1.2

Equation (1.1a) is said to be non-overdetermined if s, given by (1.3) is zero, i.e. if s = 0. Otherwise it is over-determined.

For the solution of (1.1) by the tau method (Ortiz1969,1974, Lanczos 1938, 1956), we seek an approximant

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \ n < +\infty \qquad ...(1.4)$$

of $y(x)$ which satisfies exactly the perturbed problem

$$Ly_n(x) = f(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+1}(x), a \leq x \leq b, \qquad ...(1.5a)$$

$$L^* y_n(x_{rk}) = \rho_k, k = 1(1)m \qquad (1.5b)$$

where τ_r, r = 1(1)m +s, are fixed parameters to be determined along with a_r, r = 0(1)n, in (1.4)by equating the coefficients of power of x in (1.5). The polynomial

$$T_r(x) = \cos\left\{ r\, Cos^{-1}\left[\frac{2x-2a}{b-a} - 1 \right] \right\} \equiv \sum_{k=0}^{r} C_k^{(r)} x^k \qquad ...(1.6)$$

is the r-th degree Chebyshev polynomial valid in [a, b] (see Fox and Parker 1968)

1.2 The Integrated Formulation of the Tau method

The integrated form of (1.1a) is given by

$$LY(x) = \int\int_{\dots} m \int f(x)dx + (m(x) \qquad \text{...(1.7)}$$

Where $(m(x)$ denotes an arbitrary polynomial of degree $(m - 1)$, arising from the constants of integration, and

$$I{\cdot}L = \int\int_{\dots} m \int L(\cdot)dx \qquad \text{...(1.8)}$$

is the m times indefinite integration of L (s). The integrated tau problem corresponding to (1.4) is therefore

$$I,\ (\hat{y}(x)) = \int\int_{\dots} m \int f(x)dx + (m(x)) + \sum_{r=0}^{m+s-1} \hat{\tau}_{m+s-1} T_{n-r+1}(x) \qquad \text{...(1.9a)}$$

$$L^*(\hat{Y}_n(x_{rk}) = \alpha_k,\ \text{k} = 1(1)m \qquad \text{....(1.9b)}$$

where

$$\hat{Y}_n(x) = \sum_{r=0}^{n} b_r x^r \cong y(x),\ n + \infty \qquad \text{...(1.10)}$$

Problem (1.9) often gives a more accurate approximant of $Y(x)$ than (1.4) does, due to its higher order perturbation term (see[7] and [14]).

2. ERROR ESTIMATION OF THE TAU METHOD

We review briefly here error estimation of the tau method for the variants of the preceding section and which was earlier reported (Adeniyi et al 1990, 1991 and 2007)

2.1 Error Estimation for the Differential Form

While the error function

$$e_n(x) = y(x) - y_n(x) \qquad \text{...(2.1)}$$

satisfies the error problem (see Onumanyi and Ortiz 1982)

$$Le_n(x) = \sum_{r=0}^{m+s-1} \hat{\tau}_{m+s-1}\, T_{n-m+r+1}(x) \qquad \text{...(2.2a)}$$

$$Le_n(x_{rk}) = 0,\ k = 1(1)m \qquad \text{...(2.2b)}$$

The polynomial error approximant

$$(e_n(x))_{n+1} = \frac{v_m(x)\phi_n T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \qquad \text{...(2.3)}$$

of$e_n(x)$ satisfies the perturbed error problem (Adeniyi et al 1990, 1991, Onumanyi et al 1882)

$$L(e_n(x))_{n+1} = \sum_{r=0}^{m+s-1} (-\tau_{m+s-1}\, T_{n-m+r+1}(x) + \hat{\tau}_{m+s-1}\, T_{n-m+r+2}(x)) \text{ ...(2.4a)}$$

$$L^*(e_n(x_{rk}))_{n+1} = 0 \qquad \text{(2.4b)}$$

where the extra parameters $\hat{\tau}_r\, r = 1(1)m + s$, and φ_n in (2.3) – (2.4) are to be determined and $v_m(x)$ in (2.3) is a specified polynomial of degree in which ensures that $(e_n(x))_{n+1}$ satisfies the homogenous conditions (2.4b).

