Synthesis of Nonlinear Spiral Torsion Springs

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***Abstract*—Springs are elastic machine elements that exert forces or torques and store energy. Torsion springs are loaded by torques about spring axis. Spiral springs belong to torsion springs and are characterized by the requirements that their coils have zero pitch and do not contact each other during operation. Nonlinear spiral springs have nonlinear torque-rotation relationships and are difficult to design because of their customized nonlinear features. A method is introduced in this paper for synthesizing nonlinear spiral springs. The center curve of a spiral spring is defined by an Archimedean spiral curve. Its nonlinear spring rate is realized by the nonumiform in-plane thickness of the Archimedean spiral curve. The smooth nonuniform in-plane thickness of the entire spiral spring comes from the spline interpolation on a set of in-plane thickness values on the center spiral curve. The arc length of the center spiral curve is used as interpolation parameter. The synthesis of nonlinear spiral springs is systematized as optimizing the independent design variables for the in-plane thickness interpolation. The presented method is demonstrated by the synthesis of a nonlinear stiffening spiral spring in the paper.**

***Keywords*—spiral spring; spring rate; spiral curve; synthesis; spline interpolation; interpolation parameter.**

I. Introduction

A spring is a flexible machine element used to exert a force or a torque, and store energy at the same time. Energy is stored in the spring that is bent, twisted, stretched or compressed. The stored energy is recoverable by the elastic return of the distorted spring material [1]. Springs can be divided into four general categories based on their primary functions: push, pull, twist or energy storage [2]. Within each category, there are many possible configurations. Helical compression and tension springs are commonly made of round or rectangular wire wound with constant coil diameter and uniform pitch. Pitch is the distance that is measured along the coil axis and from the center of one coil to the center of the adjacent coil. A torsion spring performs a twist function and supports a torsional load. Torsion springs are of two main types: helical and spiral [3]. The coils in a helical torsion spring are usually closely wound like an extension spring, but do not have any initial tension. The coil ends (that can have different shapes such as straight torsion, straight offset or short hook) provide lever arms to apply torsion to the coil body. The transmitted torque is applied about the axis of the helix. Spiral springs consist essentially of flat spring materials wound on themselves (zero pitch) with open space between coils. They are characterized by the requirement that their coils do not contact each other during operation [4-5]. Spiral springs have low hysteresis and are used in meters, instruments, timing mechanisms and rehabilitation devices. The inner end of a spiral spring is usually attached to an arbor and its outer end is often clamped or hinged. This paper is focused on synthesizing spiral springs with nonlinear features.

The performance of a spring is characterized by the force or torque (*F* or *T*) applied to it and the translational or angular deflection (*D* or θ) which the applied force or torque results in. The slope of the *T*-θ curve of a spiral spring is its spring rate or stiffness denoted by *k*. If the relationship between torque and angular deflection is represented by a general function $T = T(\theta)$, spring rate can then be defined as follows.

$$k(\theta) = \lim_{\Delta\theta \to 0} \frac{\Delta T}{\Delta \theta} = \frac{dT}{d\theta} \quad (1)$$

If the slope of a spring is constant, the spring is linear. Otherwise, it is nonlinear. Linear springs obey the Hooke's Law, $T = k\theta$. Conventional spring designs are mainly for linear springs. The design of linear springs has been systematized. The spring rate of a linear spiral spring can be calculated by the following equation [5].

$$k = \frac{\pi b t^3 E}{6L} \quad (2)$$

In (2), *b* is the out-of-plane width of the rectangular section of the flat strip of a spiral spring, *t* is the in-plane thickness, *L* is the active length of the strip, and *E* is the modulus of elasticity of the spring material. Equation (2) applies when both ends of a spiral spring are rigidly mounted. If the outer end is hinged, deflection for a given torque will be about 25% larger than that calculated by using equation (2) [5]. Fig. 1 shows a spiral spring with clamped outer end and constant out-of-plane width *b* and uniform in-plane thickness *t*. Its inner end is rigidly connected to the extension of an arbor. The torque delivered by the spring is proportional to the rotational angle of the arbor. Spiral springs of this type are usually used for applications that require rotation of less than 360 degrees. The strip in a spiral spring is mainly in bending like a curved beam in contrast to the torsion experienced in helical compression and tension springs. The torque applied to a spiral spring should always close the coils rather open them because the residual stresses from coil-winding are favourable against closing torque.

