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State Variable Analysis of Continuous Time Systems

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Abstract: **The concept of state relates to those physical objects whose behavior changes in time and when given an excitation or stimulus, a certain change or response can be observed. To predict the future behavior of the object under any excitation or input, a series of experiments may be done giving inputs , and observing the response or outputs. This input –output relation is put in an orderedmanner for all time t≥ t_0, where t_0 is the initial starting time, when the first input is given.**

Keywords: Concept of State and state variables, state variables model for a continuous system, solution of state equations- existence of the solution, methods of evaluation of state transition matrix.

I. INTRODUCTION

The modem trend in systems engineering is towards a greater complexity, mainly because of requirements of good accuracy in difficult and complex tasks. These systems have more than one (i.e.) multi inputs and multiple outputs and these inputs and outputs are usually time varying. The requirements on the systems are also stringent. Conventional analysis is too time consuming and some times very difficult, if not impossible. Due to the development large size computers, since 1960, a new approach to the analysis, design a control of complex systems is developed based on new concept "the state". The concept of state by itself is not new and is in existence since long in classical dynamics and other fields and is now extended to systems engineering.

A physical object or system is one which can be perceived by our senses. Its behaviour in time is abstraction of the mathematical relationships that give some expression. Here we say "some expression" because while doing the abstraction for the mathematical relations, it is possible that some relations may be lost or true behaviour may not come. Also it is not possible to realise all mathematical relations physically and vice versa.

II.CONCEPT OF STATE AND STATE VARIABLES

1. State of Physical Object

This is any property of the object which relates input to output such that knowledge of the input time function for t≥ t_0 and at time t=t_0 completely and uniquely determines the output for t≥ t_0.

Example -(1):

A simple electric circuit is shown in Fig(1). Switch S is at `A` initially and closed at t=t_0 to position `B`.

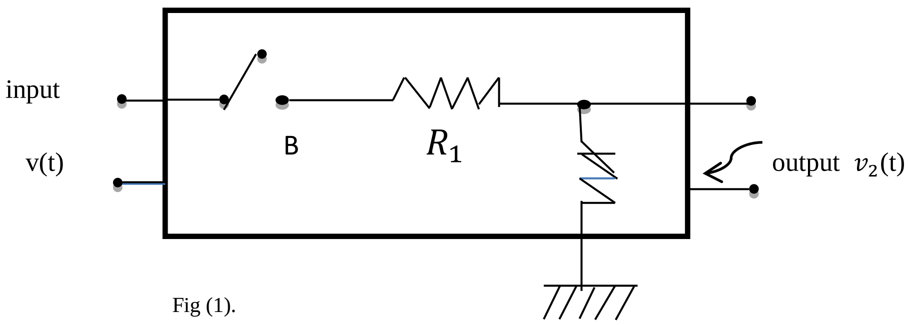

Fig (1).

2. *An Abstract Object*

An abstract object is the totality of input-output pairs that describe the output y(t) for all t≥ t_0.

In definition (1) , Define the state of physical object. This more or less refers to a mathematical model and may or may not represent a physical state. The example given below will give the difference between the two definitions cited above.

Example-(2):

Consider the R-C circuit shown in Fig(2). The physical object is the resistor-capacitor (R-C) network shown in Fig(2).

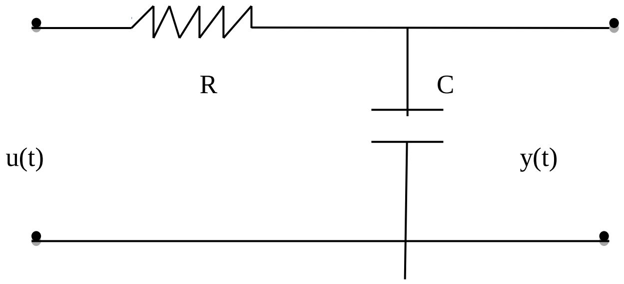

Fig (2):R-C circuit

The perform an experiment by giving different input u(t) measure the output y(t) (both being voltages in this case). A different experiment may give input y(t) and measure u(t) so that the choice of inputs and outputs are determined depending on the experiment to be conducted.