With 2.3) in (2.4) we get a linear system of $m + s +1$ equations, obtained by equating the coefficients of x^{n+s+1}, x^{n+s}, ...x^{n-m+1}, for the determination of φ_n by forward elimination, since we do not need the $\hat{\tau}$'s in (2.3) consequently, we obtain an estimate

$$\varepsilon = \max_{a\le x\le b}\left|(e_n(x))_{n+1}\right| = \frac{|\phi_n|}{\left|C_{n-m+1}^{(n-m+1)}\right|} \cong \max_{a\le x\le b}\left|(e_n(x)\right| \qquad \text{...(2.5)}$$

2.2 Error Estimation for the Integrated From

The error polynomial function

$$(\tilde{e}_n(x)) = \frac{\rho_m(x)\hat{\phi}_n T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \simeq Y(x) - \hat{y}_n(x)\, \hat{e}_n(x) \qquad \text{...(2.6)}$$

Satisfies the perturbed integrated error problem

$$I_2(\tilde{e}_n(x))_{n+1} = - \int\int_{\dots} m \int \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x)dx + (m(x) + \sum_{r=0}^{m+s-1} \hat{\tau}_{m+s-r} T_{n-r+1}(x) \quad \dots(2.7a)$$

$$L^*(\tilde{e}_n(x_{rk})) = 0 \quad \dots(2.17b)$$

Equations (2.7) together with (2.6) yield a linear system of $m + s + 1$ equations, obtained by equating the coefficients if $x^{n+s+m+1}$, x^{n+s+m}, x^{n+1} for the determination of $\varnothing_n$ subsequently we obtain

$$\bar{\varepsilon} = \max_{x \le x \le b} \left|\tilde{e}_n(x))_{n+1}\right| = \frac{|\phi_n|}{C_{n-m+1}^{(n-m+1)}} \approx \max_{a \le x \le b} \left|\tilde{e}_n(x)\right| = \varepsilon^4$$

3 A CLASS OF NON-OVERDETERMINED THIRD ORDER DIFFERENTIAL EQUATIONS

We consider here the integrated form of the tau methods and its error estimate for the class of problems:

$$Ly(x) := (\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)\, y'''(x) + (\beta_0 + \beta_1 x + \beta_2 x^2)\, y''(x)$$

$$+ (\gamma_0 + \gamma_1 x) y'(x) + \lambda o y(x) = \sum_{r=0}^{n} f_r x^r, \; a \le x \le b \quad \dots(3.1a)$$

$$y(a) = \rho_0, \; y'(a) = \rho_1, \; y''(a) = \rho_2 \quad \dots(3.1b)$$

that is, the case when $m = 3$ and $s = 0$ in (1.1)

Without loss of generality, we shall assume that $a = 0$ and $b = 1$, since the transformation

$$\frac{v = (x-a)}{(b-a)}, \; a \le x \le b \quad \dots(3.2)$$

takes (3.1) into the closed interval [0, 1].

3.1 Tau Approximant by the Integrated Form

By applying (1.9) we have

$$\int_0^x\int_0^u\int_0^t(\alpha_0+\alpha_1 v+\alpha_2 v^2+\alpha_3 v^3)\,y'''(v)dvdtdu + \int_0^x\int_0^u\int_0^t(\beta_0+\beta_1 v+\beta_2 v^2+\alpha_3 v^3)\,y''(v)dvdtdu$$

$$+\int_0^x\int_0^u\int_0^t(\gamma_0+\gamma_1 v)\,y'(v)dvdtdu + \lambda_0\int_0^x\int_0^u\int_0^t y(v)\,dvdtdu$$

$$=\int_0^x\int_0^u\int_0^t\left(\sum_{r=0}^{F} f_r v^r\right)dvdtdu + \tau_1 T_{n+3}(x) + \tau_2 T_{n+1}(x) = \tau_3 T_{n+1}(x)$$