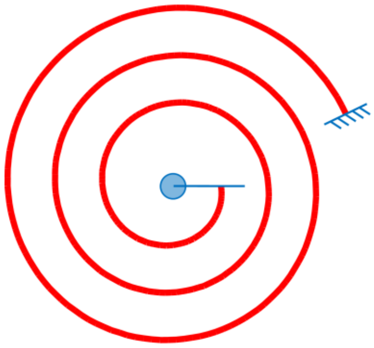

Fig. 1 A spiral spring with clamped outer end.

The applied torque in a nonlinear spiral spring is not linearly related to its angular deflection. A nonlinear stiffening spring (which is also called nonlinear progressive spring) is a spring that gradually increases its spring rate as the spring deflection progresses [6], which provides a progressively hardening reaction as the spring gets deflected. Fig. 2 shows the *T*-θ curve of a nonlinear stiffening spiral spring.

In contrast to a nonlinear stiffening spring, a nonlinear softening spring (which is also called nonlinear degressive spring) gradually decreases its spring rate as the spring deflection increases and provides a softening reaction [6]. The *T*-θ curve of a nonlinear softening spiral spring is shown in Fig. 3.

The *T*-θ relationship of a nonlinear spiral spring is based on its particular application and has its unique feature [7]. The design methods for linear springs are not applicable because of the individual nonlinear characteristics. Designing nonlinear spiral springs is more challenging and difficult than linear springs. The objective of this paper is to provide a systematic design method for nonlinear spiral springs.

The popular configuration of a spiral spring has center curve of Archimedean spiral curve with constant out-of-plane width (b) and uniform in-plane thickness (t) as shown in Fig. 1. To make the *T*-θ relationship of a spiral spring nonlinear, its in-plane thickness or out-of-plane width or both can be varied. Equation (2) points out that k has a cubic relationship with t and linear relationship with b, so changing t has higher impact on k than changing b. The t value of a synthesized spiral spring in this paper is nonuniform and optimized to make it have the desired nonlinear spring rate.

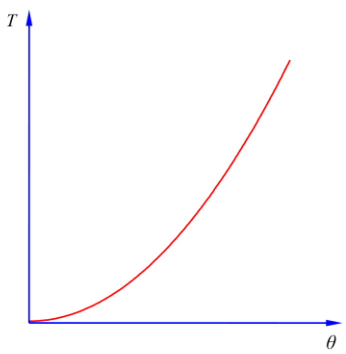

Fig. 2 The *T*-θ curve of a nonlinear stiffening spring.

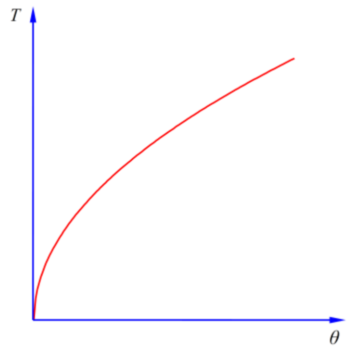

Fig. 3 The *T*-θ curve of a nonlinear softening spring.

The remainder of the paper is organized as follows. The synthesis formulation on nonlinear spiral springs are presented in section II. The optimization approach of spring parameters is provided in section III. Section IV is on the synthesis of a nonlinear stiffening spring using the method introduced in this paper. Conclusions are drawn in section V.

II. SYNTHESIS FORMULATION OF SPIRAL SPRINGS

Spiral springs are commonly made of rectangular wires wound on themselves. They are also called flat spiral springs because of their zero pitch. The center curves of spiral springs are usually Archimedean (also known as arithmetic) spiral curves [8]. There is a unique property in Archimedean spiral curves that the distance between two successive intersections of any ray from the origin and an spiral curve is constant. Because of this unique property, Archimedean spiral curves are widely used for spiral springs.