The input-output pairs in this example satisfy the mathematical relationship.

$$RC\frac{dy(t)}{dt} + y = u(t) \longrightarrow (1)$$

Equation (1) summarises the abstract object. The solution for this equation is

$$Y(t)= y(t_0)e^{(t_0-t)/RC} + \frac{1}{RC}\int_{t_0}^{t} e^{(\tau-t)/RC} (\tau)\, d\tau \longrightarrow (2)$$

3. *The state an Abstract Object*

It is the collection of numbers which together with the input u(t) for all t≥ t_0 uniquely determines the output y(t) for all t≥ t_0.

A state can be a set of any finite numbers or infinite numbers. However, in most cases, the state is a set of `n` finite numbers and hence. X(t) is a n (valued) vector function of time.

4. *State Variable*

A variable denoted by X(t) is the time function whose value at any time is the state of the abstract object at that time.

5. *The state- Space Denoted by `∑`is a set of all X(t)*

In Example (1) considered, state variable remains either state `A` or `B` whereas in example (2).

$$X(t) = y(t)$$

Note that the state representation is not unique . There can be many different ways of expressing the relationships between the inputs and outputs. In example (2) y(t) is the output across the capacitor C. Instead the current through the capacitor can be taken as y(t) and the relations can be rewritten.

6. *Dynamic System*

Dynamic system is a oriented mathematical relationship in which

(a) For a given real input u(t) for all `t`, there exists a real output y(t) for t≥ t_0 , and

(b) The output y(t) does depend on the inputs u(t) for t≥ t_0.

This can be understood by referring to the Example (2) the output y(t), the voltage across capacitor is uniquely determined once y(t_0) is given and it does not matter how it was determined in the past. All that is needed is the unique future output (the state) for the future input.

7. LINEARITY

A system is said to be linear if the following conditions are satisfied.

Given any two numbers `a` and `b` , two states $X_1(t_0), X_2(t_0)$, two inputs $u_1(t) and\ u_2$(t) and their corresponding outputs y_1(t) and y_2(t):

(1) The state $X_3(t_0) = aX_1(t_0) + bX_2(t_0)$, the output $y_3(t) = ay_1(t) + by_2$(t) for the input $u_3(t)$ = $au_1(t) + bu_2$(t) can appear in the oriented abstract object.

(2) Both $y_3(t)$ and $X_3(t)$ correspond to the state $X_3(t_0)$ *and input* u_3(t).

Consider Example (1) for same `a` and `b` there is no state corresponding to aA + bB. (A, B are switch positions).

The system does not satisfy the definition given above and hence is not linear. Consider Example (2), the R-C network.

$$y_1(t) = X_1(t) = X_1(t_0)e^{-(t_0-t)/RC} + \frac{1}{RC}\int_{t_0}^{t} e^{(\tau-t)/RC}\, u_1(\tau)d\tau$$

And $$y_2(t) = X_2(t) = X_2(t_0)e^{-(t_0-t)/RC} + \frac{1}{RC}\int_{t_0}^{t} e^{(\tau-t)/RC}\, u_2(\tau)d\tau$$

has two outputs for states $X_1(t_0)$ *and* $X_2(t_0)$ and two inputs $u_1(t)$ *and* $u_2(t)$.

Since the give any finite voltage.

$X_3(t) = aX_1(t) + bX_2(t)$ and

$u_3(t) = au_1(t) + bu_2(t)$

Let us find the output y_3(t) for this condition

$$y_3(t) = X_3(t_0)e^{-(t_0-t)/RC} + \frac{1}{RC}\int_{t_0}^{t} e^{(\tau-t)/RC}\, u_3(\tau)d\tau$$

$$= [aX_1(t_0) + bX_2(t_0)e^{-(t_0-t)/RC}] + \frac{1}{RC}\int_{t_0}^{t} e^{(\tau-t)/RC}\, [au_1(\tau) + bu_2(\tau)d\tau]$$

$$= aX_1(t_0)e^{-(t_0-t)/RC} + \frac{1}{RC}\int_{t_0}^{t} e^{-(t_0-t)/RC}\, au_1(\tau)d\tau + bX_2(t_0)e^{-(t_0-t)/RC} + \frac{1}{RC}\int_{t_0}^{t} e^{(\tau-t)/RC}\, bu_2(\tau)d\tau = ay_1(t) + by_2(t)$$

Since the conditions laid in the definition are satisfied, the system is linear.

