Strongly C*G- Continuous Maps In Topological Space

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Abstract

In this paper, we have introduced the concept of strongly c*g-continuous, perfectly c*g -continuous, c*g – locally closed, c*g –locally continuous in Topological space.

Key words: Strongly c*g-continuous, perfectly c*g -continuous, c*g-locally closed, c*g – locally continuous.
1. INTRODUCTION
Levine [3] introduced and investigated the concept of strong continuity in topological spaces. Sundaram [12] introduced strongly g – continuous maps and perfectly g – continuous maps in topological spaces. Pushpalatha [8] introduced strongly g*-continuous and perfectly g*-continuous maps in topological spaces. In this section we have introduced two strong forms of continuous maps in topological spaces, namely strongly c*g- continuous maps, perfectly c*g- continuous maps and study some of their properties.

Pushpalatha [8] introduced strongly generalized locally continuous functions & some of their properties in topological spaces. In the chapter, we have introduced the concept of c*g- locally continuous functions and study some of their properties.

2. PRELIMINARIES

DEFINITION: 2.1
A map f: X→Y from a topological space X into a topological space Y is called
i) Strongly continuous if f⁻¹(V) is both open and closed in X for each subset V in Y [3].
ii) Perfectly continuous if f⁻¹(V) is both open and closed in X for each open subset V in Y [10].
iii) generalized continuous(g-continuous) if f⁻¹(V) is g-open in X for each open set V in Y [12].
iv) Strongly g- continuous if f⁻¹(V) is both open in X for each g-open set V in Y [12].
v) Perfectly g- continuous if f⁻¹(V) is both open and closed in X for each g-open set V in Y [12].
iv) Strongly g*- continuous if \( f^{-1}(V) \) is both open in X for each g*-open set V in Y [8].

v) Perfectly g*- continuous if \( f^{-1}(V) \) is both open and closed in X for each g*-open set V in Y [8].

3. STRONGLY c*g- CONTINUOUS MAPS IN TOPOLOGICAL SPACE

**Definition: 3.1**

A map \( f : X \rightarrow Y \) from a topological space X into a topological space Y is said to be strongly c*g- continuous if the inverse image of every c*g- open set in Y is open in X.

**Theorem 3.2**

If a map \( f : X \rightarrow Y \) from a topological space X into a topological space Y in strongly c*g- continuous, then it is continuous but not conversely.

**Proof:** Assume that \( f \) is strongly c*g- continuous. Let G be any c*g open set in Y. Since \( f \) is strongly continuous, \( f^{-1}(G) \) is open in X by the definition of strongly continuous. Therefore \( f \) is strongly c*g- continuous.

The converse need not be true as seen from the following example.

**Example 3.3:** Let \( X = Y = \{a, b, c\} \) with the topologies \( \tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \) & \( \tau_2 = \{\emptyset, Y, \{a, b\}\} \). Define a map \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) be the identity. Then \( f \) is continuous. But \( f \) is not strongly c*g continuous since, for the c*g open set \( G = \{b\} \) in Y, \( f^{-1}(G) = \{G\} \) is not open in X.

**Theorem 3.4:** If \( f : X \rightarrow Y \) from a topological space X into a topological space Y is strongly continuous then it is strongly c*g- continuous but not conversely.

**Proof:** Assume that \( f \) is strongly continuous. Let G be any c*g open set in Y. Since \( f \) is strongly continuous, \( f^{-1}(G) \) open in X by the definition of strongly continuous. Therefore \( f \) is strongly c*g- continuous.

The converse need not be true as seen from the following example.

**Example 3.5:** Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, x, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{\emptyset, y, \{a\}\} \). Consider a map \( f : (x, \tau) \rightarrow (y, \sigma) \) is defined by \( f(a) = f(c) = c \) & \( f(b) = b \). Then \( f \) is strongly c*g- continuous. But not strongly continuous. For the subset \( \{a\} \) of Y \( f^{-1}(\{a\}) = \{a\} \) is open in X, but is not closed in X.
Theorem 3.6: If \( f: X \to Y \) is strongly \( c^*g \)-continuous, then it is strongly \( g^* \)-continuous but not conversely.

**Proof:** Assume that \( f \) is strongly \( c^*g \)-continuous. Let \( G \) be any strongly \( g \)-open set in \( Y \). Since every strongly \( g \)-open set is \( c^*g \)-open, \( G \) is \( c^*g \)-open in \( Y \). Since \( f \) is strongly \( c^*g \)-continuous, \( f^{-1}(G) \) is open in \( X \). Therefore \( f \) is strongly \( g^* \)-continuous.

The converse need not be true as seen from the following example.