In a polar coordinate system, an Archimedean spiral curve can be described by the following function.

$$r(\theta) = r_0 + b\theta \tag{3}$$

In (3), r_0 is the radius of the starting point of the spiral curve, θ is the rotation angle from the starting point to the current point on the spiral curve. $2\pi b$ is the distance between two successive turnings along any ray from the origin. If the orientation angle of the starting point (θ_0) is taken as 0, θ is then the orientation angle of the current point. An Archimedean spiral curve can be considered as the locus of a moving point that moves along a ray at a constant speed (v) and the ray itself also rotates with respect to the origin at a constant angular velocity (ω). In that case, $b = v/\omega$. If the radius and rotation angle of the ending point of a spiral curve are denoted by (r_e, θ_e), the function of the Archimedean spiral curve becomes:

$$r(\theta) = r_0 + \frac{(r_e - r_0)}{\theta_e}\theta \tag{4}$$

Fig. 4 shows an Archimedean spiral curve in which the ray (on which the moving point is located) rotates counter-clockwise. If the ray rotates clockwise, θ becomes negative. Equation (4) has to be changed into:

$$r(\theta) = r_0 - \frac{(r_e - r_0)}{\theta_e}\theta \tag{5}$$

Fig. 5 shows an Archimedean spiral curve in which the ray rotates clockwise. It is a mirror image of the spiral curve in Fig. 4 with respect to x-axis.

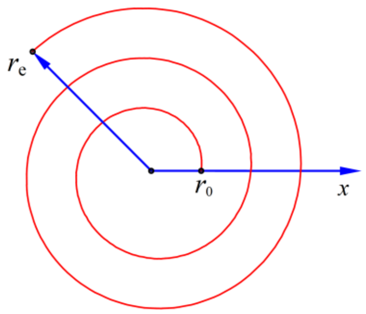

Fig. 4 An Archimedean spiral curve with positive θ.

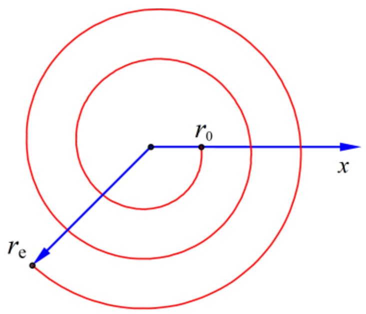

Fig. 5 An Archimedean spiral curve with negative θ.

From Figs. 4 and 5, it can be seen that an Archimedean spiral curve can have two potential rotation directions, either counter-clockwise ($\theta > 0$) or clockwise ($\theta < 0$). The two spiral curves form mirror images each other and are often called the two arms of the Archimedean spiral curve.

To simplify the calculation of the arc length of an Archimedean spiral curve, intermediate parameters p and q are introduced and equation (4) can be represented as follows.

$$r(\theta) = p + q\,\theta \qquad (6)$$

$$p = r_0$$

$$q = \frac{(r_e - r_0)}{\theta_e}$$

The arc length $s(\theta)$ is then given by the following equation.

$$\begin{aligned} s(\theta) &= \int_0^\theta \sqrt{[r(\theta)]^2 + [r'(\theta)]^2}\, d\theta \\ &= \int_0^\theta \sqrt{p^2 + q^2 + 2pq\theta + q^2\theta^2}\, d\theta \\ &= \frac{(p + q\theta)}{2q}\sqrt{p^2 + q^2 + 2pq\theta + q^2\theta^2} \\ &+ \frac{q}{2}\sinh^{-1}\left(\frac{p + q\theta}{q}\right) - \frac{p}{2q}\sqrt{p^2 + q^2} - \frac{q}{2}\sinh^{-1}\left(\frac{p}{q}\right) \end{aligned} \qquad (7)$$

