Study on Dynamic Characteristics with Differential Equations

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***Abstract*:- Versatility of differential equations cannot be denied. The area of its study is very broad. It can be applied to various fields of science and mathematics. In general, differential equations are just an equation with an unknown function and its derivative. Electrical/Electronic instruments are very widely used over the globe and there operation highly depends on its static and dynamic characteristics. Static characteristics focus on measuring an unvarying process condition. Contrary dynamic characteristics focus on measurement of quantities which vary at a faster pace. Measurement outcomes are rarely static over time. They will possess a dynamic component that must be understood for correct interpretation of the results. To properly appreciate instrumentation design and its use, it is now necessary to develop insight into the most commonly encountered types of dynamic response and to develop the basis that allows us to make concise statements about responses. If the behavior is nonlinear, then description with mathematics becomes very difficult and might be impracticable. Therefore the mathematics used to describe dynamic system can be introduced. This gives valuable insight into the expected behavior of instrumentation.**

Hence application of differential equations can be applied to understand dynamic characteristics and its various aspects such as frequency response, sensitivity etc. Thus, in the later part of study we are using differential equations to define the order for any function quantity w.r.t reference function using different responses on different electrical/electronic devices.

Keywords: Dynamic Characteristics, Differential equations and its order, Instruments etc.

INTRODUCTION

Flexibility of differential equations cannot be negotiated. The area of its study and use is very wide. In general, differential equations are just an equation with an unknown function and its derivative. The unknown function can be any ***y(x)*** which we want to determine. Differential equation categories as ordinary differential equations and partial differential equation, the difference being involvement of one independent variable or more than one independent variable functions.Thus differential equations can be applied to various electrical or electronics instruments whose operations highly depends on its static and dynamic characteristics.

Static characteristics focus on unvarying input process condition. Contrary dynamic characteristics focus on quantities which vary rapidly. Outcomes of instruments are rarely static over time. They will possess a dynamic component that should be understood for accurate and precise results. To properly appreciate instrumentation design and its use, it is necessary to develop concise statements about responses. Mathematics with nonlinear becomes very difficult; therefore the mathematics used to describe dynamic system can be introduced to get valuable insight into the behavior of instrumentation.

DIFFERENTIAL EQUATIONS

An equation containing an independent variable, dependent variable with respect to independent variable is called a differential equation.
Here we consider 'u' as dependent variable and't' independent variable then,

$$\frac{du}{dt} = \sin t + \cos t \qquad \ldots\ldots (1)$$

Here in above equation $\frac{du}{dt}$ represents the first derivative of 'u' (which is a dependent variable with respect to't'(which is an independent variable)and $\sin t$ and $\cos t$ are two functions of independent variable't'.

Above equation is a linear differential equation.

A differential equation is a**homogeneous linear differential equation** if the differential equation expressed in the form of a polynomial involves the derivatives and dependent variable in first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable.

$$C_o\frac{d^n u}{dt^n}+C_1\frac{d^{n-1}u}{dt^{n-1}}+\ldots\ldots+C_n u=0 \qquad \ldots\ldots (2)$$

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Here $C_o, C_1 \ldots\ldots\ldots C_n$ are constants or function of independent variable and $\frac{d^n t}{dt^n}$ is the n^{th} derivative of dependent variable '*u*' with respect to *t*. Similarly $\frac{d^{n-1} t}{dt^{n-1}}$ is the *n-1*th derivative of dependent variable '*u*' with respect to *t.*

The right hand sideof equation (2) is equal tozero represents an homogeneous linear differential equation for dependent variable '*u*' and independent variable'*t*'.

DEGREE OF DIFFERENTIAL EQUATION

The degree of differential equation is defined as the degree of the highest order derivatives, when differential coefficients are made from radicals and fraction.

For example in equation (1)
Where degree of the differential equation is onebecause as here highest order derivative of dependent variable '*u*' is $\frac{du}{dt}$ and for thispower of $\frac{du}{dt}$ is one.

ORDER OF DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest order derivative appearing in the equation.
For example in equation (1)
Where order of the differential equation is one . As in the above equation the highest order derivative is $\frac{du}{dt}$ whose order is one in the above equation (1).