**Example 3.7:** Let \( X = Y = \{a, b, c\} \) be topological spaces with the topologies \( \tau = \{\varnothing, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}\} \) and \( \sigma = \{\varnothing, y, \{a, c\}\} \). Let \( f: (X, \tau) \to (y, \sigma) \) be the identity map. Then \( f \) is strongly \( g^* \)-continuous, but not strongly \( c^*g \)-continuous. For, \( \{b\} \) is a \( c^*g \)-open in \( Y \), but \( f(\{b\}) = \{b\} \) is not open in \( X \).

**Theorem 3.8:** A map \( f: (X \to Y) \) from a topological spaces \( X \) into a topological space \( Y \) is strongly \( c^*g \)-continuous if and only if the inverse image of every \( c^*g \)-closed set in \( Y \) is closed in \( X \).

**Proof:** Assume that \( f \) is strongly \( c^*g \)-continuous. Let \( G \) be any \( c^*g \)-closed set in \( Y \). Then \( G^c \) is \( c^*g \)-open in \( Y \). Since \( f \) is strongly \( c^*g \)-continuous, \( f^{-1}(G^c) \) open in \( X \). But \( f^{-1}(G^c) = X - f^{-1}(G) \) and so \( f^{-1}(G) \) is closed in \( X \).

Conversely assume that the inverse image of every \( c^*g \)-closed set in \( Y \) is closed in \( X \). Let \( G \) be any \( c^*g \)-open set in \( Y \). Then \( G^c \) is \( c^*g \)-closed in \( Y \). By assumption, \( f^{-1}(G^c) \) is closed in \( X \). But \( f^{-1}(G^c) = X - f^{-1}(G) \) and so \( f^{-1}(G) \) is open in \( X \). Therefore \( f \) is strongly \( c^*g \)-continuous.

**Remarks 3.9:** From the above observation we get the following diagram.

\[
\begin{array}{ccc}
\text{Strongly continuity} & \downarrow & \text{Strongly } c^*g \text{ continuous} \\
& \downarrow & \\
& \text{Continuity.} & \\
\end{array}
\]

In the above diagram none of the implications can be reversed.

**Theorem 3.10:** If a map \( f: X \to Y \) is strongly \( c^*g \)-continuous and a map \( g: Y \to Z \) is \( c^*g \)-continuous, then the composition \( g \circ f: X \to Z \) is continuous.

**Proof:** Let \( G \) be any open set in \( Z \). Since \( g \) is \( c^*g \)-continuous, \( g^{-1}(G) \) is \( c^*g \)-open in \( Y \). Since \( f \) is strongly \( c^*g \)-continuous, \( f^{-1}(G) \) is open in \( X \). But \( f^{-1}(G^c) = X - f^{-1}(G) \) and so \( f^{-1}(G) \) is closed in \( X \).
\[ (g^{-1}(G)) \] is open in \( X \). But \( (g \circ f)^{-1}(G) = f^{-1}[g^{-1}(G)] \). Therefore \( g \circ f \) is continuous.

**Definition 3.10** A map \( f : X \rightarrow Y \) is said to be perfectly \( c^g \)-continuous if the inverse image of every \( c^g \)-open set in \( Y \) is both open and closed in \( X \).

**Theorem 3.11:** A map \( f : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) is perfectly \( c^g \)-continuous, then it is strongly \( c^g \)-continuous but not conversely.

**Proof:** Assume that \( f \) is perfectly \( c^g \)-continuous. Let \( G \) be any \( c^g \)-open set in \( Y \). Since \( f \) is perfectly \( c^g \)-continuous, \( f^{-1}(G) \) is open in \( X \). Therefore \( f \) is strongly \( c^g \)-continuous.

The converse need not be true as seen from the following example.

**Example 3.12:** Let \( X = Y = \{a, b, c\} \), with topologies \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}\} \). Define a map \( f : (X, \tau) \rightarrow (Y, \sigma) \) as the identity function. Then \( f \) is perfectly \( g^* \)-continuous, but not perfectly \( c^g \)-continuous, since for the \( c^g \) open set \( \{b\} \) in \( Y \), \( f^{-1}(\{b\}) = \{b\} \) is not both open and closed in \( X \).

**Theorem 3.13:** If a map \( f : X \rightarrow Y \) is perfectly \( c^g \)-continuous then it is perfectly \( g^* \)-continuous but not conversely.

**Proof:** Assume that \( f \) is perfectly \( c^g \)-continuous. Let \( G \) be a \( c^g \)-open set in \( Y \). Then \( G \) is \( c^g \)-open in \( Y \). Since \( f \) is perfectly \( c^g \)-continuous, \( f^{-1}(G) \) is both open and closed in \( X \). Therefore \( f \) is perfectly \( g^* \)-continuous.

