Stable Representation Theory of Infinite Discrete Groups

Mr. Sunil Bhardwaj[1]
Department of Mathematics
RNB Global University, Bikaner, Rajasthan

Ms. Aakansha Vyas[2]
Department of Mathematics
MIET, Greater Noida, Uttar pradesh

Abstract- **The goal of this paper is to study representations of infinite discrete groups from a homotopical viewpoint. Our main tool and object of study is Carlsson's deformation K-theory, which provides a homotopy theoretical analogue of the classical representation ring. Deformation K-theory is a contra variant function from discrete groups to connective Ω-spectra, and we begin by discussing a simple model for the zeroth space of this spectrum. We then investigate two related phenomena regarding deformation K-theory: Atiyah-Segal theorems, which relate the deformation K-theory of a group to the complex K-theory of its classifying space, and excision, which relates the deformation K-theory of an amalgamation to the deformation K-theory of its factors. In particular, we use Morse theory for the Yang-Mills functional to prove an Atiyah-Segal theorem for fundamental groups of compact, spherical surfaces, and we prove that deformation K-theory is excessive on all free products. We conclude this paper by considering the general relationship between deformation K-theory of a group G and complex K-theory of the classifying space BG.**

Keywords— Carlsson's deformation K-theory, ring, homotopy, Atiyah-Segal theorem

I. INTRODUCTION

In this paper, I would like to explain a relation of deformation theory to mirror symmetry. Deformation theory or theory of moduli is related to mirror symmetry in many ways. We discuss only one part of it. The part we want to explain here is related to rather abstract and formal point of the theory of moduli, which was much studied in 50's and 60's.They are related to the definition of scheme, stack and its complex analytic analogue, and also to various parts of homological and homotopical algebra. Recently those topics again call attention of several people working in areas closely related to mirror symmetry. An example of this phenomenon is as follows. Let us consider a Lagrangiansub manifold L in a simplistic manifold M. A problem, which is related to the definition of Floer homology, is to count the number of holomorphic mapsΦ: D2 → M such that Φ (∂D2) ⊂ L.2 Then the trouble is the number thus defined depends on the various choices involved. For example it is not independent of the deformation of (almost) complex structure of M. So unless clarifying in which sense the number of holomorphic disks is invariant of various choices, it does not make mathematical sense to count it. It is this essential point where we need deformation theory.

We begin by describing the excision problem for amalgamated products. Let G,H, and K be finitely generated discrete groups, with homomorphismsf$_1$:K → Gand f$_2$: K → H. Then associated to the co-Cartesian (i.e. push out) diagram of groups Kf$_1$ /f$_2$ _G_H /G ∗K H, there is a diagram of spectra Kdef (G ∗K H) /_Kdef (G) f∗1_Kdef (H) f∗2 /Kdef (K).We would say that the amalgamated product G∗KH satisfies excision (for deformation K-theory) if equation is homotopy Cartesian, i.e. if the natural map from Kdef (G∗K H) to the homotopy pullback is a weak equivalence. Note that since we are dealing with connective -spectra, this is the same as saying that the diagram of zeroth spaces is homotopy Cartesian .Excision may be thought of as the statement that deformation K-theory mapsco-cartesian equation of groups to homotopy Cartesian diagrams of spectra. Excision results are important from the point of view of computations, since associated to any homotopy Cartesian diagram of spaces there is a long exact "Mayer-Vietoris" sequence of homotopy groups. f∗⊕g∗−→ πk(W).

f∗⊕g∗−→ πk(X) ⊕πk(Y)h∗−k∗−→ πk(Z) ∂ −→ πk−1(W)

Which comes from combining the long exact sequences associated to the vertical maps; note that the homotopy fibers of the vertical maps in a homotopy Cartesian square are weakly equivalent)? It is not difficult to check that if all the spaces involved are group-like H-spaces, and the maps are homomorphisms of H-spaces, then the maps in this sequence (including the boundary maps) are homomorphisms in dimension zero. Hence when applied to (the zeroth spaces of) the deformation K-theory in an amalgamation diagram, assuming excision one obtains a long exact sequence in K∗.

The Atiyah-Segal theorem describes the relationship between the representation ring R(G) of a compact Lie group G and the complex K-theory of the classifying space BG. When G is an infinite discrete group, Carlsson´s deformation K-theory provides a homotopy-theoretical analogue of R(G) which takes into account the topology of the spaces Hom(G, U(n)). I´ll explain how Morse theory for the Yang-Mills functional may be used to prove an analogue, for surface groups, of the Atiyah-Segal theorem. Work of T. Lawson provides a deep relationship between deformation K-theory and the (stable) moduli space of flat unitary connections. I´ll explain what Lawson´s results tell us in the case of surfaces, and I´ll discuss some general conjectures relating deformation K-theory, complex K-theory, and the topology of the stable moduli space. Deformation K–theory associates to each discrete group G a spectrum built from spaces of finite dimensional unitary representations of G. In all known examples, this spectrum is 2–periodic above the rational co-homological dimension of G (minus 2), in the sense that T. Lawson's Bott map is an isomorphism on homotopy in these dimensions. We establish a periodicity theorem for crystallographic subgroups of the isometrics' of k–dimensional Euclidean space. For a certain subclass of torsion-free crystallographic groups, we prove a vanishing result for the homotopy groups of the stable moduli space of representations, and we provide examples relating these homotopy groups to the co-homology of G.

Reduction of excision to representation vari-Eties , and an example- The first goal of this section is to reduce the question of excision to representation varieties, at least when the groups in question have stably group-like representation monoids (in an appropriately compatible manner).