GENERAL FORM FOR 1ST ORDER DIFFERENTIAL EQUATION

Differential equation of 1st order involves the independent variable*t*, dependent variable *u* and $\frac{du}{dt}$ thus the general form of first order differential equation can be represented as

$$\frac{du}{dt} = f(t, u) \qquad \ldots\ldots (3)$$

Here in above general form of first order differential equation $\frac{du}{dt}$ represents first order derivative of dependent variable '*u*' with respect to'*t*' and also $f(t, u)$ represents the function of *t* on which dependent variable '*u*' depends.

For example equation (1) represents a first order differential equation where $\sin t + \cos t$ is $f(t, u)$ i.e. function of *t*.

GENERAL FORM FOR 2ND ORDER DIFFERENTIAL EQUATION

We can deduce general form of second order differential equation from general form of first order differential equation which can be represented as

$$\frac{d^2u}{dt^2} + p(t)\frac{du}{dt} + q(t)\, u = 0 \qquad \ldots\ldots(4)$$

Similarly here $\frac{d^2u}{dt^2}$ is the second order derivative of dependent variable '*u*' with respect to'*t*' and $p(t)$ as well as $q(t)$ are two different functions of '*t*'.

As for example $\frac{d^2u}{dt^2} + \sin t \frac{du}{dt} + 2u = 0 \qquad \ldots\ldots.(5)$

By comparing equations (4) and (5)

$p(t) = \sin t$

$q(t) = 2$

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HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

$$C_o\frac{d^n u}{dt^n} + C_1\frac{d^{n-1}u}{dt^{n-1}} + \ldots\ldots + C_1\frac{du}{dt} + C_n u = 0 \qquad \ldots\ldots (6)$$

Whereequation (6) represents general nth order differential equation.

SOLUTION OF DIFFERENTIAL EQUATION

The solution of a differential equation is a relation between the variables involved which satisfies the differential equation.

GENERAL SOLUTION

Solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution.

For example if any differential equation is like this $\frac{d^2u}{dt^2}+u=0$…… (7)

Hence,

$$\mathbf{u} = A \cos t + B \sin t \qquad \ldots\ldots (8)$$

Where equation (8) represents a general solution where arbitrary variables (A & B)are present which can have any values regarding provided boundary values.

PARTICULAR SOLUTION

Solution obtained by giving particular value to the arbitrary constants in the general solution of a differential equation is called particular solution.

From equation (7) and (8) at particular boundary conditions the solution of differential equation may be as

$\mathbf{u} = 3 \cos t + 2 \sin t$ (at particular boundary condition) ……. (9)

CHARACTERISTICS

Performance of any instrument is the ultimate decider for its application, utility and popularity. The two basic characteristics of performance are **static characteristics** and **dynamic characteristics**.

Static Characteristics

The instrument measure an unvarying process condition (input) with respect to time then the characteristics of instruments are termed as static characteristics.

Steady state responses of any instrument are relates in the static characteristic as because in the steady state responses of any instruments are relatively unvarying or may vary with a quite slow constant rate. Static characteristics are also named as system characteristics. These are generally specified for an instrument by the manufacturer.

Dynamic Characteristics

Due to dynamic behavior measured outcomes are rarely static and exhibit slowness due to things like mass, capacitance and delay time.The instruments measure a varying process condition with respect to time then the characteristics of instrument is called dynamic instrument.
Differential equation is used to determine the dynamic relation between the rapidly varying input and output. Thus, we are using differential equations to represent different-different dynamic responses of different order instrument.

DYNAMIC RESPONSE OF ZERO-ORDER INSTRUMENT

Using the above introduced concepts of differential equations we are trying here to find dynamic response characteristics for zero order instruments.
As zero order instrument has an output proportional to input as

$$u(t)=Kf(t) \qquad \ldots\ldots (10)$$

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Wherein above equation (10) for any instrument$\boldsymbol{u(t)}$represents dependent variable 'u' as a function of 't' which is an output function and **K** is a constant here and $\boldsymbol{f(t)}$ is another function of independent variable't 'which is an input function .

