Stability Analysis of Finite Difference Method in the Propagation of Surface Waves due to Earthquake

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***Abstract*-In this present context, mathematical modeling of the propagation of Surface waves (Love waves) in a fluid saturated poro-elastic medium has been considered using time dependent higher order finite difference method (FDM). It has been shown that the dispersion curves of Love waves are less dispersed for higher order FDM than of lower order FDM. Stability analysis has been done following conventional Eigen- value method. The variation of stability factor has been derived for *r* and *M* (for different order finite difference schemes). It has been shown that the method is always stable as we have considered $r \leq 1$. It is also shown graphically that it is more stable for lower values of *r* and for higher order finite difference schemes.**

Keywords- Love waves; Fluid saturated porous layer; Time-Space domain; Finite Difference scheme; Accuracy; Dispersion analysis; Phase velocity.

I. INTRODUCTION

The simulation of surface waves propagating in a fluid saturated poro-elastic media is of great importance to seismologists due to its possible applications in geophysical prospecting, survey techniques and reservoir engineering for understanding the cause and estimation of damage due to natural and manmade hazards.Since poroelastic theory was developed by Biot [1,2], many efforts have been made in using experimental and numerical methods to characterize elastic wave propagation in porous, liquid-saturated solids. The finite difference method is an important tool for numerical simulations of partial differential equations and has been used widely in simulating elastic waves.

This method depends on the approximation of the temporal and spatial derivatives present in these equations. Also the finite difference method has greater advantage over the other purely numerical methods for using relatively small memory and computation time. It specifies the model as a series of grid points and approximates the spatial and temporal derivatives by using the model values at nearby grid points. To approximate temporal derivatives, a second order finite difference scheme is generally used to limit the accuracy of modeling. A smaller time step or grid size may increase the modeling accuracy but also increase the computation time. But the availability of the modern digital computers has overcome this difficulty and the efficient and more general finite difference method has been smoothly developed. Therefore higher order finite difference on the space derivative appears to be a popular method to increase modeling accuracy.

Since poro-elastic theory was developed by Biot[1,2], many efforts have been made in using experimental and numerical methods to characterize elastic wave propagation in porous, liquid-saturated solids. The finite difference method (FDM) is an important tool for numerical simulations of partial differential equations and has been used widely in simulating elastic waves. Quite a good amount of information about the numerical modeling and propagation of seismic waves using FDM is available in the literature of many authors, namely, Boore [3], Alford [4]. Virieux [5] have used velocity-stress finite difference method for the propagation of SH wave in heterogeneous media. To improve the accuracy and stability of FDM many authors has used and developed different types of difference schemes. Levander [6] applied 4^{th} order approximation in space to the P-SV scheme. Kristek et al. [7] considered seismic wave propagation in visco-elastic media using 3D forth order staggered-grid finite difference scheme. Kristek et al. [8] considered 1D elastic problem on the accuracy of the finite-difference schemes. Tessmer [9] discussed Seismic finite difference modeling with spatially variable time steps. Finkelstein et al. [10] developed finite difference time domain dispersion reduction schemes. Y. Liu et al. [11, 12] considered an implicit finite difference scheme for seismic modeling and also considered a new time-space domain high-order difference method for the acoustic wave equation. Lie et al. [13] discussed finite difference numerical modeling in two phase anisotropic media with even order accuracy. Zhu et al. [14] developed finite difference modeling of the seismic response of fluid saturated, porous, elastic solid using Biot theory.

II. FORMULATION OF THE PROBLEM

We consider a model consisting of water saturated anisotropic poro-elastic layer of finite thickness H; the z axis are taken vertically downward. The x axis are chosen parallel to the layer in the direction of propagation of surface wave (Fig.1).

