The Bounds of Crossing Number in Complete Bipartite Graphs

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Abstract—**We compare the lower bound of crossing number of bipartite and complete bipartite graph with Zarankiewicz conjecture and we illustrate the possible upper bound by a modified Zarankiewicz conjecture.**

Keywords—complete bipartite graphs, crossing numbers

I. Introduction

Let G = (V,E) be a simple connected graph with vertex set V(G) and edge set E(G).

The crossing number of a graph G, denoted by Cr(G), is the minimum number of crossings in a drawing of G in the plane[2,3,4].

The crossing number of the complete bipartite graph [7] was first introduced by Paul Turan, by his brick factory problem.

In 1954, Zarankiewicz conjectured [8] that,

$$Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

Where m and n are vertices.

Later, Richard Guy shown that the conjecture doesnot holds for all m,n. Then in 1970 D.J.Kleitman proved that Zarankiewicz conjecture holds for Min(m,n) ≤ 6.

A good drawing of a graph G is a drawing where the edges are non-self-intersecting in which any two edges have atmost one point in common other than end vertex. That is, a crossing is a point of intersection of two edges and no three edges intersect at a common point. So a good drawing is a crossing free drawing by arriving at a planar graph.

The crossing number is an important measure of the non-planarity of a graph. Therefore this application can be widely applied in all real time problems.

II A MODIFIED ZARANKIEWICZ CONJECTURE:

For any complete bipartite graphs with 'n' vertices,

$$Z(n,n) = \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor^2 \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right]$$

By using this conjecture, we can get all the possible number of crossings between every vertices for a given 'n', without any good drawing D. This reverse way of finding the crossings facilitates for large 'n' by without drawing the graph, we can get all possible crossings between every edges.

The best known lower bound on general case for all m,n ϵ N which was proved by D.J.Kleitman [1] in the following theorem. That is,

Theorem1[6]:

$$cr(K_{5,n}) \geq 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$cr(K_{6,n}) \geq 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

From this he deduced that

$$cr(K_{m,n}) \geq \frac{1}{5} m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

Theorem2:

For m > n, $cr\left(Z(m,n)\right) \geq cr\left(K_{m,n}\right) \leq cr\left(K_{n,m}\right)$.

Proof:

From theorem 1,

$$cr(K_{m,n}) \geq \frac{1}{5} m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

By definition,

$$Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

We can prove the theorem byinduction. Since in $cr\left(K_{m,n}\right)$, there are ${}^{m}c_{5}K_{5,n}$ subgraphs of $K_{m,n}$ with the partite with 'n' vertices in $K_{5,n}$. So we shall obtain the lower bound of $cr\left(K_{m,n}\right)$ for m ≥ 5 and n ≥ 3.

Case(i): Let n=3.

Subcase(i): m=5,

$$cr\left(Z(5,3)\right) = 2.2.1.1 = 4$$

$$cr\left(K_{5,3}\right) = \frac{1}{5}.5.4.1.1 = 4$$

$$cr\left(K_{3,5}\right) = \frac{1}{5}.3.2.2.2 = \frac{24}{5} = 4.8$$

$$\therefore cr\left(Z(5,3)\right) \geq cr\left(K_{5,3}\right) \leq cr\left(K_{3,5}\right)$$

Subcase(ii): m=6,

$$cr\left(Z(6,3)\right) = 3.2.1.1 = 6$$

$$cr\left(K_{6,3}\right) = \frac{1}{5}.6.5.1.1 = 6$$

$$cr\left(K_{3,6}\right) = \frac{1}{5}.3.2.3.2 = \frac{36}{5} = 7.2$$

$$\therefore cr\left(Z(6,3)\right) \geq cr\left(K_{6,3}\right) \leq cr\left(K_{3,6}\right)$$

Subcase(ii): m=7,

$$cr\left(Z(7,3)\right) = 3.3.1.1 = 9$$

$$cr\left(K_{7,3}\right) = \frac{1}{5}.7.6.1.1 = \frac{42}{5} = 8.4$$

$$cr\left(K_{3,7}\right) = \frac{1}{5}.3.2.3.3 = \frac{54}{5} = 10.8$$

$$\therefore cr\left(Z(7,3)\right) \geq cr\left(K_{7,3}\right) \leq cr\left(K_{3,7}\right)$$

$$\Rightarrow cr\left(Z(m,3)\right) \geq cr\left(K_{m,3}\right) \leq cr\left(K_{3,m}\right)$$

Case(ii): Let n=4.