8. Time Invariance

A system is said to be "time-invariant" , if the time axis can be translated, and an equivalent system results in?

The test any system with this definition by comparing the original output with the shifted output. First shift the input function by `T` seconds starting from the same initial state X_0 at time t_0+T.Does y(t +T) of the shifted system is equal to y(t) of the original system? If yes, system is time-invariant.

III. STATE VARIABLE MODEL FOR A CONTINUOUS SYSTEM

A time-variant linear system can be described by a nth order ordinary differential equation provided it has one input and output y(t).Denoting p= $\frac{d}{dt}$ the system equations be written as

$$(p^n + a_1p^{n-1} + a_2p^{n-2} + \cdots + a_n)y =$$

$$(b_0p^n + b_1p^{n-1} + b_2p^{n-2} + \cdots . + b_n)u \longrightarrow (1)$$

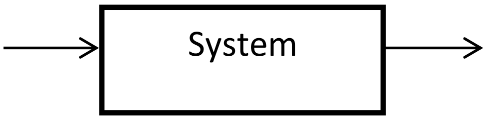

Fig (1)

Where $a_1, a_2, \ldots\ldots, a_n$ *and* $b_0, b_1, \ldots\ldots, b_n$ are constants and y and u are functions of `t` (Fig(1)).

In a similar manner, a time varying system can be described by an nth order differential equation (provided it has one input and one output only) with varying coefficients as follows.

$$\frac{d^n y}{dt^n} + \alpha_1(t)\frac{d^{n-1}y(t)}{dt^{n-1}} + \alpha_2(t)\frac{d^{n-2}y(t)}{dt^{n-2}} \ldots \alpha_n(t)y(t) = \beta_0(t)\frac{d^n}{dt^n}u(t) + \beta_1(t)\frac{d^{n-1}}{dt^{n-1}}u(t) + \cdots + \beta_n(t)u(t) \quad (2)$$

Here the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ *and* $\beta_0, \beta_1, \beta_2, \ldots\ldots, \beta_n$ are all the functions of time.

The above system considered, in general can be put into a compact from as

$$\frac{dX}{dt} = A(t)\,X(t) + B(t)u(t) \quad \longrightarrow (3)$$

And y = C(t) X(t) +D(t) u(t) ⟶ (4)

Where X(t) is an `n` vector called state vector

u(t) is an m vector corresponding , to m inputs (if exist)

y(t) is a k vector corresponding to k outputs (if the system have)

A(t), B(t), C(t) and D(t) are functions of time in matrix form as

A(t) ….n×n matrix

B(t) ….n×m matrix

C(t) ….k×n matrix

D(t) ….k×m matrix

For a single input, single output system, however,

$$\frac{dX(t)}{dt} = A(t)X + B(t)u$$

And y = C(t) + D(t)u

(Where B(t) and C(t) are column vectors and D(t) a single value.)

The state variable analysis mainly concerns with the choice of the states or state variable X(t) for the given system and then obtaining the solution for y by solving the eqns (3) and (4).

Before going into general methods of choice of states and state variables, a few examples will discussed. In the state variable analysis, the try to reduce a nth order differential equation in one independent variable into `n` number of 1st order differential equations and try to get the solution. The advantage of the method is that solve the 1st order equations easily either by classical method or obtain numerical solution. Also computers and numerical machines can handle such equations easily and efficiently. It may be noted that in this analysis the variables chosen may or may not correspond to real physical states and as such in the initial introduction and definitions, a clear cut explanation is made between real (physical) and abstract states.

Example -(1):

A system is described by the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = u(t)$$

Reduce the equations to state variable from.