The equations of motion for the fluid-saturated anisotropic porous layer without body force and neglecting the viscosity of the fluid are

$$\sigma_{ij,j} = \frac{\partial^2}{\partial t^2}\left(\rho_{11}u_i + \rho_{12}U_i\right) \tag{1a}$$

$$\sigma_{,i} = \frac{\partial^2}{\partial t^2}\left(\rho_{12}u_i + \rho_{22}U_i\right) \tag{1b}$$

where σ_{ij} are the components of stress tensor in the solid skeleton, $\sigma(=-f\ p)$ is the reduced pressure of the fluid (p is the pressure in the fluid , and f is the porosity of the porous layer), u_i are the components of the displacement vector of the solid and U_i are those of fluid. The dynamic coefficients, ρ_{11}, ρ_{12}, ρ_{22} take into account of the inertia effects of the moving fluid and are related to the densities of the solid ρ_s , the fluid ρ_f and the layer by ρ'.

Using the conventional Love wave conditions,

ie. $\vec{u} = (0, u_y, 0)$ and $\vec{U} = (0, U_y, 0)$

where $u_y = v(x,z,t)$ and $U_y = V(x,z,t)$

,

and solving we have

$$N\left(\frac{\partial^2 v}{\partial x^2}\right) + L\left(\frac{\partial^2 v}{\partial z^2}\right) = d'\frac{\partial^2 v}{\partial t^2} \tag{2}$$

where $d' = \rho_{11} - \rho_{12}^2/\rho_{22}$ (3)

where N, L correspond to the familiar Lame's constants, From the equation (2) it can be seen that the velocities of shear waves in the porous medium in x and z directions are $\sqrt{N/d'}$ and $\sqrt{L/d'}$ respectively .

III. Finite Difference Approximation

Equation (2) can be written as

$$\gamma\left(\frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial^2 v}{\partial z^2}\right) = \frac{\gamma}{\alpha^2}\frac{\partial^2 v}{\partial t^2} \tag{4}$$

Where, $\gamma = N/L$, the anisotropic parameter and $\alpha = \sqrt{N/d'}$, the velocity of shear wave in the porous medium in x direction.

The shear wave velocity in the x-direction may be expressed as

$$\alpha = \beta_a\sqrt{1/d} \tag{5}$$

where, $d = \gamma_{11} - \gamma_{12}^2/\gamma_{22}$ and $\beta_a = \sqrt{N/\rho'}$, the velocity of the shear wave in the corresponding non-porous anisotropic elastic medium along the direction of x.

Also, $\gamma_{11} = \rho_{11}/\rho',\ \ \gamma_{12} = \rho_{12}/\rho,\ \ \gamma_{22} = \rho_{22}/\rho'.$ (6)

are the non-dimensional parameters for the material of the porous layer .

To improve the accuracy, we have considered the higher order finite difference scheme for spatial derivatives as

$$\frac{\partial^2 v}{\partial x^2} \approx \frac{1}{h^2}\left[a_0 v_{0,0}^0 + \sum_{m=1}^{M} a_m (v_{m,0}^0 + v_{-m,0}^0)\right] \tag{6a}$$

$$\frac{\partial^2 v}{\partial z^2} \approx \frac{1}{h^2}\left[a_0 v_{0,0}^0 + \sum_{m=1}^{M} a_m (v_{0,m}^0 + v_{0,-m}^0)\right] \tag{6b}$$

As generally higher order finite difference scheme on temporal derivatives requires large space in the computer memory and usually unstable, 2nd order finite difference scheme is used for temporal derivatives as:

$$\frac{\partial^2 v}{\partial t^2} \approx \frac{1}{\tau^2}\left[-2v_{0,0}^0 + (v_{0,0}^1 + v_{0,0}^{-1})\right] \tag{6c}$$

where $v_{m,j}^n = v(x+mh, z+jh, t+n\tau)$,

h is the grid size and τ is the time step.