The converse need not be true as seen from the following example.

**Example 3.14:** Let \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}\} \). Define a map \( f : (X, \tau) \rightarrow (Y, \sigma) \) as the identity function. Then \( f \) is perfectly \( g^* \)-continuous, but not perfectly \( c^g \)-continuous, since for the \( c^g \) open set \( \{b\} \) in \( Y \), \( f^{-1}(\{b\}) = \{b\} \) is not both open and closed in \( X \).

**Theorem 3.15:** If a map \( f : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) is perfectly \( c^g \)-continuous if and only if \( f^{-1}(G) \) is both open and closed set in \( X \) for every \( c^g \)-closed set \( G \) in \( Y \).

**Proof:** Assume that \( f \) is perfectly \( c^g \)-continuous. Let \( F \) be any \( c^g \)-closed set in \( Y \). Then \( F^c \) is \( c^g \)-open set in \( Y \). Since \( f \) is perfectly \( c^g \)-continuous, \( f^{-1}(F^c) \) is...
both open & closed in $X$. But $f^{-1}(F^c) = X - f^{-1}(F)$ and also $f^{-1}(F)$ is both open and closed in $X$.

Conversely assume that the inverse image of every $c^g$-closed set in $Y$ is both open and closed in $X$. Let $G$ be any $c^g$-open set in $Y$. Then $G^c$ is $c^g$-closed in $Y$. By assumption $f^{-1}(G^c)$ is both open and closed in $Y$. But $f^{-1}(G^c) = X - f^{-1}(G)$ and so $f^{-1}(G)$ is both open and closed in $Y$. Therefore $f$ is perfectly $c^g$-continuous.

**Remark 3.16:** From the above observations we have the following implications and none of them are reversible.

Perfectly $c^g$-continuity
↓
Strongly $c^g$-continuity
↓
Strongly $g^*$-continuous
↓
Continuity.

### 4. $c^g$ – locally continuous function in topological spaces

**Definition 4.1:**
A subset $S$ of $X$ is called $c^g$-locally closed set if $S = A \cap B$, where $A$ is $c^g$-open in $X$ and $B$ is $c^g$-closed in $X$. $C^gGLC(X)$ denotes the class of all $c^g$-sets in $X$.

**Theorem 4.2:** If a subset $S$ of $X$ is locally closed then it is $c^g$-locally closed but not conversely.

**Proof:** Let $S = P \cap Q$, where $P$ is open in $X$ and $Q$ is closed in $X$. Since every open set is $c^g$-open and every closed, $S$ is $c^g$-locally closed in $X$.

The converse need not be true as seen from the following examples.

**Example 4.3:** Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{ \emptyset, X, \{a\}\}$. Then the set $\{a, c\}$ is $c^g$-locally closed but is not locally closed.

**Theorem 4.4:** If a subset $S$ of $X$ is strongly generalized locally closed in $X$ then $S$ is $c^g$-locally closed but not conversely.

**Proof:** Let $S = P \cap Q$, where $P$ is strongly $g$-open and $Q$ is strongly $g$-closed in $X$. Since strongly $g$-open implies $c^g$-open and strongly $g$-closed implies $c^g$-closed, $S$ is $c^g$-locally closed in $X$. 

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Example 4.4: Consider the topological space $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{b\}\}$. Then the set \{a,b\} c*g - locally closed but is not strongly generalized locally closed.

**Theorem 4.5:** If a subset $S$ of $X$ is c*g-locally closed in $X$, then $S$ is regular generalized locally closed but not conversely.

**Proof:** Let $S = P \cap Q$, Where $P$ is c*g-locally closed and $Q$ is c*g-locally closed in $X$. Since c*g-locally closed implies rg-closed and c*g-locally open implies rg-open. Therefore $S$ is regular generalized locally closed.

Example 4.6: Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then \{d\} is rg-locally closed but is not c*g-locally closed set in $X$.

**Theorem 4.7:** If $A$ is c*g-locally closed in $X$ and $B$ is c*g-open (respectively closed) in $X$, then $A \cap B$ is c*g-locally closed in $X$.

**Proof:** There exist a c*g-open set $P$ and a c*g-closed set $Q$ such that $A = P \cap Q$. Now, $A \cap B = (P \cap Q) \cap B = (P \cap B) \cap Q$. Since $P \cap Q$ is c*g-open and $Q$ is closed, $A \cap B$ is c*g-locally closed.

