Taylor Series Method Coupled with Shooting Technique for Solving Boundary Value Problems

M. A. Sandoval-Hernandez[1,3], U. A. Filobello-Nino[2], H. Vazquez-Leal[3], J. Huerta-Chua[4], R. Castaneda-Sheissa[2], G. U. Diaz-Arango[4], R. A. Callejas-Molina[5], S. E. Torreblanca-Bouchan[1], J. Chong-Duran[1], B. Pablo-Lopez[6]

[1] Centro de Bachillerato Tecnológico industrial y de servicios No. 190, Av. 15 Col. Venustiano Carranza 2da Sección, Boca del Río, 94297, Veracruz, México.

[2] Facultad de Instrumentación Electrónica, Universidad Veracruzana, Circuito Gonzalo Aguirre Beltrán S/N, Xalapa, 91000, Veracruz, México.

[3] Consejo Veracruzano de Investigación Científica y Desarrollo, Av. Rafael Murillo Vidal No. 1735, Cuauhtémoc, Xalapa, 91069, Veracruz, México.

[4] Instituto Tecnológico de Poza Rica TNM, Luis Donaldo Colosio Murrieta S/N, Arroyo Del Maíz, Poza Rica, 93230, Veracruz, México.

[5] Instituto Tecnológico de Celaya TNM, Antonio García Cubas 600, Col. Fovissste, Celaya, 38010, Guanajuato, México.

[6] Instituto Tecnológico de Veracruz TNM, Av. Miguel Ángel de Quevedo 2779, Col. Formando Hogar, 91897, Veracruz, Veracruz, México.

***Abstract*— In this work, we propose the Taylor series method (TSM) coupled with a shooting technique to obtain solutions of boundary value problems (BVP). In order to assess the benefits of this proposal, four different kinds of nonlinear BVP problems of different kind are proximately solved and compared versus their numerical solutions: two Neumann boundary condition problems, a fifth order mixed boundary conditions equation with an exponential term and the governing equation of the steady diffusion-reaction regime in a porous slab with parallel plane boundaries. The obtained results show that TSM generates highly accurate handy approximations, requiring only a few steps.**

Keywords*— Taylor series method, Shooting technique, Boundary valued problems.

I. INTRODUCTION

Nonlinear differential equations are applied to model a wide scope of phenomena in almost all branches of sciences. Unfortunately, it is not common to find the exact solution of such equations. Therefore, approximative methods [1,2,3,4,5,6,7,8,9] are a good alternative when it is required to know more about the nature of the phenomenon and the influence of its parameters.

Among the approximative methods highlights the series method [10,11,12,13] or its equivalent Taylor series method (TSM) [14,15,16] due to its simplicity and power. Such methods are mathematical tools applied to obtain power series approximations of linear and nonlinear equations. Although both methods can generate equivalent results, TSM can be easier to implement due to its capability to obtain the coefficients of the power series by a straight forward procedure that involves derivatives of the differential equation. Besides, both methods are designed to solve problems governed by Dirichlet conditions. Nonetheless, the boundary valued problems (BVP) are very common and important in all the branches of sciences, from thermodynamics to biology and many more. Therefore, we propose the apply the combination of the TSM method with a shooting technique [16,17,18] to solve BVP problems as reported in [16,19,20]. The shooting technique aids to circumvent the issue of TSM method with boundary conditions by converting the BVP problem into a Dirichlet type problem. In this work, we will denote this procedure as shooting Taylor series method (STSM). The main idea behind the proposed technique is:

1. First, the boundary conditions are substituted for the equivalent Dirichlet conditions. During this procedure the Dirichlet conditions not provided by the boundary conditions are replaced by shooting constants to be determined later by the STSM method.
2. Next, we apply the TSM method to obtain the coefficients of the series solution using derivatives. The derivatives are obtained from the nonlinear differential equation.
3. Then, the approximate solution is obtained by substituting the calculated coefficients from last step into the Taylor series expression.
4. Finally, the values of the shooting constants are obtained by evaluating the approximate solutions at the boundary conditions and solving the resulting system of equations.

In order to show the potential of the proposed procedure, four nonlinear BVP problems are solved and compared versus numerical methods: two Neumann boundary condition problems [21,22], a fifth order equation [23] and the governing equation of the steady diffusion-reaction regime in a porous slab with parallel plane boundaries [24].

This paper is organized as follows. In Section II, we introduce the basic concept of STSM method. In Section III, we find the approximated solutions of four BVP problems of different kind. Numerical simulations and a discussion about the results are provided in Section IV. Finally, a concluding remark is given in Section V.

