STRONGLY UNIQUE BEST COAPPROXIMATION IN LINEAR 2-NORMED SPACES

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Abstract

This paper deals with some fundamental properties of the set of strongly unique best coapproximation in a linear 2-normed space.

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1. INTRODUCTION

The problem of best coapproximation was first introduced by Franchetti and Furi [2] to study some characteristic properties of real Hilbert spaces and was followed up by Papini and Singer [12]. Subsequently, Geetha S.Rao and coworkers have developed this theory to a considerable extent [4,5,6,7,8,9]. This theory is largely concerned with the questions of existence, uniqueness and characterization of best coapproximation. Newman and Shapiro [11] studied the problems of strongly unique best approximation in the space of continuous functions under supremum norm. Geetha S.Rao, et al. [3,10] established many significant results in strongly unique best coapproximation in normed linear spaces. The notion of strongly unique best coapproximation in the context of linear 2-normed spaces is introduced in this paper. Section 2 provides some important definitions and results that are required. Sections 3 delineates some fundamental properties of the set of strongly unique best coapproximation with respect to 2-norm.

2. PRELIMINARIES

Definition 2.1. [1] Let X be a linear space over real numbers with dimension greater than one and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following properties for every x, y, z in X .

(i) $\| x, y \| = 0$ if and only if x and y are linearly dependent,

(ii) $\| x, y \| = \| y, x \|$,

(iii) $\| \alpha x, y \| = |\alpha| \, \| x, y \|$, where α is a real number,

(iv) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$.

Then $\| .,. \|$ is called a 2-norm and the linear space X equipped with the 2-norm is called a linear 2-normed space. It is clear that 2-norm is non negative.

The following important property of 2-norm was established by Cho [1].

Theorem 2.2. [1] For any points $a, b \in X$ and any $\alpha \in \mathbb{R}$,

$$\| a, b \| = \| a, b + \alpha a \| .$$

Definition 2.3. Let G be a non-empty subset of a linear 2-normed space X . An element $g_0 \in G$ is called a best coapproximation to $x \in X$ from G if for every $g \in G$,

$$\| g - g_0, k \| \leq \| x - g, k \|, \text{ for every } k \in X \setminus [G, x],$$

where $[G, x]$ represents a linear space spanned by elements of G and x .

The definition of strongly unique best coapproximation in the context of linear 2-normed space is introduced here for the first time as follows.

Definition 2.4. Let G be a non-empty subset of a linear 2-normed space X . An element $g_0 \in G$ is called a strongly unique best coapproximation to $x \in X$ from G, if there exists a constant $t > 0$ such that for every $g \in G$,

$$\| g - g_0, k \| \leq \| x - g, k \| - t \| x - g_0, k \|, \text{ for every } k \in X \setminus [G, x].$$

The set of all elements of strongly unique best coapproximations to $x \in X$ from G is denoted by $T_G(x)$.

The subset G is called an existence set if $T_G(x)$ contains at least one element for every $x \in X$. G is called a uniqueness set if $T_G(x)$ contains at most one element for every $x \in X$. G is called an existence and uniqueness set if $T_G(x)$ contains exactly one element for every $x \in X$.

For the sake of brevity, the terminology subspace is used instead of a linear 2-normed subspace. Unless otherwise stated all linear 2-normed spaces considered in this paper are real linear 2-normed spaces and all subsets and subspaces considered in this paper are existence subsets and existence subspaces with respect to strongly unique best coapproximation.

3. SOME FUNDAMENTAL PROPERTIES OF $T_G(x)$

Some basic properties of strongly unique best coapproximation are obtained in the following Theorems.

Theorem 3.1. Let G be a subset of a linear 2-normed space X and $x \in X$. Then the following statements hold.

(i) $T_G(x)$ is closed if G is closed.

(ii) $T_G(x)$ is convex if G is convex.

(iii) $T_G(x)$ is bounded.

Proof. (i). Let G be closed.

Let $\{g_m\}$ be a sequence in $T_G(x)$ such that $g_m \to \tilde{g}$.

To prove that $T_G(x)$ is closed, it is enough to prove that $\tilde{g} \in T_G(x)$.