The curvature radius $R(\theta)$ can be calculated as follows.

$$\begin{aligned} R(\theta) &= \frac{\left\{[r(\theta)]^2 + [r'(\theta)]^2\right\}^{\frac{3}{2}}}{\left|[r(\theta)]^2 + 2[r'(\theta)]^2 - r(\theta)\, r''(\theta)\right|} \\ &= \frac{\left\{p^2 + q^2 + 2pq\theta + q^2\theta^2\right\}^{\frac{3}{2}}}{\left(p^2 + 2q^2 + 2pq\theta + q^2\theta^2\right)} \end{aligned} \qquad (8)$$

When both the in-plane thickness (t) and out-of-plane width (b) of a spiral spring are uniform, the spring is linear. To make the spiral spring have the desired nonlinear spring rate, the in-plane thickness is made nonuniform in this paper. To optimally design the nonuniform in-plane thickness of a nonlinear spiral spring, a set of interpolation points are selected on the center Archimedean spiral curve of the spiral spring. At each interpolation point, an independent design variable on the in-plane thickness at the point is introduced. The entire spiral spring has then smooth nonuniform in-plane thickness through the interpolation of in-plane thickness design variables. The arc length of the center Archimedean spiral curve is used as the interpolation parameter.

There are several popular interpolation approaches such as Lagrange and spline interpolation. The individual Lagrange polynomials depend on the locations of the parameter values and all of them have to be recalculated when a parameter value is changed. Besides, the degree of Lagrange polynomials becomes very high when there are many interpolation points since the Lagrange polynomial degree is the total number of interpolation points minus 1. High degree polynomials are prone to oscillate [9]. Because of these disadvantages, Lagrange interpolation is not used in this paper.

In the synthesis of spiral springs, it is undesirable for the in-plane thickness to change suddenly or sharply. Synthesizers are more interested in smooth interpolation of the in-plane thickness and tight interpolation curves that path through all the interpolation points in order. These needs can be met by spline interpolation. A set of polynomials of degree 3 that are smoothly connected at given interpolation points form a cubic spline interpolation curve. The slope and curvature at internal points are continuous between any two adjacent polynomials. At the two end points, their conditions can be chosen differently which include natural end conditions (two end curvatures are set as zero), not-a-knot end conditions (the third derivative is continuous at both the first and last internal nodes) or clamped end conditions (two end slopes are specified).

If there are n+1 in-plane thickness values (t_0, t_1, $\cdots$, t_n) to be interpolated for a spiral spring and their corresponding arc length values on the center Archimedean spiral curve are (s_0, s_1, $\cdots$, s_n). Then, the piecewise cubic spline interpolation is given by n cubic polynomials between each successive pair of points [10]. Here is an interpolation example. Suppose 9 thickness values to be interpolated, which are (0.5, 0.7, 1.0, 1.5, 2.3, 1.5, 1.0, 0.7, 0.5). The corresponding arc lengths are (0, 2, 4, 6, 8, 10, 12, 14, 16). The spline interpolation is shown in Fig. 6 in which the horizontal axis represents arc length and the vertical axis is for thickness. The 9 thickness values are smoothly interpolated by the spline curve. If the 9 thickness values are interpolated by Lagrange interpolation, its result is shown in Fig. 7. The interpolation curve swings a lot. The interpolated thickness value even becomes negative, which does not make any sense.

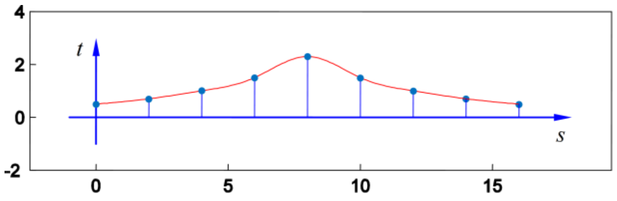

Fig. 6 A spline interpolation.