For any order instrument, the relation between input and output can be represented as (with the concept of differential equation)

$$C_n\frac{d^n u}{dt^n}+ C_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \ldots\ldots + C_0 u = K_m\frac{d^m u}{dt^m} + K_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + K_o u_i \quad \ldots\ldots(11)$$

Where we assumed an m^{th} order differential equation in $\boldsymbol{u}_i$with K_m, K_{m-1}…….K_1, K_0 as constants and an another n^{th} order differential equation in $\boldsymbol{u}$with C_n,C_{n-1},……..,C_1, C_0 as constants.

Now to verify the equation (10) assume

C_n, C_{n-1} …….. C_1=0 and K_m, K_{m-1}……. K_1= 0 ...…. (12)

ButC_o,$K_o \neq 0$ ……. (13)

Thus by putting equation (12) and (13) in equation (11)we can reduce equation (11) as

$C_o u_o = K_o u_i$ …… (14)

Divide $\mathbf{C_o}$in both sides of equation

$$u_o = \left(\frac{Ko}{Co}\right) u_i \quad \ldots\ldots (15)$$

Where B $=\left(\frac{Ko}{Co}\right)$ …… (16)

Where equation (15) represent general form.

Example-1 We have taken here a thermometer that measures the temperature of an isolated box at room temperature. The thermometer indicates the temperature of the box.

Following zero order system, we can interpret outputwhich exhibit exact identical to input.

If$\boldsymbol{g(t)}$is input response then there response output $\boldsymbol{f(t)}$proportional to input response $\boldsymbol{g(t)}$.

$\boldsymbol{f(t)}$=**B** $\boldsymbol{g(t)}$ ……. (17)

Comparing to equation (15) and (17)

B =Static sensitivity

DYNAMIC RESPONSE OF A FIRST ORDER INSTRUMENT

Similarly using the above introduced concepts of differential equations we are trying here to find dynamic response characteristics for first order instruments.

Let we assume in equation (11) as

C_n, C_{n-1} …… C_2=0=K_m, K_{m-1} ……. K_1……. (18)

C_o, C_1&$K_o \neq 0$ ……. (19)

Thus by putting equation (18) and (19) in equation (11) we can reduce equation (11) as

$C_1\frac{du_0}{dt} + C_o u_o = K_o u_i$ ……. (20)

The above differential equation is for first order instrument.

Divide equation (20) both sides by C_0then we obtain equations as

$\frac{C_1}{C_0}\frac{du_0}{dt} + u_o = \frac{K_0}{C_0}u_i$ …… (21)

$$A\frac{du_0}{dt} + u_o = Bu_i \quad \ldots\ldots\ldots (22)$$

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Where, $A=\frac{C_1}{C_0}$ =Time constant

$B=\frac{K_0}{C_0}$=Static sensitivity

Thus, equation can be modified as

$$\frac{u_0}{u_i}=\frac{B}{(A\frac{du}{dt}+1)} \qquad \text{...... (23)}$$

Where equations (18) & (19) are the general forms for first order instruments.

EXAMPLE-2 Here we have taken a temperature transducer thermometer, used to measure body temperature. When the thermometer is put in the mouth it experiences a sudden increase in temperature.To obtain results, the mercury should be heated up to T_2(initial temperature for mercury to respond) however, here present a time delay for temperature value is equal to T.

Now to determine the time constant and static sensitivity for mercury we have used the concept of differential equation and tried to find the dynamic response for mercury.

Let thermometer temperature is at T_0 (which is the room temperature) and temperature $T_1$is the temperature inside the mouth.

As the law of conservation of energy stats that

Rate of change of energy =Rate of heat flow

$$\frac{dE}{dt}=\frac{dQ}{dt} \quad \text{where E is the energy and Q is heat flowing} \qquad \text{...... (24)}$$

Now, as for liquids by introducing concept of specific heat flow [2]

$$\frac{dE}{dt}=mC_v\frac{dQ}{dt} \qquad \text{...... (25)}$$

Where,

m=Mass of mercury

C_v=Specific heat of the mercury

To obtain reading heat must flow, as flow of heat from a closed chamber, is depends on three major factors which are

1. **Heat transfer coefficient** of that material**(h)**
2. **Surface areaof the thermometer knob(S)** and
3. The **temperature on which liquid starts to respond** (as here because a delay is present so **(T_2-T)** is responding time for mercury)[2]

Thus,

$$\frac{dQ}{dt}=hS\,(T_2-T) \qquad \text{...... (26)}$$

Substitute equation (26) and (25) in equation (24)then we obtained as

$$mC_v\frac{dT}{dt}=hS\,(T_2 - T) \qquad \text{....... (27)}$$

$$\frac{mCv}{hS}\frac{dT}{dt}+T=T_2 \qquad \text{....... (28)}$$

The above equation (28) which is a differential equation, governs the temperature of the mercury.