Since G is closed, $\{g_m\} \in G$ and $g_m \to \tilde{g}$, we have $\tilde{g} \in G$. Since $\{g_m\} \in T_G(x)$, we have

$$\| g - g_m, k \| \leq \| x - g, k \| - t \| x - g_m, k \|, \text{ for every } k \in X \backslash [G, x] \text{ and for some } t > 0$$

$$\Rightarrow \| g - g_m + \tilde{g} - \tilde{g}, k \| \leq \| x - g, k \| - t \| x - g_m, k \|$$

$$\Rightarrow \| g - \tilde{g}, k \| - \| g_m - \tilde{g}, k \| \leq \| x - g, k \| - t \| x - g_m, k \|, \text{ for every } g \in G \quad (3.1)$$

Since $g_m \to \tilde{g}$, $g_m - \tilde{g} \to 0$. So $\| g_m - \tilde{g}, k \| \to 0$, as 0 and k are linearly dependent.

Therefore, it follows from (3.1) that

$$\| g - \tilde{g}, k \| \leq \| x - g, k \| - t \| x - \tilde{g}, k \|,$$

for every $g \in G$ and for some $t > 0$.

Thus $\tilde{g} \in T_G(x)$. Hence $T_G(x)$ is closed.

(ii). Let G be convex, $g_1, g_2 \in T_G(x)$ and $\alpha \in (0,1)$. To prove that $\alpha g_1 + (1-\alpha) g_2 \in T_G(x)$,

let $k \in X \setminus [G, x]$.

Then

$$\begin{aligned}
&\| g-(\alpha g_1+(1-\alpha)g_2, k \| \\
&= \| \alpha(g-g_1)+(1-\alpha)(g-g_2), k \| \\
&\leq \alpha \| g-g_1, k \| +(1-\alpha) \| g-g_2, k \| \\
&\leq \alpha \| x-g, k \| -\alpha t \| x-g_1, k \| \\
&\quad +(1-\alpha) \| x-g, k \| -(1-\alpha)t \| x-g_2, k \|, \\
&\qquad \text{for every } g \in G \text{ and for some } t>0. \\
&= \| x-g, k \| -t(\| \alpha x-\alpha g_1, k \| + \| (1-\alpha)x-(1-\alpha)g_2, k \|) \\
&\leq \| x-g, k \| -t \| \alpha x-\alpha g_1+(1-\alpha)x-(1-\alpha)g_2, k \| \\
&= \| x-g, k \| -t \| x-(\alpha g_1+(1-\alpha)g_2), k \| .
\end{aligned}$$

Thus $\alpha g_1+(1-\alpha)g_2 \in T_G(x)$. Hence $T_G(x)$ is convex.

(iii). To prove that $T_G(x)$ is bounded, it is enough to prove for arbitrary $g_0, \tilde{g}_0 \in T_G(x)$ that $\| g_0-\tilde{g}_0, k \|< c$ for some $c>0$, since $\| g_0-\tilde{g}_0, k \|< c$ implies that $\sup_{g_0, \tilde{g}_0 \in T_G(x)} \| g_0, \tilde{g}_0, k \|$ is finite and hence the diameter of $T_G(x)$ is finite.

Let $g_0, \tilde{g}_0 \in T_G(x)$. Then there exists a constant $t>0$ such that for every $g \in G$ and $k \in X \setminus [G, x]$,

$\| g-g_0, k \| \leq \| x-g, k \| -t \| x-g_0, k \|$

and

$\| g-\tilde{g}_0, k \| \leq \| x-g, k \| -t \| x-\tilde{g}_0, k \|$.

Now,

$$\begin{aligned}
\| x-g_0, k \| &\leq \| x-g, k \| + \| g-g_0, k \| \\
&\leq 2 \| x-g, k \| -t \| x-g_0, k \| .
\end{aligned}$$

Thus $\| x-g_0, k \| \leq \frac{2}{1+t} \| x-g, k \|$, for every $g \in G$.

Hence $\| x-g_0, k \| \leq \frac{2}{1+t}d$, where $d=\inf_{g \in G} \| x-g, k \|$.

Similarly, $\| x-\tilde{g}_0, k \| \leq \dfrac{2}{1+t}d.$

Therefore, it follows that

$$\begin{aligned}
\| g_0-\tilde{g}_0, k \| &\leq \| g_0-x, k \| + \| x-\tilde{g}_0, k \| \\
&\leq \frac{4}{1+t}d \\
&= C.
\end{aligned}$$

Whence $T_G(x)$ is bounded.

Let X be a linear2-normed space, $x \in X$ and $[x]$ denote the set of all scalar multiplications of x

$$\text{i.e.,} \quad [x] = \{\alpha x : \alpha \in \mathbb{R}\}\,.$$

Theorem 3.2. Let G be a subset of a linear 2-normed space $X, x \in X$ and $k \in X \setminus [G, x]$. Then the following statements are equivalent for every $y \in [k]$.