That is

$$\alpha_0\sum_{r=0}^{n} a_r x^r + \alpha_1\sum_{r=0}^{n} a_r x^{r+1} + \alpha_2\sum_{r=0}^{n} a_r x^{r+2} + \alpha_3\sum_{r=0}^{n} a_r x^{r+3} - \alpha_0\rho_0 - 3\alpha_1\sum_{r=0}^{n}\frac{arx^{r+1}}{r+1}$$

$$-6\alpha_2\sum_{r=0}^{n}\frac{arx^{r+2}}{r+2} - 9\alpha_3\sum_{r=0}^{n}\frac{arx^{r+3}}{r+3} + x(2\alpha_1\rho_0-\alpha_0\rho_1) + \frac{x^2}{2}(\alpha_1\rho_1-\alpha_0\rho_2-2\alpha_2\rho_0)$$

$$+6\alpha_2\sum_{r=0}^{n}\frac{arx^{r+2}}{(r+1)(r+2)} + 18\alpha_3\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+2)(r+3)}$$

$$-6\alpha_3\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+1)(r+2)(r+3)} + \beta_0\sum_{r=0}^{n}\frac{arx^{r+1}}{r+1} + \beta_1\sum_{r=0}^{n}\frac{arx^{r+2}}{r+2} + \beta_2\sum_{r=0}^{n}\frac{arx^{r+3}}{r+3} - (\beta_0\rho_0)u$$

$$-\quad 2\beta_2\sum_{r=0}^{n}\frac{arx^{r+2}}{(r+1)(r+2)} - 4\beta_2\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+2)(r+3)} + \frac{x^2}{2}(\beta_1\rho_0-\beta_0\rho_1)u$$

$$+2\beta_2\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+1)(r+2)(r+3)} + \gamma_0\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+1)(r+2)(r+3)} + \gamma_1\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+2)(r+3)}$$

$$-\frac{x^2}{2}(\gamma_0\rho_0) + \lambda_0\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+1)(r+2)(r+3)} - \gamma_1\sum_{r=0}^{n}\frac{arx^{r+3}}{(r+)(r+2)(r+3)}$$

$$=\sum_{r=0}^{n}\frac{f_r x^{r+3}}{(r+1)(r+2)(r+3)} + \tau_1\sum_{r=0}^{n+3}C_r^{(n+2)}x^r + \tau_2\sum_{r=0}^{n+2}C_r^{(n+2)}x^r + \tau_3\sum_{r=0}^{n+1}C_r^{(n+1)}x^r$$

We equate corresponding coefficients of powers of x to obtain the system

$$\alpha_0 a_0 - \tau_1 C_0^{(n+3)} - \tau_2 C_0^{(n+2)} - \tau_3 C_0^{(n+1)} = \alpha_0\rho_0\;(\alpha_0-2\alpha_1+\beta_0)a_1 - \tau_1 C_1^{(n+3)}$$

$$-\ \tau_2 C_1^{(n+2)} \quad - \tau_3 C_1^{(n+1)} = 2\alpha_1\rho_0 - \alpha_0\gamma_1 - \beta_0\gamma_1$$

$$\frac{1}{2}[(2\alpha_2 - \beta_1 + \gamma_0)a_0 + (\beta_0 - \alpha_1]a_1 + 2\alpha_0 a_2 - 2\tau_1 C_2^{(n+3)} - 2\tau_2 C_2^{(n+2)}$$

$$-2\tau_3 C_2^{(n+1)} = \alpha_1\rho_0 - \alpha_0\rho_2 - 2\alpha_2\rho_0 - \gamma_0\rho_0 + \beta_1\rho_0 - \beta_1\rho_1 - \frac{[(k-3)\alpha_1 + \beta_0]}{k_1}a_{k-1}$$

$$+ \frac{[(k-2)^2 + 3(k-2) + 2)\alpha_2 + \beta_1(k-3) - 6(k-3) - 6(k-2)\alpha_2 + \gamma_0]}{(k-1)k}a_{k-2}$$

$$\frac{+ [(k-3)^3 - 9(k-3)^2 + 2(k-3) - 18)\alpha_3 - (k-3)^2 + 7(k-3+4)\beta_2 + (k-2)\gamma_1 + \lambda_0 - \gamma_0]a_{k-3}}{(k-2)(k-1)k}$$