II. INTRODUCTION OF STSM METHOD

We consider a nonlinear differential equation of the following form

$$u^{(n)} = N(u) - f(x) \qquad x \in \Omega, \tag{1}$$

with the boundary conditions

$$B(u, \partial u/\partial n) = 0, \qquad x \in \Gamma, \tag{2}$$

where n is the order of the differential equation, N is a general operator, $f(x)$ a known analytical function, B is a boundary operator, Γ is the boundary of domain Ω, and $\partial u/\partial n$ denotes differentiation along the normal drawn outwards from Ω.

In order to apply STSM, we express the solution of (1) as a Taylor series

$$u = u(x_0) + \frac{u'(x_0)}{1!}(x-x_0)^1 + \frac{u''(x_0)}{2!}(x-x_0)^2 + \frac{u'''(x_0)}{3!}(x-x_0)^3 + \cdots, \tag{3}$$

where (x_0) is the expansion point and derivatives $u^{(i)}(x_0), (i = 0,1,2,\cdots)$ are expressed in terms of the parameters and boundary conditions of (1) .

As we require to solve BVP problems, the boundary conditions not located at the expansion point (x_0) will be replaced by shooting constants giving as result traditional Dirichlet conditions. Next, in order to obtain the coefficients of (3) $u^{(i)}(x_0), (i = 0,1,2,\cdots)$, STSM requires (I) calculate the successive derivatives of (1) and (II) evaluate each derivative using the Dirichlet Conditions.

Finally, in order to fulfil the boundary conditions originally replaced by the shotting constants, it is necessary to evaluate (3) at such boundary conditions; then, the resulting system of equations is solved to obtain the values of the shooting constants.

III. CASE STUDIES

In the present section, we will solve four case studies to show the accuracy and usefulness of the approximated solutions obtained by STSM.

3.1 Bratu's problem with Newmann boundary conditions

Bratu's differential equation [21,22] arises in problems to fuel ignition in thermal combustions theory and also in the Chandrasekhar model of the expansion of the universe. Now, let's consider the following *Bratu's* equations with Newman boundary conditions

$$u'' - 2\exp(u) = 0, \qquad u'(0) = 0, \quad u'(1) = 2\tan(1), \tag{4}$$

where the exact solution is $u(x) = -2\log(\cos(x))$.

Considering the expansion point $x_0 = 0$, it yields to the following Taylor series.

$$u(x) = u(0) + \frac{u'(0)}{1!}(0)^1 + \frac{u''(0)}{2!}(0)^2 + \cdots \tag{5}$$

where derivatives $u^{(m)}, (m = 0,1,2,\cdots)$ are unknowns to be determined by Taylor series method.

Next, we derive successively (4), resulting

$$\begin{aligned} u'' &= 2\exp(u), \\ u''' &= 2u'\exp(u), \\ u^{(4)} &= 2\exp(u)(u'^2 + u''), \\ u^{(5)} &= 2\exp(u)(u'^3 + 3u''u' + u'''), \quad \vdots \end{aligned} \tag{6}$$

Now, the boundary conditions of (4) are transformed into $[u(0) = c, u'(0) = 0\,]$, and replaced it into (6) to obtain

$$\begin{aligned} u''(0) &= 2\exp(c), & u'''(0) &= 0, \\ u^{(4)}(0) &= 4\exp(2c), & u^{(5)}(0) &= 0, \\ u^{(6)}(0) &= 32\exp(3c), & u^{(7)}(0) &= 0, \\ u^{(8)}(0) &= 544\exp(4c), & u^{(9)}(0) &= 0, \\ u^{(10)}(0) &= 15872\exp(5c), & u^{(10)}(0) &= 0, \end{aligned} \tag{7}$$

Finally, substituting (7) into (5), yields the series

$$u(x) = \exp(c)x^2 + \frac{1}{6}\exp(2c)x^4 + \frac{2}{45}\exp(3c)x^6 + \frac{17}{1260}\exp(4c)x^8 + \frac{62}{14175}\exp(5c)x^{10}, \qquad 0 \le x \le x1. \tag{8}$$

Finally, if we substitute the second boundary condition $u'(1) = 2\tan(1)$ into (8) and solve for the shooting constants, it results that $c = 0.006190945532$.

3.2 Nonlinear Burguers' equation

Now, we consider the following Newmann boundary conditions Burguers' equations [21]

$$u'' = -uu' - u + \left(\frac{1}{2}\right)\sin(2x),$$

$$u'(0) = 1, \qquad u'\left(\frac{\pi}{2}\right) = 0,$$

(9)

where the prime denotes differentiation with respect x, and the exact solution is $u(x) = \sin(x)$.