**Definition 4.8:** A subset $S$ of a topological space $X$ is called c*gclc*- set if $S = P \cap Q$ where $P$ is c*g-open in $X$ and $Q$ is closed in $X$.

**Definition 4.9:** A subset $S$ of a topological space $X$ is called c*gclc**- set if $S = P \cap Q$ where $P$ is open in $X$ and $Q$ is c*g-closed in $X$.

**Theorem 4.10:**

i) If $A$ is c*gclc* --set in $X$ and $B$ is c*g-open (or closed), then $A \cap B$ is c*gclc* - set in $X$.

ii) If $A$ is c*gclc**- set in $X$ and $B$ is closed then $A \cap B$ is c*gclc**.

**Proof:**

i) Since $A$ is c*gclc*- set, there exist a c*g-open set $P$ and a closed set $Q$. Such that $A = P \cap Q$. Now $A \cap B = (P \cap Q) \cap B = (P \cap B) \cap Q$. Since $P \cap Q$ is c*g-open and $Q$ is closed, $A \cap B$ is c*gclc*- set. In the case of $B$ being a closed set, we have $A \cap B = (P \cap Q) \cap B = P \cap (Q \cap B)$. Since $P$ is c*g-open and $Q \cap B$ is closed, $A \cap B$ is c*gclc*- set.

ii) Since $A$ is c*gclc**, there exist an open set $P$ and a c*g-closed set $Q$ such that $A = (P \cap Q)$. Now $A \cap B = (P \cap Q) \cap B = P \cap (Q \cap B)$. Since $Q$ is c*g-closed and $B$ is closed, $Q \cap B$ is c*g-closed. Therefore, $A \cap B$ is c*gclc**- set.
Theorem 4.11: A subset A of a topological space X is c*glc*-set if and only if there exists a c*g-open set P such that A = P ∩ cl(A).

Proof : Assume that A is c*glc*-set. There exists a c*g-open set P and a closed set Q such that A = P ∩ Q. Since A ⊆ Q and Q is closed, A ⊆ cl(A) ⊆ Q. Then A ⊆ P and A ⊆ cl(A), and hence A ⊆ P ∩ cl(A). To prove the reverse inclusion let X ∈ P ∩ cl(A). Then X ∈ P and X ∈ cl(A) ∩ Q and so X ∈ P ∩ Q = A. Hence P ∩ cl(A) ⊆ A. Therefore A = P ∩ cl(A).

Conversely assume that there exist a c*g-open set P such that A = P ∩ cl(A). Now P is c*g-open set and cl(A) is closed. Therefore A is c*glc*-set.

Theorem 4.12: If a subset A of a topological space X is c*glc** - set then there exists an open set P such that A = P ∩ cl*(A), where cl*(A) is the closure of A as defined by Dunham [19].

Proof : By definition there exist an open set P and a c*g-closed set Q such that A = P ∩ Q. Then, since A ⊆ cl*(A) ⊆ Q, We have A ⊆ P ∩ cl*(A). Conversely, if X ∈ P ∩ cl*(A), then X ∈ Q and X ∈ P. Then, X ∈ Q ∩ P = A and hence P ∩ cl*(A) ⊆ A. Therefore A = P ∩ cl*(A).

Theorem 4.13: If A and B are c*glc*- set in a topological space X then A ∩ B is c*glc*-set in X.

Proof : From the assumptions there exist c*g-open sets P and Q such that A = P ∩ cl(A) and B = Q ∩ cl(B). Then A ∩ B = (P ∩ Q) ∩ cl(A) ∩ cl(B). Since P ∩ Q is c*g-open and cl(A) ∩ cl(B) is closed, A ∩ B is c*glc* - set.

5. c*g - locally closed continuous functions

Notations: - LC(X) denotes the class of all locally closed sets in a topological space X and C*GLC(X) denotes the class of all c*glc- sets in X.

Similarly, C*GLC**(X) [respectively C*GLC***(X)] denotes the class of all c*glc*-sets [respectively c*glc** - sets ]

Ganster and Reilly [2] have proved that

Continuity

↓

LC- irresolute

↓

LC- continuity

Pushpalatha [17] has proved that

LC – continuity

↓

S*GLC- irresoluteness

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S*GLC- continuity

GLC – continuous.

But none of these implications can be reversed. Also they observed that the composition of two S*GLC- irresolute functions is S*GLC- irresolute and the composition of a S*GLC – continuous function is S*GLC- continuous.

Definition 5.1.