Thus by comparing equation (28) with general equation (22) (dynamic response equations for first order) we can deduce as

$u=T, f(t)=T_1$

Where,
Time constant for mercury, **A=** $\frac{mCv}{hS}$

Static sensitivity for mercury,**B=1**

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DYNAMIC RESPONSE OF SECOND ORDER INSTRUMENTS

Similarly using the above introduced concepts of differential equations we are trying here to find dynamic response characteristics for second order instruments.

Let we assume in equation (11) as

$C_n, C_{n-1} \ldots\ldots C_3=0=K_m, K_{m-1} \ldots\ldots K_1$ ...... (30)

$C_o, C_1, C_2 \& K_o \neq 0$ ...... (31)

Then equation (11) can be reduces to dynamic response equation for second order instruments and instrument follows differential equation as follows

$$C_2\frac{d^2u_0}{dt^2}+C_1\frac{du_0}{dt}+C_o u_o=K_o u_i \quad \ldots\ldots(32)$$

Divide whole equation by C_2 to develop general equation thuscan be reduced to

$$\left\{\frac{D^2}{W_n^2}+\frac{2\zeta D}{Wn}+1\right\}u_o = Bu_i \quad \ldots\ldots.(33)$$

Where,

$D =\frac{d}{dt}$,

$W_n= (\frac{C_0}{C_2})^{1/2}$= natural frequency in rad/sec

$\zeta =\frac{\frac{C_1}{(C_0C_2)^{1/2}}}{2}$=Damping ratio

$B=\frac{K_0}{C_0}$=static sensitivity

Example-3 Here we have taken a mechanical assembly where a **massm** is hinged with help of a spring whose **spring constant is k** .We are using the concept of differential equation to balance the mechanical forces of this arrangement to findnatural frequency and Damping ratio of this arrangement. [3]

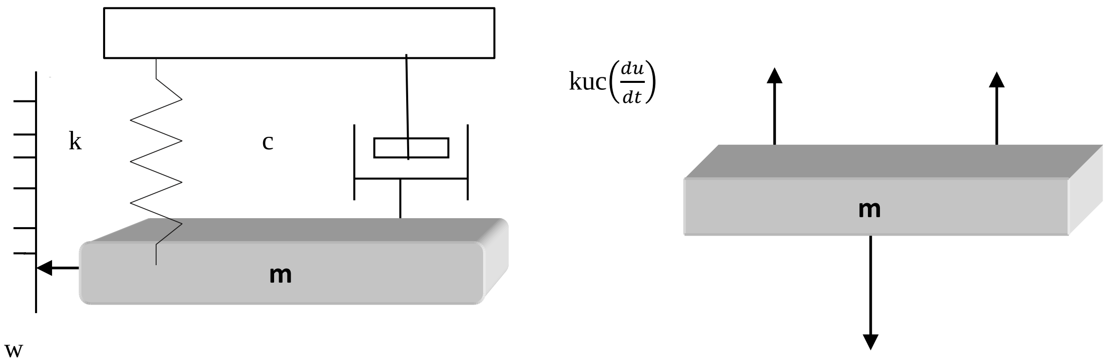

Fig-1
(Mechanical arrangement)Fig-2 (Free Body Diagram)

- Spring is a mechanical element which develops force in proportional to level of stressed/strain. Spring is widely used in Electrical/Electronic instrument.