$$\tau_1 C_k^{(n+3)} + \tau_2 C_k^{(n+2)} + \tau_3 C_k^{(n+1)} = \frac{f_{(k-3)}}{k(k-1)(k-2)}, k = 3(1)n \qquad \text{... (3.11)}$$

$$\frac{+ \frac{[\alpha_1(n+1) - 3\alpha_1 + \beta_0]}{(n+1)}a_n + [(n-2)^3 - 9(n-2)^2 + 2(n-2) - 18)\alpha_3 - (n-2)^2 + 7(n-2) + 4)\beta^2 + (n-1)\gamma_1 + \lambda_0 + \gamma_0]a_{(n-2)} - \tau_1 C_{n+1}^{(n+3)}}{n(n-1)(n+1)}$$

$$-\tau_2 C_{n+1}^{(n+2)} + \tau_3 C_{n+1}^{(n+1)} = \frac{f_{n-2}}{(n+1)(n)(n-1)}$$

$$\frac{[n^3 + 3n + 2)\alpha_2 + (n-1)\beta_1 - 6n\alpha_2 + \gamma_0]a_n}{(n+2)(n+1)}$$

$$\frac{+ [(n-1)^3 - (n-1)^2 + 2(n-1) - 18)\alpha_3 - ((n-1)^2 + 7(n-1) + 4)\beta_2]a_{n-1}}{n(n+1)(n+2)}$$

$$-\tau_1 C_{n+2}^{(n+3)} + \tau_2 C_{n+2}^{(n+2)} = \frac{f_{n-1}}{(n+2)(n+1)n}$$

$$\frac{+ [(n^3 - 9n^2 + 2n - 18]\alpha_3 - (n^2 + 7n + 4)\beta_2 + (n+1)\gamma_1 + \lambda_0 + \gamma_0]a_n}{(n+1)(n+2)(n+3)}$$

$$\tau_3 C_{n+3}^{(n+3)} = \frac{f_n}{(n+1)(n+2)(n+3)}$$

We solve this to subsequently obtain the approximant $\hat{Y}_n(x)$.

3.2.1 Error Estimation for the Integrated From

For problem (3.1) we have, from (2.7)

$$\int_0^x\int_0^u\int_0^t(\alpha_0 + \alpha_1 v + \alpha_2 v^2 + \alpha_3 v^3)(e_n(v))_{n+1}\,dvdtdu + \int_0^x\int_0^u\int_0^t(\beta_0 + \beta_1 v + \beta_2 v^2)$$

$$(e_n(v))_{n+1}\,dvdtdu + \int_0^x\int_0^u\int_0^t(\gamma_0 + \gamma_1 v)(e_n(v))_{n+1}\,dvdtdu + \int_0^x\int_0^u\int_0^t\lambda_0\,(e_n(v))_{n+1}\,dvdtdu$$

$$= -\int_0^x\int_0^u\int_0^t(\tau_1\sum_{r=0}^{n}C_r^{(n)}V^r + \tau_2\sum_{r=0}^{n-1}C_r^{(n-1)}V^r + \tau_3\sum_{r=0}^{n-2}C_r^{(n-2)}V^r)\,dvdtdu$$

$$+ \hat{\tau}_1\sum_{r=0}^{n+4}C_r^{(n+4)}x^r + \hat{\tau}_2\sum_{r=0}^{n+3}C_r^{(n+3)}x^r + \hat{\tau}_3\sum_{r=0}^{n+2}C_r^{(n+2)}x^r$$