Solution:

Let $y = X_1 \; and \; \frac{dy}{dt} = \frac{dX_1}{dt} = X_2$

The equation reduces to

$$\frac{dX_1}{dt} = -6X_1 - 5X_2 + u(t)$$

$$\frac{dX_1}{dt} = 0.X_1 + X_2 + 0.u(t)$$

Denoting $\frac{dX_1}{dt} = \dot{X}_1$ and $\frac{dX_2}{dt} = \dot{X}_2$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$Y = X_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [0]u(t)$$

Here X_1 and X_2 are two state variables corresponding to the output y and its derivative $\frac{dy}{dt}$.

Both X_1 and X_2 are described by 1st order differential equations only. Also the original differential equation is reduced to standard form of the type in eqns (3) and (4).

Here $[A] = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$, $[B] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$[C] = [1 \quad 0]$, $[D] = [0]$

All are matrices with numbers since the equations is a time invariant equation with single input and single output.

Choice of state variables for continuous system for Time-Invariant system:

Let the time-invariant system be represented by the differential equation already given in eqn (1).

$$(p^n + a_1 p^{n-1} + \cdots + a_n)y = (b_0 p^n + b_1 p^{n-1} + \cdots + b_n)u(5) \longrightarrow (5)$$

Where p is the differential operator $\frac{d}{dt}$; y = output, u = the input to the system.

Rearranging the terms

$$p^n(y - b_0u) + p^{n-1}(a_1y - b_1u) + \cdots + (a_ny - b_nu) = 0$$

Dividing throughout by p^n and rearranging,

$$= b_0u + \frac{1}{p}(b_1u - a_1y) + \cdots + \frac{1}{p^n}(b_nu - a_ny)$$

Let y $= b_0u$ + X

(1) $X_1 = \frac{1}{p}(b_1u - a_1y) + \frac{1}{p^2}(b_2u - a_2y) + \ldots..$

Multiplying by p, pX_1=$\dot{X}_1$

(2)$\dot{X}_1 = b_1u - a_1y + X_2$ or $pX_1 = b_1u - a_1y + \frac{1}{p}(b_2u - a_2y) + \ldots\ldots.$

(3) $\dot{X}_2 = b_2u - a_2y + X_3$ where $X_2 = \frac{1}{p}(b_2u - a_2y) + \ldots\ldots.$

$\dot{X}_2 = b_2u - a_2y + \frac{1}{p}(b_3u - a_3y) + \ldots\ldots.$

$\vdots$

$\vdots$

$\vdots$

$\dot{X}_n = b_nu - a_ny$

Rearranging in matrix form

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_{n-1} \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & 0 & \ldots & 1 \\ -a_n & 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix}$$

$$+ \begin{bmatrix} b_1 & \ldots & a_1 & b_0 \\ b_2 & \ldots & a_2 & b_0 \\ \vdots & \vdots & \vdots & \vdots \\ b_{n-1} & \ldots & -a_{n-1} & b_0 \\ b_n & \ldots & -a_n & b_0 \end{bmatrix} [u]$$

Or $[\dot{X}]$ = AX + Bu

Also y = $X_1 + b_0$u

$$=[1 \quad 0 \quad 0 \cdots \quad 0]\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} + b_0 u \longrightarrow (6)$$

or Y = CX + Du

The nth order differential equation is reduced to `n` 1st differential equations in `n` state variables $X_1, X_2, \ldots\ldots, X_n$.

In a physical system, the first write the equation of the system (the differential equation) and then the may those abstract state variables as is done in section. This form is called OBSERVABLE or CANONICAL form.

IV. SOLUTION OF STATE EQUATIONS-EXISTENCE OF SOLUTION:

Here the discuss some methods of solving the state equations. The equations obtained are vector-matrix differential equations. Firstly, the existence and uniqueness of the solution will be discussed before the actual solution is presented.

For the state equation $\dot{X}$(t) = AX + Bu $\longrightarrow$ (a)

And output equation Y = CX + Du $\longrightarrow$ (b)

Since the state of a zero input system dose not depend on the input u(t), it can be Written that X(t) = ϕ (t; X_0, t_0) where ϕ (t; X_0, t_0) is the state trajectory in its state space. Now we shall discuss:

(1) Does a solution of that kind exist and unique?
(2) Under what conditions does that solution exist?