We derive successively (9), resulting

$$u'' = -uu' - u + \left(\frac{1}{2}\right)\sin(2x),$$

$$u''' = -u'^2 - uu'' - u' + \cos(2x),$$

$$u^{(4)} = -uu''' + (-3u' - 1)u'' - 2\sin(2x),$$
$$\vdots \qquad (10)$$

Now, the boundary conditions of (9) are replacing by $u(0) = c, u'(0) = 1$

$$u''(0) = -2c, \quad u'''(0) = 2c^2 - 1,$$
$$u^{(4)}(0) = 2c^3 + 9c, \quad u^{(5)}(0) = 2c^4 - 31c^2 + 1,$$
$$u^{(6)}(0) = -2c^5 + 83c^3 - 75c, \qquad (11)$$

Finally, using (11) and (3) (considering $x_0 = 0$). We obtain the following power series

$$u(x) = \left(-\frac{1}{360}c^5 + \frac{83}{720}c^3 - \frac{5}{48}c\right)x^6 + \left(\frac{1}{60}c^4 - \frac{31}{120}c^2 + \frac{1}{120}\right)x^5 + \left(-\frac{1}{12}c^3 + \frac{3}{8}c\right)x^4 + \left(\frac{1}{3}c^2 - \frac{1}{6}\right)x^3 + -cx^2 + x,$$
$$0 \le x \le \frac{\pi}{2}. \qquad (12)$$

Finally, if we substitute the second boundary condition $u'\left(\frac{\pi}{2}\right) = 0$ into (12) and solve for the shooting constant, it results that $c = 0.00598416801101$.

3.3 Fifth order BVP equation

Let us the following problem [23]

$$u^{(5)} = u^2\exp(-x),$$
$$u(0) = u'(0) = u''(0) = 1,$$
$$u(1) = u'(1) = \exp(1), \qquad (13)$$

where the prime denotes differentiation with respect to x, and the exact solution is $u(x) = \exp(x)$.

As aforementioned procedure for the first two case studies, we replace the boundary conditions (13) by their Dirichlet equivalent $u(0) = u'(0) = u''(0) = 1$, $u'''(0) = c_1$, $u^{(4)}(0) = c_2$ to obtain the coefficients of the following ninth-order Taylor series

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}c_1x^3 + \frac{1}{24}c_2x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \frac{1}{4030}(-1 + 2c_1)x^8 + \frac{1}{362880}(-1 + 2c_2)x^9, \quad 0 \le x \le 1. \qquad (14)$$

Finally, if we substitute the boundary conditions $u(1) = u'(1) = \exp(1)$ into (14) and solve the system of linear equations, it results that the shooting constants are $c_1 = 0.999889130$, $c_2 = 1.000051624$.

3.4 Steady diffusion-reaction regime in a porous slab with parallel plane boundaries

The governing equation of the steady diffusion-reaction regime in a porous slab with parallel plane boundaries [24] cam be expressed as

$$\varphi'' = \varphi^n, u'(0) = 0, u(1) = 1,$$
$$0 \le x \le 1, \qquad (15)$$

Where u is the dimensionless concentration of the reactant, the primes denote differentiation with respect to the dimensionless transverse coordinate x, φ stands for the Thiele modulus, and n is the reaction order with range $n \ge -1$.

As aforementioned, we replace the boundary conditions of (15) by Dirichlet equivalent $u(0) = c, u'(0) = 0$ to obtain the coeficients of the following eight-order Taylor series

$$u(x) = \frac{1}{40320c}(34\varphi^4n^3c^{4n-2} + 30\varphi^4nc^{4n-2} - 63\varphi^4n^2c^{4n-2})x^8 + \frac{1}{40320c}(224c^{-1+3n}\varphi^3n^2 - 168c^{-1+3n}\varphi^3n)x^6 + \frac{1}{24c}(\varphi^2c^{2n}n)x^4 + \frac{1}{2c}(\varphi c^{n+1}n)x^2 + c,$$
$$0 \le x \le 1. \qquad (16)$$

Finally, if we choose as a particular case $n = 3$ and $\varphi = 0.7$ and substituting $u(1) = 1$ into (16), it results that the shooting constants that fulfil the boundary condition is $c = 0.7987274733$.

IV. Numerical Simulation And Discussion

From figures 1-3, we observe the high accuracy for the STSM approximations for the first three case studies. The exact solution was used por comparison purposes. The last case study does not possess a known solution for $n = 3$; the, we employed as reference the built-in numerical routine for BVPs from Maple 17. The command was configured to use a tolerance of absolute error of 10^{-12}. There upon, the high accuracy of STSM approximation is depicted on figure 4.