Subcase(i): m=5,

$$cr\left(Z(5,4)\right) = 2.2.2.1 = 8$$

$$cr\left(K_{5,4}\right) = \frac{1}{5}.5.4.2.1 = 8$$

$$cr\left(K_{4,5}\right) = \frac{1}{5}.4.3.2.2 = \frac{48}{5} = 9.8$$

$$\therefore cr\left(Z(5,4)\right) \geq cr\left(K_{5,4}\right) \leq cr\left(K_{4,5}\right)$$

Subcase(ii): m=6,

$$cr\left(Z(6,4)\right) = 3.2.2.1 = 12$$

$$cr\left(K_{6,4}\right) = \frac{1}{5}.6.5.2.1 = 12$$

$$cr\left(K_{4,6}\right) = \frac{1}{5}.4.3.3.2 = \frac{72}{5} = 14.4$$

$$\therefore cr\left(Z(6,4)\right) \geq cr\left(K_{6,4}\right) \leq cr\left(K_{4,6}\right)$$

Subcase(ii): m=7,

$$cr\left(Z(7,4)\right) = 3.3.2.1 = 18$$

$$cr\left(K_{7,4}\right) = \frac{1}{5}.7.6.2.1 = \frac{84}{5} = 16.8$$

$$cr\left(K_{4,7}\right) = \frac{1}{5}.4.3.3.3 = \frac{108}{5} = 21.6$$

$$\therefore cr\left(Z(7,4)\right) \geq cr\left(K_{7,4}\right) \leq cr\left(K_{4,7}\right)$$

$$\Rightarrow cr\left(Z(m,4)\right) \geq cr\left(K_{m,4}\right) \leq cr\left(K_{4,m}\right)$$

In general,

$$cr\left(Z(m,n)\right) \geq cr\left(K_{m,n}\right) \leq cr\left(K_{n,m}\right).$$

We also observe that the following inequality ,

$$2cr\left(Z(m,n+1)\right) \geq 2cr\left(K_{m,n+1}\right) \leq 2cr\left(K_{n+1,m}\right)$$

Also holds good for the above cases.
Hence the proof.

Theorem 2:

For m = n, $cr\left(Z(m,n)\right) \geq cr\left(K_{m,n}\right)$.That is,

$$\frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor^{2} \left[\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \right] =$$

$$\left\lfloor \frac{n}{2} \right\rfloor^{4} \geq \frac{1}{5} n(n-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

Proof:

When m = n, $Z(n,n)$ is a complete bipartite graph.

By theorem 1,

$$cr\left(K_{m,n}\right) \geq \frac{1}{5} n(n-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

We shall prove the theorem for large sufficiently larger 'n' and hence deducing the result for subsequent small 'n'.

Case(i):

$$cr\left(K_{11,11}\right)=625$$

$$=\frac{1}{2}\begin{bmatrix}5(5)^2+4(5)^2+4(5)^2+4(5)^2+4(5)^2+4(5)^2\\+5(5)^2+5(5)^2+5(5)^2+5(5)^2+5(5)^2\end{bmatrix}$$

$$=\frac{1}{2}(5)^2\left[6.5+5.4\right]$$

$$=\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^2\left[\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)+\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right]$$

$$=\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^3\left[2\left\lfloor\frac{n}{2}\right\rfloor\right]$$

$$=\left\lfloor\frac{n}{2}\right\rfloor^4$$

$$cr\left(Z(11,11)\right)=550=\frac{1}{5}.11.10.5.5$$

$$=\frac{1}{5}.11(11-1)\left\lfloor\frac{11}{2}\right\rfloor\left\lfloor\frac{11-1}{2}\right\rfloor$$

$$=\frac{1}{5}.n(n-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

$$\Rightarrow cr\left(Z(11,11)\right)\geq cr\left(K_{11,11}\right)$$

Case(ii):

$$cr\left(Z(9,9)\right)=256=\frac{1}{5}.9.8.4.4$$

$$=\frac{1}{5}.9(9-1)\left\lfloor\frac{9}{2}\right\rfloor\left\lfloor\frac{9-1}{2}\right\rfloor$$

$$=\frac{1}{5}.n(n-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

$$cr\left(K_{9,9}\right)=256$$

$$=\frac{1}{2}\begin{bmatrix}4(4)^2+3(4)^2+3(4)^2+3(4)^2+3(4)^2+\\4(4)^2+4(4)^2+4(4)^2+4(4)^2\end{bmatrix}$$

$$=\frac{1}{2}(4)^2\left[5.4+4.3\right]$$

$$=\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^2\left[\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)+\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\right]$$

$$=\frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor^3\left[2\left\lfloor\frac{n}{2}\right\rfloor\right]$$

$$=\left\lfloor\frac{n}{2}\right\rfloor^4$$

$$\Rightarrow cr\left(Z(9,9)\right)\geq cr\left(K_{9,9}\right)$$

In general,

$$cr\left(Z(m,n)\right)\geq cr\left(K_{m,n}\right)$$

Hence the proof.

III CONCLUSION

We have given an alternate way of finding crossings in complete bipartite graphs. We also proved in bipartite graphs, the best lower bound of $cr\left(K_{m,n}\right)$ will always be a lower bound until 'm' and 'n' are altered.

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