A function f : X→Y from a space X into a space Y is called
(i) LC- irresolute [2] if f⁻¹(V) ∈ LC(X) for each V in LC(Y).
(ii) S*GLC- irresolute [17] if f⁻¹(V) ∈ S*GLC(X) for each V ∈ S*GLC(X).
(iii) LC-continuous [2] if f⁻¹(V) ∈ LC(X) for each open set V in Y.
(iv) S*GLC - continuous [17] if f⁻¹(V) ∈ LC(X) for each open set V in Y.

Definition 5.2.

A function f : X→Y from a space X into a space Y is called
i) C*GLC-irresolute if f⁻¹(V) ∈ C*GLC(X) for each V ∈ C*GLC(X).
ii) C*GLC-continuous if f⁻¹(V) ∈ C*GLC(X) for each open set V in Y.
iii) C*GLC* – irresolute (respectively C*GLC**- irresolute) if f⁻¹(V) ∈ C*GLC*(X) (respectively f⁻¹(V) ∈ C*GLC**(X)) for each V ∈ C*GLC*(Y) (respectively V ∈ C*GLC**(X)).
iv) C*GLC* - continuous (respectively C*GLC**- continuous) if f⁻¹(V) ∈ C*GLC*(X) (respectively f⁻¹ (V) ∈ C*GLC**(X) ) for each open set V in Y.

Theorem 5.3: If a function f : X→Y from a space X into a space Y is LC- continuous then it is C*GLC- continuous but not conversely.

Proof: - Assume that f is LC -continuous. Let V be an open set in Y. Then f⁻¹(V) is locally closed in Y. But locally closed sets are c*g- locally closed sets. Therefore f⁻¹(V) ∈ C*GLC(X) and so f is C*GLC-continuous. The converse need not be true as seen from the following example.

Example 5.4: Let X = Y = {a,b,c}, τ = { φ,x, {a} } and σ be the discrete topology. Define f : (X,τ) →(Y,σ) as the identity function .Then f is not LC-continuous. Because {b} is open in Y but f⁻¹( {b} ) = {b} is not locally closed in y, clearly f is C*GLC-continuous.

Theorem 5.5: If function f: X→Y from a space X into a space Y is C*GLC- irresolute then it is C*GLC- continuous.
Proof: - Let V be open in Y. Since every open set is c*g-open set and every c*g-set open set is c*g-locally closed, V ∈ C*GLC(Y). Since f is C*GLC-irresolute, f^{-1}(V) ∈ C*GLC(X). Therefore f is C*GLC-continuous. Thus we have the following implications

i) Continuity  
   \[ \downarrow \]  
   LC-irresolute  
   \[ \downarrow \]  
   LC-continuity  
   \[ \downarrow \]  
   C*GLC-continuous

ii) C*GLC-irresoluteness  
   \[ \downarrow \]  
   C*GLC-continuity

However none of the above implications can be reversed.

Theorem 5.6: If function f: X→Y from a space X into a space Y be C*GLC-continuous and A be a c*g-open subset of X (respectively closed). Then the restriction f/A: A→Y is C*GLC-continuous.

Proof: - Let V be open in Y.Let f^{-1}(V) = W. Then W is c*glc in X.Since f is C*GLC-continuous .Let W= P∩Q where P is c*g-open in X and Q is c*g-closed in X. Now (f/A)^{-1}(V) = W∩A = (P∩Q) ∩A = (P∩A)∩Q.

But P∩A [respectively A∩G] is c*g-closed by [18] is c*g-open in X and so the restriction f/A is C*GLC-continuous.

Theorem 5.7: (i) Let f : X→Y be C*GLC-continuous and B be an open subset of Y containing f(X). Then f : X→B is C*GLC-continuous.
(ii) If f : X→Y and g : Y→Z are both C*GLC-irresolute then the composition g◦f: X→Z is C*GLC-irresolute.
(iii) If f : X→Y is C*GLC-continuous and g : Y→Z is continuous then the composition g◦f : X→Z is C*GLC-continuous.

Proof: - (i) Let V be open in B.Since B is open in Y, the set V is open in Y .Therefore f^{-1}(V) is c*glc in X. Hence f : X→B is C*GLC-continuous.
(ii) Let V be c*glc-set in Z. Since g is C*GLC-irresolute, g^{-1}(V) is c*glc in Y. Since f is C*GLC-irresolute, f^{-1}(g^{-1}(V)) is c*glc in X .But f^{-1}(g^{-1}(V)) = (g◦f)^{-1}(V) and so g◦f is C*GLC-irresolute.
(iii) Let V be open in Z. Since g is continuous g^{-1}(V) is open in Y. Since f is C*GLC-continuous, f^{-1}(g^{-1}(V)) is c*glc in X. But f^{-1}(g^{-1}(V)) = (g◦f)^{-1}(V) and so g◦f is C*GLC-continuous.
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