From the coefficients if x^{n+3}, x^{n+2}, x^{n+1}, we get the system

$$\hat{\tau}_1 C_{n+4}^{(n+4)} = \frac{\theta[18\alpha_3(n+2)(1-(n+3)+10\beta_2(n+2)(n+3)+6(2\beta_2 - 6\alpha_3 + \lambda_0]\,C_{n-2}^{(n-2)}}{6(n+2)(n+3)(n+4)}$$

$$\hat{\tau}_1 C_{n+3}^{(n+4)} + \hat{\tau}_2 C_{n+3}^{(n+3)} - \frac{\tau_1 C_n^n}{(n+1)(n+2)(n+3)} =$$

$$\frac{\theta\{12(n+5) - 6\alpha_2(n+2)(n+3) + \beta_1(n+5)((n+2+4) - \gamma_1(n+2)(n+5))\,C_{n-2}^{n-2}}{2(n+2)(n+3)(n+5)}$$

$$\frac{[9\alpha_3(n+1) - (n+1)(n+2)(18\alpha_3 + 10\beta_2)) + 6(2\beta_2 - 6\alpha_3 + \lambda_0 - \gamma_1)\,C_{n-3}^{(n-2)}\}}{6(n+1)(n+2)(n+3)}$$

$$\hat{\tau}_1 C_{n+2}^{(n+4)} + \hat{\tau}_2 C_{n+2}^{(n+3)} + \hat{\tau}_3 C_{n+2}^{(n+2)} - \frac{\tau_1 C_{n-1}^{n}}{n(n+1)(n+2)} - \frac{\tau_2 C_{n-1}^{(n-1)}}{n(n+1)(n+2)}$$

$$\theta\{[\frac{\beta_0 - 3\alpha_1 - 2\beta_1 + \gamma_0}{(n+2)} C_{n=2}^{(n-2)} +$$

$$\frac{(6\alpha_2(2(n+5) - (n+1)(n+2)) + ((n+1)(n+5)(\beta_1 - \gamma_1))\, C_{n-2}^{n-2}\,]}{2(n+1)(n+2)(n+5)}$$

$$+ \frac{[(9\alpha_3 n\,(1 - 2(n+1) + 10\beta_2 n\,(n+1) + 6(12\beta_2 - 6\alpha_3 + \lambda_0 - \gamma_1))\; C_{n-4}^{(n-2)}}{6n(n+1)(n+2)}$$

$$\hat{\tau}_3 C_{n+1}^{(n+2)} - \frac{\tau_1 C_{n-2}^{n}}{n(n-1)(n+1)} - \frac{\tau_2 C_{n-2}^{(n-1)}}{(n-1)(n)(n+1)} - \frac{\tau_3 C_{n-2}^{(n-2)}}{(n-1)(n)(n+1)} =$$

$$\theta\{[\frac{6\alpha_2(2-n) + \beta_1(n-4) - n\gamma_1}{2n(n+1)} C_{n-4}^{(n-2)} + \frac{\beta_0 - 3\alpha_1 - 2\beta_1 + \gamma_0}{(n+2)} C_{n-3}^{(n-2)}$$

$$+ \frac{9\alpha_3(n-1)(n+2)(1-2n) + 10\beta_2 n(n+1)(n-1) + 6\,((2\beta_2 - 6\alpha_3 + \lambda_0 - \gamma_1)\, C_{n-5}^{(n-2)}}{6n(n+1)(n-1)\,(n+2)} \qquad ...(3.12)$$

where $\theta = \varnothing_n(C_{n-2}^{(n-2)})^{-1}$

we solve this system by forward elimination of $\varnothing_n$ and subsequently obtain from (2.8) the error estimate

$$\bar{\varepsilon} = \frac{2^{2n-4}|\tau_3| - |\tau_1| 2\, C_{n-2}^{(n)} + |\tau_2|(n-1)\, C_{n-2}^{(n-1)} - 2n\, C_{n-2}^{(n-1)}}{2n(n+1)(n-1)2^{2n-4}\,|p_4|} \qquad (3.13)$$

where

$$p_4 = \frac{(6\alpha_2(2-n) + \beta_1(n+4) - n\gamma_1)}{2n(n+1)} \frac{C_{n-4}^{(n-2)}}{2^{2n-5}} - \frac{(n-2)}{2} \frac{(\beta_0 - 3\alpha_1 + \gamma_0)}{(n-1)}$$

$$+ \frac{(9\alpha_3(n-1)(n+2)(1-2n) + 10\beta_2 n(n+1)(n-1) + 6\,(n+2)(2\beta_2 - 6\alpha_3 + \lambda_0 - \gamma_1)\, C_{n-5}^{(n-2)}}{6n(n+1)(n-1)\,(n+2) \qquad 2^{2n-5}}$$