As$\mathbf{F_{sp} = ku}$ ...... (34)

Where, $\mathbf{F_{sp}}$is force of spring

u = displacement and k = spring constant

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The balance needs a way to damper the oscillation of pointer after a weight is dropped. A force is needed to move the piston. This result to produce a force F_{dp} which is proportional to the speed of the piston relative to the cylinder

Therefore,

$$F_{dp} = c \left(\frac{du}{dt}\right) \quad \text{...... (35)}$$

Where,

c is called the damping co-efficient

$\left(\frac{du}{dt}\right)$= velocity of piston.

From free body diagram, by balancing mechanical forces including oscillations with including differential equation.

$$m \frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = w \quad \text{....... (36)}$$

$$\frac{m}{k}\frac{d^2u}{dt^2} + \frac{c}{k}\frac{du}{dt} + u = k^{-1}w \quad \text{....... (37)}$$

Thus by comparing equation (37) with general equation (33) (dynamic response equations for first order) we can deduce as

Natural frequency, $W_n = \left(\frac{k}{m}\right)^{½}$,

Damping ratio, $\zeta = \frac{c}{2(km)^{½}}$,

Static sensitivity (B) $=k^{-1/2}$

EXAMPLE-4 We consider a here R-L-C series circuit with a DC excitation V now applying differential equation on this network with some initial condition as switch is closed at t=0sec. and before that switch was open and now if we want to find out current response in the network the we can use mainly two methods .

One is simply based on the differential equation, by this we try to find second order differential equation for this network and by using initial conditions we are able to derive the particular solution with different –different damping ratios following different condition for response

Second method uses Laplace transformation and inverse Laplace transformation to find the particular response as in this first we transform the domains as'*t*' **domain** to '*s*' **domain**, using **Laplace transformation.** Then we try to find simply the response due to excitation and then we again transform the domains as from '**s'domain**to**'*t*' domain,** using **Inverse Laplace transformation.**

Here to find response we are using differential equations.

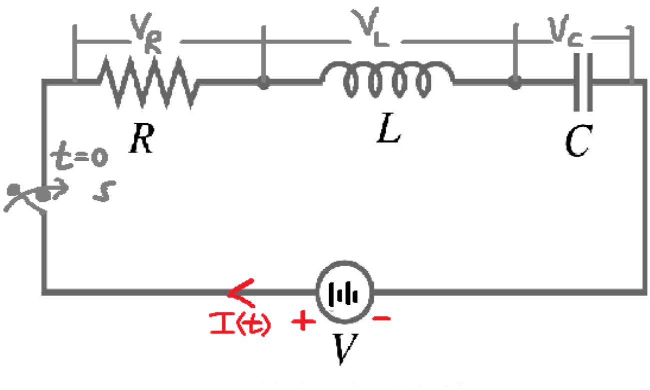

Fig-3 (R-L-C series circuit)

As at $t=0^-$ sec for this I (t) =0because for this condition switch was opened so no current will flow from the circuit for open circuit condition.

Now applying for $t=0^+$ sec. at this switch will be closed so,

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For this conditionof above general series RLC circuit, we try to write basic **KVL** equation

Let I (t) current as we know Voltage across resister here =V_R= I (t)R

Similarly Voltage across inductor =V_L=L $(\frac{dI(t)}{dt})$

And Voltage across capacitor=V_C=$\frac{1}{C}\int idt$

Thus by KVL:-

V= V_R+ V_{L+} V_C ……. (38)

V= I (t) R+ L $(\frac{dI(t)}{dt})$ **+**$\frac{1}{C}\int idt$ ……. (39)

To obtain second order differential equation differentiate equation (32) with respect to't'

$$0=R\frac{dI(t)}{dt}+L\frac{d^2I(t)}{dt^2}+\frac{I(t)}{C} \quad \text{……. (40)}$$

Divide by L both sides of the equation (33)

$$\frac{d^2I(t)}{dt^2}+\left(\frac{R}{L}\right)\left(\frac{dI(t)}{dt}\right)+\frac{I(t)}{LC}=0 \quad \text{……. (41)}$$

$$D=\frac{d}{dt} \quad \text{……. (42)}$$

$$\boxed{[D^2+(\frac{R}{L})D+(\frac{1}{LC})]\ I\ (t)=0} \quad \text{……. (43)}$$

From equation (43) we found as

$$[\frac{1}{(\frac{1}{LC})}D^2+\frac{(\frac{R}{L})}{(\frac{1}{LC})}D+1]I(t)=0 \quad \text{……. (44)}$$