The following assumptions are made:

(a) AX the first term of eqn (a) , A is continuous function of X for all X ∈ R^n. (∈ is for "belongs to" and R^n is the set of n real numbered vector space)
(b) AX is a bounded function.
(c) Also $|X(X_1) - X(X_2)| \leq L\ |(n_1 - n_2)|$ where L is a constant called Lipschitz constant. This condition is satisfied by all functions which are differentiable with respect to X(X= $X_1, X_{2,\ldots\ldots}$)

Existence of the Solution:

Let $\dot{X}(\psi)$ = f $|X(\psi)|$. Where f(X) satisfies the above conditions. Integrating from `$t_0$` to `t`.

$\int_{t_0}^{t} \dot{X}(\psi)\, d\psi$ = X(t) = $X_0 + \int_{t_0}^{t} f[\dot{X}(\psi)]\, d\psi$ $\longrightarrow$ (7)

Now ϕ(t) be the solution so that

$$\phi(t) = X_0 + \int_{t_0}^{t} f[(\psi)]\, d\psi \longrightarrow (8)$$

Let $\phi_0(t), \phi_1(t)$,….. be the sequence of time functions that satisfy the above relation

$\phi_0(t)$ = 0

$\phi_1(t) = X_0 + \int_{t_0}^{t} f[\phi_0(\psi)]\, d\psi$ etc.

…. …..

…. .....

…. …..

So that $\phi_{j+t}(t) = X_0 + \int_{t_0}^{t} f[\phi_n(\psi)]\, d\psi$ ⟶ (9)

And $\lim_{j\to\infty} \phi_j(t)$ for all finite \`j\` to exists and is equal $\phi(t)$

Since $\phi_0(t)$ is bounded, f(x) is continuous and bounded and its integrals are well defined for finite \`t\`.

$$\phi(t) = \phi_0(t) + [\phi_1(t) - \phi_0(t)] + [\phi_2(t) - \phi_1(t)] + \ldots.. + [\phi_{j-1}(t) - \phi_{j-2}(t)] + [\phi_j(t) - \phi_{j-1}(t)]$$

$$= \phi_0(t) + \sum_{j=0}^{j-1} |\phi_{j+1}(t) - \phi_j(t)|$$

$\phi_0(t)$ converges to a limit if we can show that the R.H.S. converges.

From definition,

$$\phi_1(t) - \phi_0(t) = \int_{t_0}^{t} f[\phi_0(\psi)]\, d\psi \qquad |\phi_1(t) - \phi_0(t)| = \left|\int_{t_0}^{t} f[\phi_0(\psi)]\, d\psi\right|$$

$$\le \int_{t_0}^{t} |f[\phi_0(\psi)]\, d\psi| \le \int_{t_0}^{t} F\, d\psi$$

$$\le F(t - t_0) \qquad \longrightarrow (10)$$

Next compute

$$\phi_2(t) - \phi_1(t) = \int_{t_0}^{t} F[\phi_1(\psi) - \phi_0(\psi)]\, d\psi$$

Using Lipschitz condition,

$$|\phi_2(t) - \phi_1(t)| \le \int_{t_0}^{t} F^2(\psi - t_0)\, d\psi$$

$\le \frac{1}{2l} F^2 (t - t_0)^2$ where \`2l\` is the interval taken.

Proceeding in similar manner,

$$|\phi_j(t) - \phi_{j-1}(t)| \le \frac{1}{jl} F^j (t - t_0)^j$$

$$\lim_{j\to\infty} \sum_{j=0}^{j-1} \phi\, |\phi_{j+1}(t) - \phi_j(t)|$$

$$\le \lim_{j\to\infty} \sum_{j=0}^{j-1} \frac{1}{(j+1)l} F^{j+1} (t - t_0)^{j+1}$$

The series on R.H.S. converges to $[\, e^{F(t-t_0)} - 1]$ for all finite values of $(t - t_0)$.

Hence, the conclude that $\phi_j(t)$ converges uniformly to limit function $\phi_1(t)$ in the time interval \`t\`$_0$ to \`t\`.