II. K-THEORY AS A HOMOLOGY THEORY ON BANACH ALGEBRAS

So far, we have defined K-theory for unital rings only. A slight modification allows us togeneralize this definition to non unital rings A as follows. We assume that A is a kalgebrawhere k is any commutative ring with unit (for instance Z). We define a new unitalring A+ as A x k with the obvious addition and the following "twisted" multiplication.

(a, λ) (a', λ') = (aa' + λ a' + λ'a, λ λ')

There is an obvious "augmentation" A+ zc k and the K-theory of A is then defined as the kernel of the induced map (A+) zc K (k). It can be shown (not quite easily) that this definition is in fact independent of k.An interesting and motivating example is the ring of k-valued continuous functions f on a locally compact space X (k = R or C) such that f(x) goes to 0 when x goes to ∞. Then A+ is the ring of continuous functions on the "one point compactification" of X. For instance, if X = Rp , K(A) is isomorphic to ~ K(C(Sp)).Also note that this method enables us to define a morphism K(A) zc K(B) each time we have a general ring map A zc B (we no longer assume that f(1) = 1, even if A and B are unital rings).A sequence of rings and maps 0 zc A' zc A zc A" zc 0 is called exact if the underlying sequence of abelian groups is exact. In other words A' is a two sided ideal in A, whereas A" may be identified with the quotient ring A/A'.

*Theorem***.** The sequence induces an exact sequence of K-groups

K(A') zzcα K(A) zzcβ K(A")

that is Im(α) = Ker(β) [for a proof see for instance [M] or [KV]).

It is natural to ask what is Ker(α) and Coker(β). If one is familiar with homologic alalgebra, one should define "derived functors" Kn, n [Z, of the K-group in order to extend the previous exact sequence to the left and to the right (the group K0 (A) being K (A)). A partial solution (to the left) is given by the theorem a few lines below. There are at least two ways to solve this problem of derived functors. The first one is to put some topology on the rings involved (i.e. consider Banach algebras as explained below),

The other is to stay in pure algebra, which is paradoxically much harder. We shall begin with the first approach. We recall that a Banach algebra (over k = R or C) is a k-algebra A (not necessarily unital) with the following properties:

1) A norm a € || a|| is defined on the vector space A in such a way that A is complete forthe distance d(a, b) = || a-b||

2) We have the inequality ||ab|| ≤ ||a|| .||b||

A typical example is the ring of continuous functions f on a locally compact space X suchthat f(x) goes to 0 when x goes to ∞. The norm ||f|| of f is then the maximum of the valuesof |f(x)| when x [X] .More generally, if A is a Branch algebra and X a locally compact space, we define a new branch algebra A(X) as the ring of continuous functions f on X with values in A with the same condition at infinity. For the definition of the norm, we just replace |f(x)| in the previous example by || f(x)|| . With obvious notations, we have the following isomorphism

A(X) (Y) –> A(X × Y).

III. K-THEORY AS A HOMOLOGY THEORY ON DISCRETE RINGS

For various reasons, especially the applications of K-theory to Algebraic Geometry and Number Theory, the definition of the functors Kn we have just given is not very satisfactory. We would rather not use the topology of the ring A. For such a purpose, adefinition has been proposed by Quillen in 1970. Unfortunately, Quillen's definition requires some sophistication in Algebraic Topology. Therefore, we shall use another one,introduced slightly before by Villamayor and the author which coincides withQuillen's definition in many favorable cases. It is much easier to define, in the same spiritas the higher K-groups for Branch algebras. Caution: those algebraic K-groups will now be called Kn(A). In order to avoid confusion, the topological K-groups for Branch algebras defined earlier will be denoted by Kntop(A) .

As we have seen previously, homotopy groups play an important role in the definition and properties of the topological K-groups. Therefore, it is natural to look for an "algebraic"definition of them, especially for spaces like the general linear groups GLr(A) or their limit

GL(A) = ∪GLr(A)

The first thing is to define algebraically π0 (GL(A)). In other words, one should say when two invertible matrices α0 and α1 are homotopic (or are in the same "algebraic" connected path component).

IV. RELATION BETWEEN K-THEORY AND BOTT PERIODICITY

We shall limit ourselves to the complex case for simplicity and describe an explicit map From πp-1(GL(C)) to ~K(A), where A is CC(Sp), the ring of complex continuous functions defined on the sphere Sp. For this purpose, we decompose the sphere Sp into two "fat" hemispheres S+pand Sp ,each hemisphere being defined by xp+1 > -1/2 (resp xp+1 < 1/2). Using the last coordinate xp+1 as a parameter, we define two continuous functions α+ and α- from the sphere Sp to the unit interval [0, 1] such that α+ (resp. α-) is zero outside S+p (resp. Sp) and such 8 that α+ + α- = 1. We define β = (α+)2 + (α−) 2where α = α+ or α- and β = β+ or β- accordingly. After these preliminaries let us consider an element of πp-1(GL(C)) represented by a continuous map (for large enough n) f : Sp-1 zcGLn(C) We extend this map to S+p∩ Spby meridian projection on the equator ~f :S+p ∩ SpzcGLn(C).The matrix Q = (β+)2 β+β−~f β+β− ~ f-1 (β−)2)) (where (β+)2 , (β-) 2 or (β+β-) means this scalar times the identity matrix of order n) is a 2n x 2n matrix whose entries are continuous functions on Sp with values in M2n(C). It is easy to see that Q2 = Q. Therefore, the image of Q defines a projective module over the ring A = CC (Sp), hence an element of ~K (A) which is independent of the choice of the partition of unity (β+, β-).

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