$$+ \frac{(n+2)}{2} p_3]$$

$$p_3 = \frac{(\beta_0 - 3\alpha_1 - 2\beta_1 + \gamma_0)}{(n+2)}$$

$$- \frac{(n-1)((6\alpha_2(2n+10) - (n+1)(n+2) + (n+1)(n+5)(\beta_2 - \gamma_1)}{4(n+1)(n+2)(n+5)}$$

$$+ \frac{(9\alpha_3 n(-2n-1) + 10\beta_2 n(n+1) + 6(2\beta_2 - 6\alpha_3 + \lambda_0 + \gamma_1 \frac{C_{n-5}^{(n-2)}}{2^{2n-5}} - \frac{C_{n-2}^{(n+4)}}{2^{2n-7}} p_1 + \frac{(n+3)}{2} p_2}{6n(n+1)(n+2)}$$

$$p_2 = \frac{6(n+1)(n+2)(n+3)(12(n+5)) - 6\alpha_2(n+2)(n+3) + \beta_1(n+5)((n+2)+4) - \gamma_1((n+2)(n+5) - (n-2)(18\alpha_2(n+1) - (n+1)(n+2)(18\alpha_2 - 10\beta_2) + 3(2\beta_2 - 6\alpha_3 + \lambda_0 - \gamma_1) + 3(n+1)(n+2)(n+3)(n+4)p_1}{6(n+1)(n+2)(n+3)}$$

$$p_1 = \frac{18\alpha_3(n+2)(1-(n+3) + 10\beta_2(n+2)(n+3) + 6(2\beta_2 - 6\alpha_3 + \lambda_0 - \gamma_1)}{6(n+2)(n+3)(n+4)}$$

In an earlier work (Ojo and Adeniyi (2011), we obtained the corresponding error estimate for the differential form as

$$\varepsilon = \frac{2^{2-10n}|\tau_3|}{p_7} \qquad \ldots (3.14)$$

where

$$p_7 = \{[(n+1)(n-1)\alpha_0 - (n-1)(n-3)\alpha_2](n-1)(n-2)\beta_1 + (n-1)\gamma_0)C_{n-4}^{(n-2)}$$
$$+ ((n-2)(n-3)(n-4)\alpha_3 + (n-2)\gamma_1 + \lambda_0)C_{n-5}^{(n-2)} - n(n-1)(n-2)^2\alpha_1$$
$$+ n(n-1)(n-2)\beta_0]$$

Computed results from this are contained in Table 4.2 below.

4 A NUMERICAL EXPERIMENT

We consider here the following problem for experimentation with our results of the preceding sections. The exact error is defined by

$$\varepsilon^* = \max_{0 \leq x \leq 1} \{|y(x_k)\text{-}y_n(x_k)|\},\ 0 \leq x \leq 1\ \ \{x_k\} = \{0.01\text{k}\},\ \text{for } k = 0\ (1)100$$

Example

$$Ly(x) = y'''(x) - 5y''(x) + 6y'(x) = 0\ ,\ 0 \leq x \leq 1$$
$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$$

The exact solution is

$$y(x) = \frac{5}{6} + \frac{3}{2^{e2x}} - \frac{2}{3^{e3x}}$$

The numerical results are presented in Table 4.1 below.

Table 4.1: Error and Error Estimates using (3.13)

Degree(n) / Error	5	6	7
$\tilde{\varepsilon}$	9.88×10^{-5}	5.99×10^{-5}	3.85×10^{-5}
ε^*	1.96×10^{-4}	2.29×10^{-5}	4.25×10^{-7}

5 CONCLUSION

The integrated form of the tau method for the solution of Initial Value Problems (IVPs) in a class of third order differential equations with non-over-determination has been presented. The error estimate is good as it closely captures the order of the error. This is better achieved than for the case of the direct series substitution approach otherwise referred to as the differential form thus lending credence to the preference of the former. This may be attributed to the higher order perturbation term which the integrated form involves.

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