And as such

$$\phi(t) = X_0 + \lim_{j\to\infty} \int_{t_0}^{t} f[\phi_j(\psi)]\, d\psi \qquad \longrightarrow (11)$$

Taking the limit to inside of the integral,

$$\phi(t) = X_0 + \int_{t_0}^{t} f[\phi_j(\psi)]\, d\psi \qquad \longrightarrow (12)$$

Thus, the existence of solution ϕ(t) has been established for all finite values of time.

With the earlier made assumptions, prove that ϕ(t) is unique by taking another function ψ(t) as solution and showing that ψ(t) will become equal to ϕ(t). Hence, the solution for $\dot{X}$(t) is ϕ(t) (both X and ϕ are vectors of n dimensions).

For any input function u(t), the solution is obviously given by the equation

$$X = \phi[t, u(t)] \longrightarrow (13)$$

The State Transition Matrix:

The function that obtained as a solution (viz.) ϕ(t) with its n values transformed into a matrix ϕ(t, t_0), an n × n matrix is called state transition matrix so that the solution for X(t) for all its n values will be

$$X(t) = \phi(t; X_0, t_0) = \phi(t, t_0)X_0, \longrightarrow (14)$$

This is true for any t_0; (i.e) X(t) = ϕ(t; ψ) X(ψ)

For all ψ > t as well as for ψ ≤ t …. Substituting this in the state equation for zero input condition

(i.e) $\frac{dX}{dt}$ = A(t)X gives the matrix equation.

$$\frac{d\phi(t,t_0)}{dt} = A(t)\,\phi(t, t_0) \longrightarrow (15)$$

For any X_0 = X(t_0) = ϕ(t, t_0)X_0,

The initial condition on ϕ(t, t_0) is ϕ(t_0, t_0) =1 (unit matrix)

If this transition matrix is found, the solution to a time-varying linear differential equation. In the previous section, shown that ϕ(t) is obviously of the form of an exponential function so that the transition matrix ϕ(t), ψ = $e^{A(t-\psi)}$ for a time invariant linear differential equation.

Since time invariant systems are most common and important, the know how to evaluate ϕ(t), i.e.e^{At}. Here all possible methods of evaluating e^{At} are listed and the last method Resolvant Matrix or Laplace transform method is given in detail.

1. Power series Method

$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$, i.e.$e^{At}$is expanded as power series in At. ⟶ (16)

2. Eigenvalue Method

e^{At} = p$e^{Jt}p^{-1}$ where p is the model matrix and J is the diagonal matrix with diagonal elements as the eigenvalues of [A] which are distinct

3. By using Cayley - Hamilton theorem

$$e^{A(t)} = \sum_{k=0}^{n-1} r_k(t)A^k \; where \; r_k(t) is \; obtained \; form$$

$$e^{jt} = \sum_{k=0}^{n-1} r_k(t)J^k;\ e^{jt} \text{is the diagonal matrix with diagonal elements } e^{\lambda t}$$

Where λs are the eigenvalues of the matrix [A]. ⟶ (17)

4. Resolvant Matrix or Laplace transform method:

The Laplace transform of e^{At} is given by

$$L[e^{At}] = L\left[\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k\right]$$

$$= \sum_{k=0}^{\infty} A^k s^{-(k+1)} = s^{-1} \sum_{k=0}^{\infty} (As^{-1})^k$$

$= s^{-1}[1 - As^{-1}]$

$since \sum_{k=0}^{\infty} (As^{-1})^k$ *is a geometrical series with common ratio* As^{-1}

and hence their sum is $\frac{1}{1-common\ ratio}$

$\frac{1}{1-As^{-1}}$ *since I is unit matrix.*

$L(e^{At}) = L^{-1}[I - As^{-1}]^{-1} = (sI - A)^{-1} = \phi(s)$ ⟶ (18)

The matrix ϕ(s) is called "Resolvant matrix".

$(sI - A)^{-1} = \frac{\phi(s)}{\Delta(s)} = \frac{1}{\Delta(s)}[\phi_1 s^{n-1} + \phi_2 s^{n-2} + \cdots + \phi_n]$ ⟶ (19)

Where Δ(s) is characteristic polynomial of matrix A and $\phi_1, \phi_2, \ldots, \phi_n$ are constant matrices

The inverse transform of $\frac{\phi(s)}{\Delta(s)}$ or $(sI - A)^{-1}$ gives the transition matrix e^{At}.