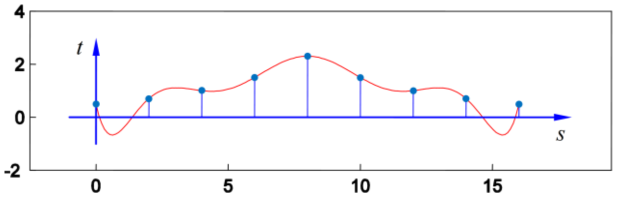

Fig. 7 A Lagrange interpolation.

To ensure against the existence of any cusp in a spiral spring, the curvature radius of the center Archimedean spiral curve at any point has to be greater than half of the in-plane thickness at that point [11]. Cusp can be avoided through the following constraint.

$$R(\theta) > 0.5t(\theta) \tag{9}$$

In (9), $R(\theta)$ is the curvature radius calculated by (8) and $t(\theta)$ is the in-plane thickness at point θ on the center Archimedean spiral curve.

III. OPTIMIZATION OF SPRING PARAMETERS

The center curve of a spiral spring is an Archimedean spiral curve. The nonlinear spring rate of the spiral spring is realized by the nonuniform in-place thickness in this paper. The smooth nonuniform in-plane thickness of the entire spiral spring comes from the spline interpolation on a set of in-plane thickness values on the center spiral curve. The arc length of the center spiral curve is used as the interpolation parameter. The synthesis of a nonlinear spiral spring is systematized as optimizing the independent design variables for the in-plane thickness interpolation. The values of the independent design variables of a synthesized nonlinear spiral spring are optimized in this paper by using the Global Optimization Toolbox of MATLAB [12-13]. Global Optimization Toolbox provides methods that find optimal solutions to synthesis problems with multiple local optima. Global Search Solver in MATLAB's Global Optimization Toolbox is used in the paper to search for the optimal spring design parameters.

Finite element analysis software ANSYS is used in the paper to evaluate the performance of a synthesized nonlinear spiral spring [14-15]. Given the parameters for the in-plane thickness interpolation, the torque, deflection and stress of the related nonlinear spiral spring are analyzed by ANSYS. An ANSYS batch file is first generated in MATLAB on elements, material properties, boundary conditions and input information. ANSYS is then called from MATLAB to execute the batch file. After going through preprocessor, processor and postprocessor processes, ANSYS creates an output file on the performance information of the synthesized nonlinear spiral spring. MATLAB reads the ANSYS output file and computes the objective and constraint functions for optimization. ANSYS Parametric Design Language is employed in the paper as a data transfer bridge between MATLAB optimization and ANSYS finite element analysis.

IV. SYNTHESIS OF A STIFFENING SPIRAL SPRING

A stiffening spiral spring is a nonlinear spiral spring that gradually increases its spring rate as the spring deflection progresses. The torque (T) and angular deflection (θ) ranges of the synthesized spiral spring are from 0 to 75 N mm and from 0 to 60 degrees, respectively. The T-θ relationship is defined by 5 desired (T, θ) target points: (T_0, θ_0), (T_1, θ_1), (T_2, θ_2), (T_3, θ_3) and (T_4, θ_4). The deflection range of 60 degrees is divided into 5 equal-distance points, i.e. $\theta_j = 15j$, j = 0, 1, 2, 3, 4. T_0 and T_4 correspond to the two end points of the torque range, so that T_0 = 0 and T_4 = 75 N mm. The 3 internal torque points are set to $T_1 = 0.10T_4$, $T_2 = 0.27T_4$ and $T_3 = 0.55T_4$. The spline curve that interpolates the 5 desired target points is shown in Fig. 8.