(i) $g_0 \in T_G(x)$.

(ii) $g_0 \in T_G(x + y)$.

(iii) $g_0 \in T_G(x - y)$.

(iv) $g_0 + y \in T_G(x + y)$.

(v) $g_0 + y \in T_G(x - y)$.

(vi) $g_0 - y \in T_G(x + y)$.

(vii) $g_0 - y \in T_G(x - y)$.

(viii) $g_0 + y \in T_G(x)$.

(ix) $g_0 - y \in T_G(x)$.

Proof. The proof follows immediately by using Theorem 2.2.

Theorem 3.3. Let G be a subspace of a linear 2-normed space X, $x \in X$ and $k \in X \setminus [G, x]$. Then $g_0 \in T_G(x) \Leftrightarrow g_0 \in T_G(\alpha^m x + (1 - \alpha^m) g_0)$, for all $\alpha \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$.

Proof. Claim: $g_0 \in T_G(x) \Leftrightarrow g_0 \in T_G(\alpha x + (1-\alpha) g_0)$, for every $\alpha \in \mathbb{R}$. Let $g_0 \in T_G(x)$. Then

$$\| g - g_0, k \| \leq \| x - g, k \| - t \| x - g_0, k \| \text{ , for all } g \in G \text{ and for some } t > 0\,.$$

$$\begin{aligned}
&\Rightarrow \quad \| \alpha g - \alpha g_0, k \| \leq \| \alpha x - \alpha g, k \| - t \| \alpha x - \alpha g_0, k \|, \text{ for all } g \in G. \\
&\Rightarrow \quad \| \alpha \left(\frac{(\alpha - 1) g_0 + g}{\alpha} \right) - \alpha g_0, k \| \leq \| \alpha x - \alpha \left(\frac{(\alpha - 1) g_0 + g}{\alpha} \right), k \| \\
&\qquad - t \| \alpha x - \alpha g_0, k \|, \text{ for all } g \in G \text{ and } \alpha \neq 0, \text{ since } \frac{(\alpha - 1) g_0 + g}{\alpha} \in G. \\
&\Rightarrow \quad \| g - g_0, k \| \leq \| \alpha x + (1 - \alpha) g_0 - g, k \| - t \| \alpha x + (1 - \alpha) g_0 - g_0, k \| \\
&\Rightarrow \quad g_0 \in T_G(\alpha x + (1 - \alpha) g_0, \text{ when } \alpha \neq 0.
\end{aligned}$$

If $\alpha = 0$, then it is clear that $g_0 \in T_G(\alpha x + (1 - \alpha) g_0)$.

The converse is obvious by taking $\alpha = 1$. Hence the claim is true.
By repeated application of the claim the result follows.

Corollary 3.4. Let G be a subspace of a linear 2-normed space X, $x \in X$ and $k \in X \setminus [G, x]$. Then the following statements are equivalent for every $y \in [k], \alpha \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$

(i) $g_0 \in T_G(x)$.

(ii) $g_0 \in T_G(\alpha^m x + (1 - \alpha^m)g_0 + y)$.

(iii) $g_0 \in T_G(\alpha^m x + (1 - \alpha^m)g_0 - y)$.

(iv) $g_0 + y \in T_G(\alpha^m x + (1 - \alpha^m)g_0 + y)$.

(v) $g_0 + y \in T_G(\alpha^m x + (1 - \alpha^m)g_0 - y)$.

(vi) $g_0 - y \in T_G(\alpha^m x + (1 - \alpha^m)g_0 + y)$.

(vii) $g_0 - y \in T_G(\alpha^m x + (1 - \alpha^m)g_0 - y)$.

(viii) $g_0 + y \in T_G(\alpha^m x + (1 - \alpha^m)g_0)$.

(ix) $g_0 - y \in T_G(\alpha^m x + (1 - \alpha^m)g_0)$.

Proof. The proof follows from simple application of Theorem 2.2 and the Theorem 3.3.

Theorem 3.5. Let G be a subset of a linear 2-normed space X, $x \in X$ and $k \in X \setminus [G, x]$. Then $g_0 \in T_G(x) \Leftrightarrow g_0 \in T_{G+[k]}(x)$.

Proof. The proof follows from simple application of Theorem 3.2.