Now compare equations (43) and (33)

Natural frequency (W_n) $=\frac{1}{\sqrt{LC}}$,

ζ = damping ratio=$\left(\frac{R}{2}\right)\sqrt{\frac{C}{L}}$

Above equation founds quadratic thus roots can be found using **ShriDharacharya method** as

Let D_1 and D_2 are two roots thus

$$D_1, D_2=\frac{-(\frac{R}{L})\pm\sqrt{(\frac{R}{L})^2-4\frac{1}{Lc}}}{2} \quad \text{…… (45)}$$

$$D_1, D_2=-(\frac{R}{2L})\pm\sqrt{(\frac{R}{2L})^2-\frac{1}{Lc}} \quad \text{..…. (46)}$$

Now the solution will differ and there exist 4 no. of cases and a concept of damping ratio occurs over here as

Case 1:-if $(\frac{R}{2L})^2>\frac{1}{Lc}$Over Damping → **2ζ>1**i.e. **damping ratio>1** →

Case 2:- if $(\frac{R}{2L})^2=\frac{1}{Lc}$Critical Damping → **2ζ=1**i.e. **damping ratio=1** →

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Case 3:- if $(\frac{R}{2L})^2 < \frac{1}{Lc}$ Under Damping ➡ **2ζ<1**i.e. **damping ratio<1** ➡