$e^{At} = L^{-1}\left[\frac{\phi(s)}{\Delta(s)}\right]$ ⟶ (20)

When the order of the matrix A is 3 or less this method is the quickest and most convenient.

V.METHODS OF EVALUATION OF STATE TRANSITION MATRIX

In the state-space and state-space trajectory, While discussing the existence of solution for state equations, it has been presented out that the solution for $\dot{X}$ (t) = [A][X] is X =ϕ[t, u(t)] where `ϕ` is the state transition matrix. Methods of evaluating `ϕ` are mentioned. In this section, a detailed discussion of the methods of obtaining the state transition matrix and its properties are presented.

The state transition matrix for a time-invariant linear differential system

$\phi(t,\psi) = e^{A(t-\phi)}$.

Proof:

Since we know that

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

is uniformly convergent, it can be differentiated term by term and get

$$\frac{de^{At}}{dt} = \sum_{k=0}^{\infty} \frac{A^{k+1} t^k}{k}$$

Substituting this is the original equation

$$\frac{\partial \phi(t,t_0)}{\partial t} = A(t)\phi(t, t_0) \left[i.e. \frac{\partial X(t)}{\partial t} = A(t)X \right] \longrightarrow (1)$$

The verify that $e^{A(t-\phi)}$ is the solution for $\phi(t, \phi)$. Also see that $\phi(t, t_0)$ for t = ϕ is $e^{A(t-\phi)} = I$.,unit matrix

POWER SERIES EXPANSION METHOD:

One common method to find e^{At} is to expand it as power series i.e.

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k}$$

and compute the powers of A and sum then.

USING EIGENVALUES OF CHARACTERISTIC EQUATION:

If (A) is a n × n matrix. The equation obtained by $\det|A - \lambda I|$ =0 is called the "characteristic equation" and the roots of this equation are called "eigenvalues".

Case-(1):

If all the roots of the characteristic equation are distinct then e^{At} is given by $e^{At} = p\, e^{\lambda_i t} p^{-1}$

Where λ_i are the eigenvalues of matrix A and p is the transformation matrix.

For distinct value of λ_i

$$e^{At} \text{ becomes } \sum_{i=0}^{n} e^{\lambda i} X_i r_i^{-1}$$

Where (X_i) is the eigenvector and r_i^{-1} is the reciprocal basic vector.

Case -(2):

When the eigenvalues are repeated, i.e. some of the eigenvalues are equal, the matrix `A` cannot be diagonalized by a transformation and as such e^{At} = [T] [e^{Jt}] [T^{-1}] where T is a transformation matrix, which makes the a nearly diagonal matrix. Matrix [T] is called Jordon matrix denoted as `J` and evaluation is discussed in next section.

Using Cayley- Hamilton Theorem:

For any arbitrary n × n matrix `A` the characteristic polynomial is $\phi(\lambda) = \det(A - \lambda I)$, Matrix `A` satisfies the characteristic equation $\phi(A) = 0$. This result is called Cayley- Hamilyton theorem. Since `A` satisfies the characteristic equation and e^{At} is a function of A, e^{At} can be evaluated from the characteristic equation of A itself.

Using Laplace Transform or Resolvant Matrix Method:

It has been stated in the state-space and state-space trajectory that Laplace transform can be used for obtaining e^{At}. The shown that ϕ(t) the state transform matrix is given by e^{At} for time invariant case and that it is $L^{-1}(sI - A)^{-1}$

VI.CONCLUSION

In mechanical systems involving displacement of masses due to forces, a choice may be an independent set of displacements and velocities.In mechanical rotational systems, with inertia, torsional spring , damber or frictional elements, etc. an independent set of angular velocities or displacements associated with rotational inertial elements or torsional springs may form the state variable set.

Likewise depending on the system understanding and the knowledge or the quantities that are independently varying and the quantities to be found will decide the choice of the state variables.

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