The design domain is a circle with radius of 50 mm which is shown Fig. 9. The inner end of the spiral spring is rigidly connected to the extension of an arbor while its outer end is clamped. When the arbor rotates through 60 degrees, the torque on the arbor from the spiral spring is required to meet the desired T-θ relationship. The center curve of the spiral spring is an Archimedean spiral curve described by equation (6), $r(\theta) = p + q\,\theta$. The starting point of the spiral curve is set at $r_0 = p = 10$ mm. The radius of the ending point is set at $r_e = p + q\,\theta_e = 45$ mm based on the radius of the design domain of 50 mm. The rotation angle (θ_e) of the ending point is a design variable. The material for the spring is engineering plastic with yield strength of 71 MPa and modulus of elasticity of 2200 MPa. The out-of-plane width of the spring is of constant value of 10 mm. The in-place thickness is nonuniform and interpolated by 7 thickness values (t_0 to t_6 that are marked in Fig. 9). t_j's (j= 0, 1, 2, 3, 4, 5, 6) are independent design variables and vary from 1.0 mm to 4.0 mm. They are evenly distributed along the center curve of the spiral spring. As shown in Fig. 9, the arc lengths between any two consecutive t_j's are equal. Design variables to be optimized are θ_e and the 7 thickness values. Thus, there are totally 8 independent parameters to be optimized, which can be represented as a design variable vector X.

$$X = [\theta_e \quad t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6] \tag{10}$$

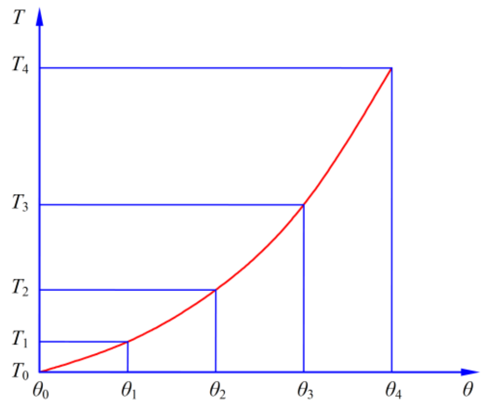

Fig. 8 The desired T-θ curve of the synthesized spiral spring.

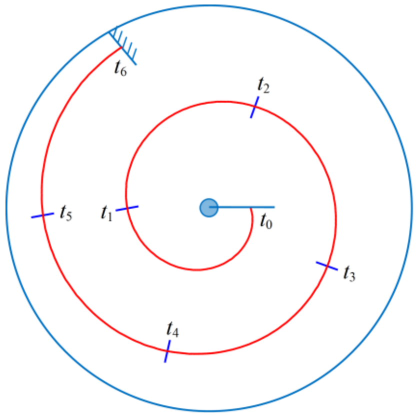

Fig. 9 The design domain of the synthesized spiral spring.

The spring is synthesized to minimize the error between the actual torque from the spring and the desired torque when a certain deflection is input to the spring. This error is measured by the average deviation at the 4 target deflections (θ_1 to θ_4) as follows.

$$TE = \frac{1}{n}\sum_{j=1}^{n}\left|T_{a,\,j} - T_{d,\,j}\right| \tag{11}$$

TE is the average torque error. $T_{a,\,j}$ is the actual torque generated by the spring when the target deflection θ_j is input to the spring while $T_{d,\,j}$ is the desired spring torque and equals the target torque T_j. The maximum stress in the spiral spring is constrained to be below its allowable value. Cusp and coil contact are not allowed to happen to the synthesized spiral spring.

The synthesis result is shown in Fig. 10. The design variable vector *X* for this solution is

$$\begin{aligned} X = [&7.05 \quad 3.93 \quad 3.76 \quad 1.00 \\ &2.98 \quad 4.00 \quad 1.36 \quad 1.05] \end{aligned} \tag{12}$$

The unit of θ_e with value of 7.05 in (12) is radian. In this solution, the desired and actual spring torques at 5 targets are: (0, 0), (7.50, 10.80), (20.25, 24.41), (41.25, 44.52) and (75, 74.55). The spline curves that interpolate the two sets of spring torques are shown in Fig. 11. The red spline is from the desired target torques while the blue one interpolates the actual spring torques. Except the starting and ending points, the actual spring torque is a little bit above the desired torque. The maximum stress in the spring is 34.28 MPa, which occurs when the spiral spring has deflection of 60 degrees.