Case 4:- if R=0 ➡ Un damping ➡ **2ζ=0** i.e. **damping ratio=0**

Thus we can draw responses as

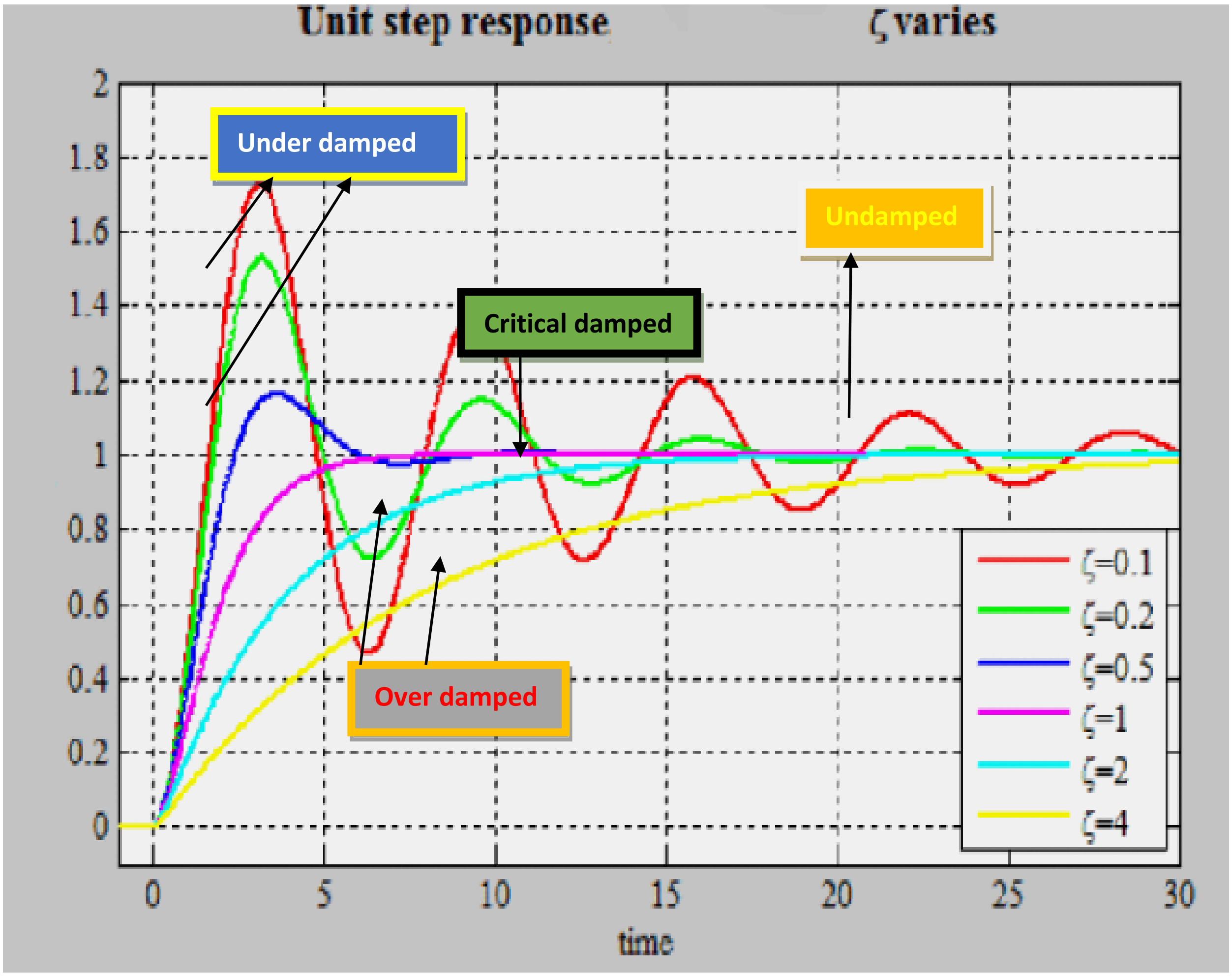

Fig-4 (Dynamic response curves for RLC Series circuit)

RESPONSE OF INSTRUMENT TO UNIT STEP FUNCTION

Consider a function x(t)=0 for t≤0, and x(t)=1 for t>0.Then the function x(t) is called unit step function.

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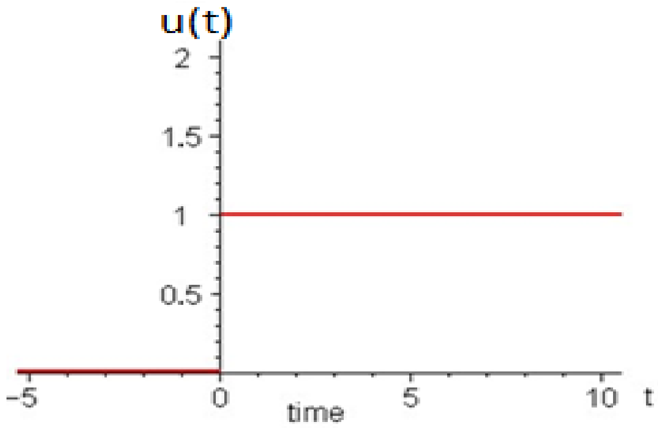

Fig-5 (Unit step function x (t)) [4]

The practical curves for different order differential form of equations we can obtain different-different responses for a input of unit step function as

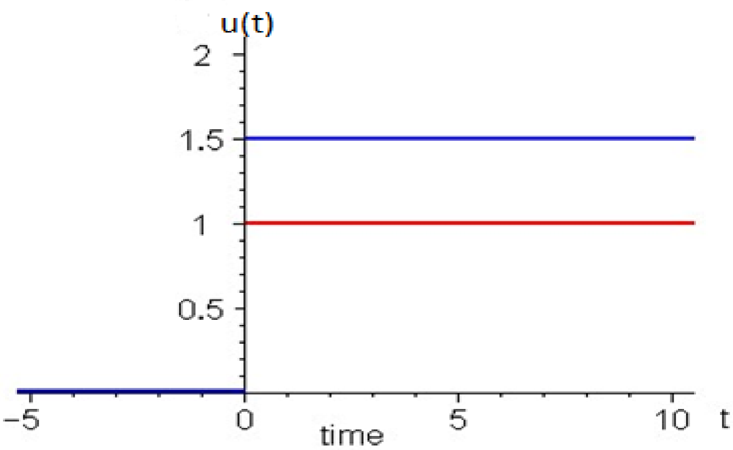

Fig-6 (Zero order response to unit step input)[4]