The power of coupling a shooting method [17,18,16,19,20] with the TSM method was exhibited by the solution of two highly nonlinear problems with Neumann boundary conditions, a fifth order nonlinear BVP problem with exponential term and a second order nonlinear BVP with cubic non-linearity. What is more, due to the straightforward procedure for the application of STSM method and the high

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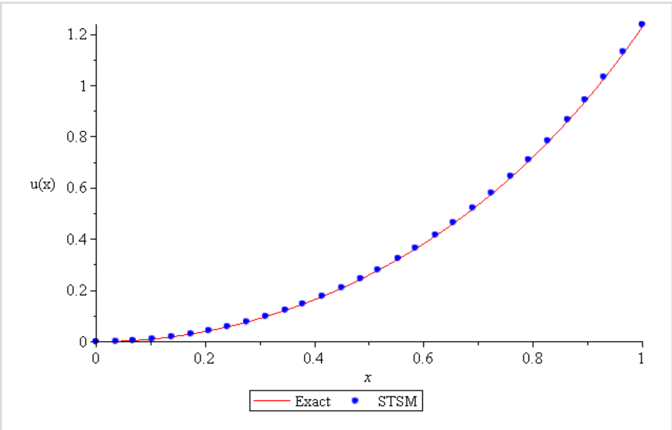
accurate handy approximations obtained, it can be an attractive math tool for engineers interested in the field of modelling.



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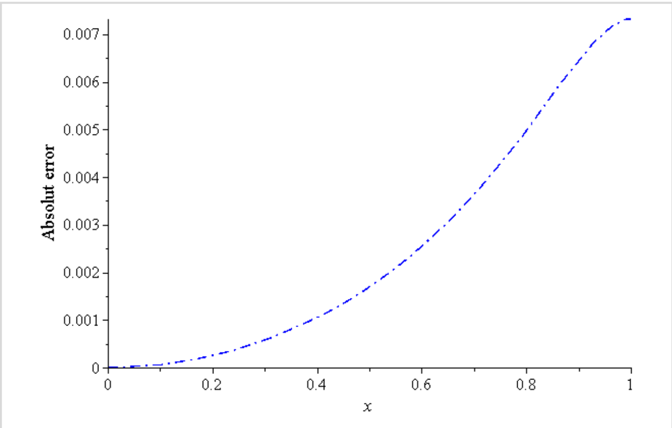

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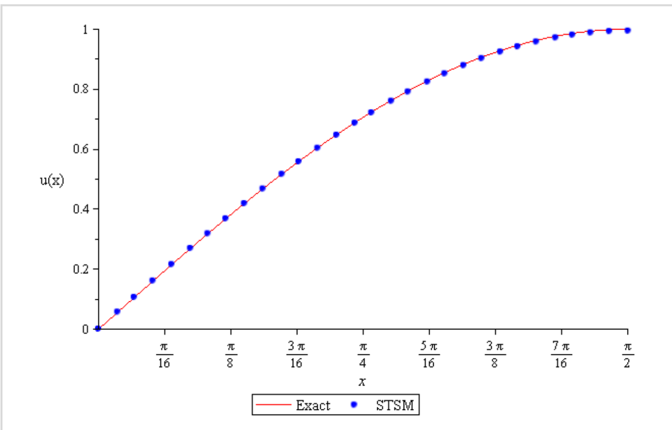
Figure 1. (a) Exact solution for (4) (solid line) and its approximate STSM solution (8) (solid circles). (b) Absolute error of approximation with respect to exact solution.



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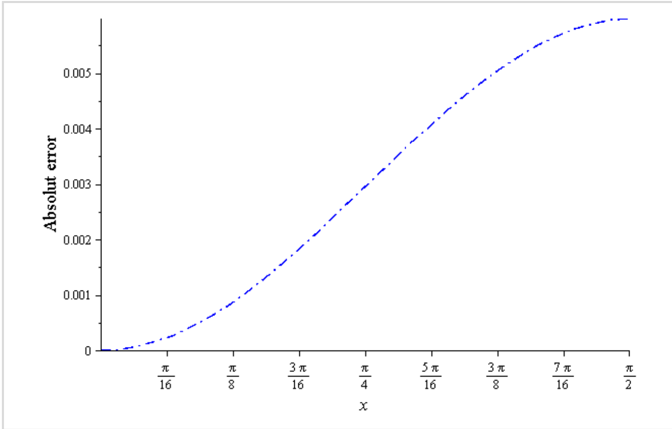