A corollary similar to that of Corollary 3.4 is established next as follows:

Corollary 3.6. Let G be a subspace of a linear 2-normed space X, $x \in X$ and $k \in X \setminus [G, x]$. Then the following statements are equivalent for every $y \in [k], \alpha \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$

(i) $g_0 \in T_{G+[k]}(x)$.

(ii) $g_0 \in T_{G+[k]}(\alpha^m x + (1 - \alpha^m)g_0 + y)$.

(iii) $g_0 \in T_{G+[k]}(\alpha^m x + (1 - \alpha^m)g_0 - y)$.

(iv) $g_0 + y \in T_{G+[k]}(\alpha^m x + (1 - \alpha^m)g_0 + y)$.

(v) $g_0 + y \in T_{G+[k]}(\alpha^m x + (1-\alpha^m)g_0 - y)$.

(vi) $g_0 - y \in T_{G+[k]}(\alpha^m x + (1-\alpha^m)g_0 + y)$.

(vii) $g_0 - y \in T_{G+[k]}(\alpha^m x + (1-\alpha^m)g_0 - y)$.

(viii) $g_0 + y \in T_{G+[k]}(\alpha^m x + (1-\alpha^m)g_0)$.

(ix) $g_0 - y \in T_{G+[k]}(\alpha^m x + (1-\alpha^m)g_0)$.

Proof. The proof easily follows from Theorem 3.5 and Corollary 3.4.

Proposition 3.7. Let G be a subset of a linear 2-normed space X, $x \in X$, $k \in X \setminus [G, x]$ and $0 \in G$. If $g_0 \in T_G(x)$, then there exists a constant $t > 0$ such that $\| g_0, k \| \leq \| x, k \| - t \| x - g_0, k \|$.

Proof. The proof is obvious.

Proposition 3.8. Let G be a subset of a linear 2-normed space X, $x \in X$ and $k \in X \setminus [G, x]$. If $g_0 \in T_G(x)$, then there exists a constant $t > 0$ such that for all $g \in G$,

$$||x - g_0, k|| \leq 2||x - g, k|| - t||x - g_0, k||.$$

Proof. The proof is trivial.

Theorem 3.9. Let G be a subspace of a linear 2-normed space X and $x \in X$. Then the following statements hold.

(i) $T_G(x + g) = T_G(x) + g$, for every $g \in G$.

(ii) $T_G(\alpha x) = \alpha T_G(x)$, for every $\alpha \in \mathbb{R}$.

Proof. (i). Let $\tilde{g}$ be an arbitrary but fixed element of G .
Let $g_0 \in T_G(x)$. It is clear that $g_0 + \tilde{g} \in T_G(x) + \tilde{g}$.

To prove that $T_G(x) + \tilde{g} \subseteq T_G(x + \tilde{g})$, it is enough to prove that $g_0 + \tilde{g} \in T_G(x + \tilde{g})$.

Now,

$\| g + \tilde{g} - g_0 - \tilde{g}, k \| \leq \| x - g, k \| - t \| x - g_0, k \|$, for all $g \in G$
and for some $t > 0$.

$\Rightarrow \| g + \tilde{g} - (g_0 + \tilde{g}), k \| \leq \| x + \tilde{g} - (g + \tilde{g}), k \| - t \| x + \tilde{g} - (g_0 + \tilde{g}), k \|$,
for all $g \in G$.

$\Rightarrow \; g_0 + \tilde{g} \in T_G(x + \tilde{g}),$ since $g - \tilde{g} \in G.$

Conversely, let $g_0 + \tilde{g} \in T_G(x + \tilde{g})$.

To prove that $T_G(x + \tilde{g}) \subseteq T_G(x) + \tilde{g}$, it is enough to prove that $g_0 \in T_G(x)$.

Now,

$$\begin{aligned} \| \, g - g_0, k \, \| &= \| \, g + \tilde{g} - (g_0 + \tilde{g}), k \, \| \\ &\leq \| \, x + \tilde{g} - (g + \tilde{g}), k \, \| - t \, \| \, x + \tilde{g} - (g_0 + \tilde{g}), k \, \|, \\ &\qquad \text{for all } g \in G \text{ and for some } t > 0. \\ \Rightarrow \quad & g_0 \in T_G(x). \text{ Thus the result follows.} \end{aligned}$$

(ii). The proof is similar to that of (i).

Remark 3.10. Theorem 3.9 can be restated as

$$T_G(\alpha x + g) = \alpha T_G(x) + g, \text{ for every } g \in G.$$

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