Fig. 12 shows the undeformed and deformed beam elements of the spiral spring, which is from ANSYS with the input angular deflection of 60 degrees.

When the torque, deflection and stress of the spiral spring are analyzed in ANSYS, the input angular deflection of 60 degrees is divided into 4 even load steps and geometric nonlinearity command "NLGEOM" is turned on. The spiral spring is discretized into beam elements and modeled by BEAM188 that allows tapered beam cross-sections.

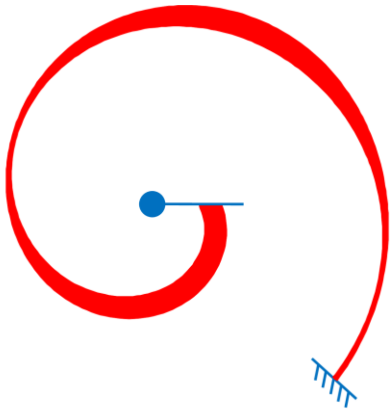

Fig. 10 The synthesized spiral spring.

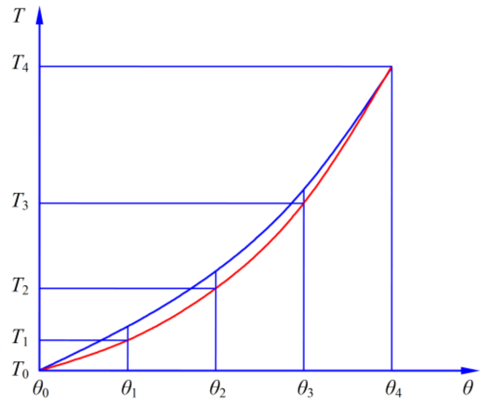

Fig. 11 The desired and actual *T*-θ curves of the spring.

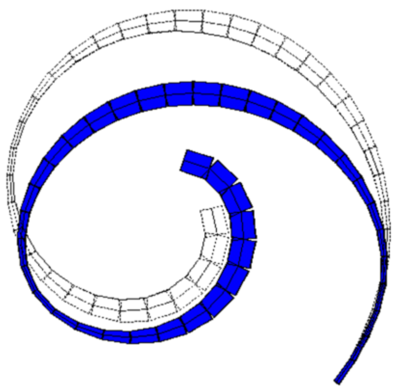

Fig. 12 The deformed spiral spring.

V. CONCLUSIONS

A synthesis method of nonlinear spiral springs is introduced in the paper. The nonlinear spring rate of a spiral spring is realized by its nonuniform in-plane thickness. The desired nonlinear torque-rotation relationship comes from the optimization of the design variables for the in-plane thickness interpolation. The center curve of a synthesized nonlinear spiral spring is an Archimedean spiral curve with constant out-of-plane width. The nonuniform in-plane thickness is interpolated by using cubic splines. A set of in-plane thickness values are employed for interpolation. They are evenly distributed along the arc length of the center spiral curve. The interpolation parameter is the arc length of the center spiral curve. The rotation angle (θ_e) of the ending point of the spiral curve and the set of in-plane thickness values form the independent design variables. The synthesis of nonlinear spiral springs is systematized as optimizing the independent design variables for nonuniform in-plane thickness interpolation.

The optimization of the independent design variables for the nonuniform in-plane thickness interpolation is conducted in the paper by using MATLAB's Global Optimization Toolbox. The deviation between the desired torque- rotation relationship and the actual torque- rotation relationship is minimized. The deviation is measured by the average difference between the desired spring torques and the actual spring torques under certain spring deflections. The maximum stress in the spiral spring is constrained to be below the yield strength of the spring material. Cusp and coil contact are not allowed to happen to the synthesized spiral spring. The deflection, torque and stress of the synthesized spring are analyzed by using ANSYS. The communication between MATLAB and ANSYS is based on ANSYS Parametric Design Language. A nonlinear stiffening spiral spring is synthesized in the paper to demonstrate the synthesis procedure and verify the effectiveness of the introduced synthesis method.

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