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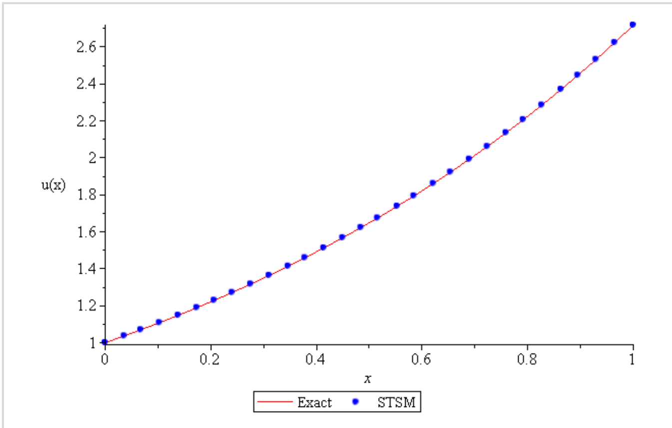
Figure 2. (a) Exact solution for (9) (solid circles) and its approximate STSM solution (12) (solid circles). (b) Absolute error of approximation with respect to exact solution.



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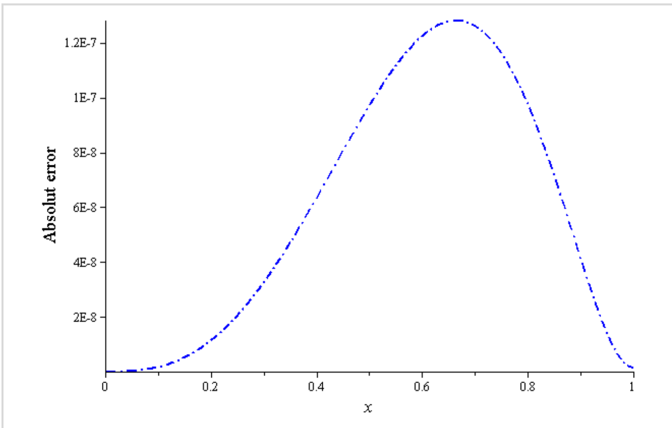

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Figure 3. (a) Exact solution for (13) (solid circles) and its approximate STSM solution (14) (solid circles). (b) Absolute error of approximation with respect to exact solution.

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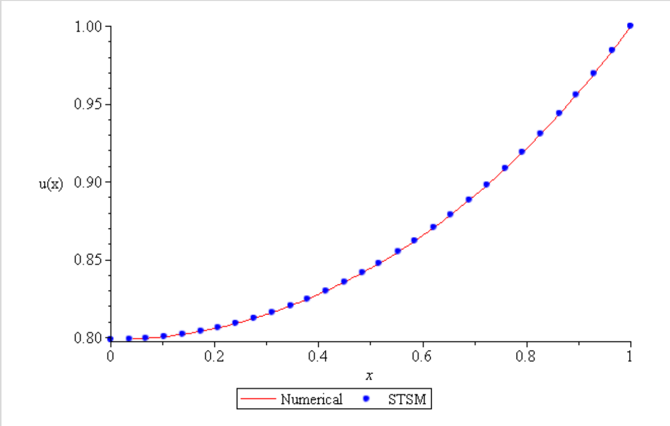

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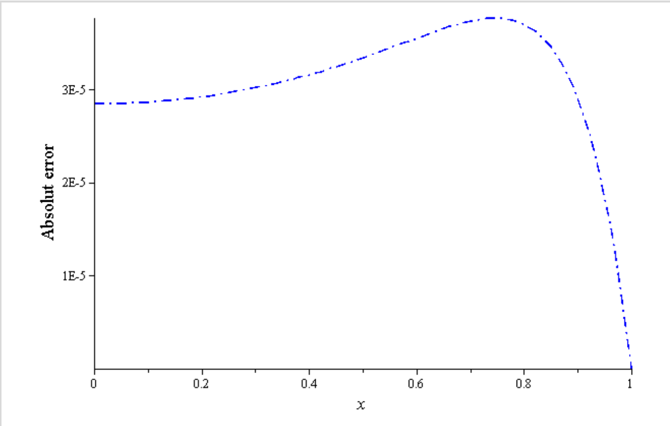

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Figure 4. (a) Numerical solution for (15) (solid line) and its approximate STSM solution (16) (solid circles). (b) Absolute error of approximation with respect to numerical solution.

V1. CONCLUSION

This work introduced the shooting Taylor method STSM as a powerful tool to solve boundary problems (BVPs) in nonlinear differential equations. We were able to obtain accurate and handy approximations for different types of highly non-linear BVP problems due to the shooting constants strategy. Therefore work can be addressed to employ STSM for the approximation of Robin boundary conditions problems, among others.

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