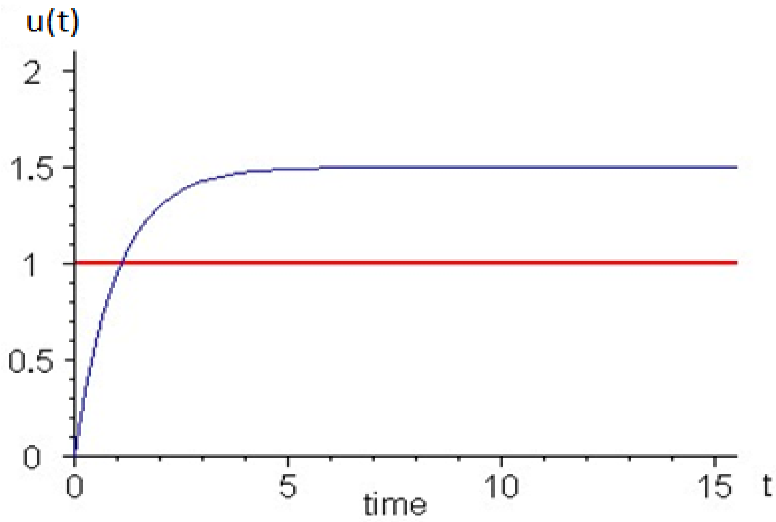

Fig-7 (First order response to unit step input at B=1.5)[4]

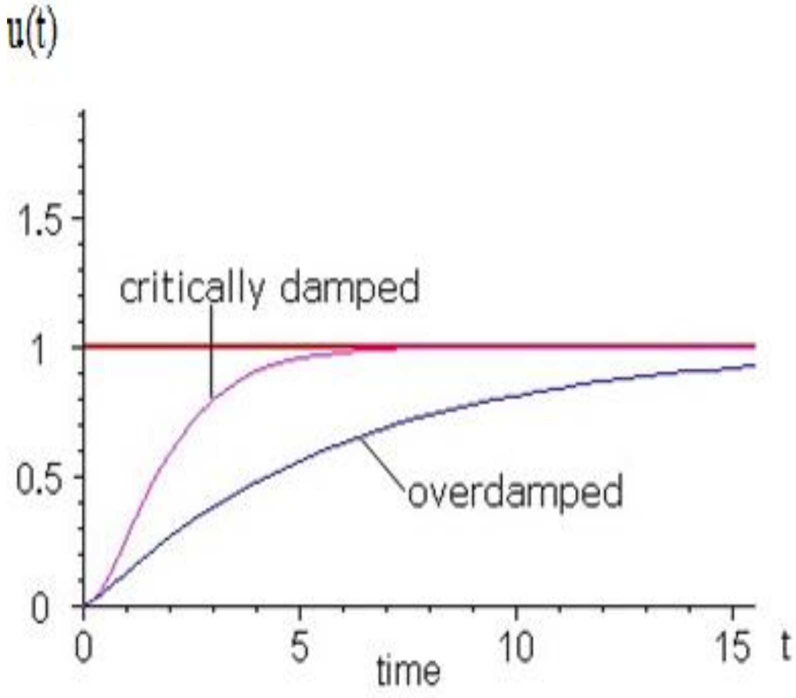

Fig-8 (Second order response to unit step input at E=3 and E=1) [4]

CONCLUSION

In this paper, we have given an overview related to the static and dynamic characteristics, although in this paper we focused mostly on dynamic characteristics. We have studied and observed the relation and effect of differential equations on the dynamic responses of different-different electronicinstrument, material and circuits of different orders. Moreover we believe that in the coming time, there will be less focus on the static responses and instead of this the more focus will be on relatively complex dynamic responses because in real time system static responses does not exist , where in the field for dynamic responses the above study of differential will be very useful.

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REFERENCES

[1] Measurement, Instrumentation and Sensors Handbook (John G Webster)
[2] Survey of Instrumentation and measurement, Stephen a. Dyer
[3] Mathematics by RD Sharma class 12^{th}.
[4] NCERT class 11^{th} Physics chapter fluids.
[5] The Dynamic Response of Measuring Instruments, R. H. B. Exell, 2003. King Mongkut's University of Technology, Thonburi
[6] Review of First- and Second-Order System Response, Massachusetts Institute Of Technology Department Of Mechanical Engineering
[7] An Application of Differential Equations in the Study of Elastic Columns, Krystal Caronongan Southern Illinois University Carbondale.