Using equations (6a), (6b), (6c) into equation (4) we have

$$a_0(1+\gamma)v_{0,0}^0 + \sum_{m=1}^{M} a_m\left[\gamma(v_{m,0}^0 + v_{-m,0}^0) + (v_{0,m}^0 + v_{0,-m}^0)\right] = \frac{\gamma h^2}{\tau^2\alpha^2}\left[-2v_{0,0}^0 + (v_{0,0}^1 + v_{0,0}^{-1})\right] \tag{7}$$

Using the plane wave theory, let us consider

$$v_{m,j}^n = e^{i[k_x(x+mh)+k_z(z+jh)-\omega(t+n\tau)]} \tag{8}$$

Substituting equation (8) into equation (7) and simplifying, we have

$$\frac{1}{2}a_0(1+\gamma) + \sum_{m=1}^{M} a_m\left[\gamma\cos(mkh\cos\theta) + \cos(mkh\sin\theta)\right] = \gamma r^{-2}\left[-1 + \cos(\omega\tau)\right] \tag{9}$$

where, $r = \dfrac{\tau\alpha}{h}$ and

$k_x = k\cos\theta,\quad k_z = k\sin\theta$, and θ, being the propagation direction angle of the plane wave.

Using the Taylor series expansion for cosine functions, we have from the equation (9)

$$\frac{1}{2}a_0(1+\gamma)+\sum_{m=1}^{M}a_m\left\{(1+\gamma)+\sum_{j=1}^{\infty}(-1)^j\frac{[\gamma\cos^{2j}\theta+\sin^{2j}\theta](mkh)^{2j}}{(2j)!}\right\}$$
$$=\gamma\sum_{j=1}^{\infty}(-1)^j\frac{r^{2j-2}(kh)^{2j}}{(2j)!} \quad (10)$$

Comparing the coefficients of k^{2j}, we get,

$$a_0+2\sum_{m=1}^{M}a_m=0 \quad (11)$$

$$\sum_{m=1}^{M}a_m(\gamma\cos^{2j}\theta+\sin^{2j}\theta)(m)^{2j}=\gamma r^{2j-2}, \qquad j=1,2.......M \quad (12)$$

This equation indicates that the coefficients a_m are the function of θ. To obtain a single set of coefficients, an optimal angle has to be chosen. We solve the equation (12) to get a_m by using $\theta=\pi/4$ and then a_0 can be obtained from equation (11).

IV. STABILITY ANALYSIS

The recursion equation of finite difference scheme can be obtained from the equation (7) as follows:

$$v_{0,0}^{1}=\left(\frac{(1+\gamma)\,r^2}{\gamma}a_0+2\right)v_{0,0}^{0}+\frac{r^2}{\gamma}\sum_{m=1}^{M}a_m\left[\gamma(v_{m,0}^{0}+v_{-m,0}^{0})+(v_{0,m}^{0}+v_{0,-m}^{0})\right]-v_{0,0}^{-1} \quad (13)$$

Using the conventional eigen-value method of stability analysis, let's consider:

$$p_{m,m}^{0}=v_{m,m}^{0};\quad q_{m,m}^{0}=v_{m,m}^{-1};$$
$$U_{m,m}^{0}=\left(p_{m,m}^{0}\;,\;q_{m,m}^{0}\right)^T=W^0e^{i(k_x mh+k_z mh)};$$
$$U_{m,m}^{1}=\left(p_{m,m}^{1}\;,\;q_{m,m}^{1}\right)^T=W^1e^{i(k_x mh+k_z mh)} \quad (14)$$

Using equation (14) in equation (13), we obtain

$$W^1=GW^0=\begin{bmatrix} g & -1\\ 1 & 0\end{bmatrix}W^0 \quad (15)$$

Where G is the transition matrix and

$$g=2+\frac{2r^2}{\gamma}\sum_{m=1}^{M}a_m[\gamma\cos(k_x mh)+\cos(k_z mh)-(\gamma+1)] \quad (16)$$

The recursion relations of finite difference scheme will be stable if the absolute values of the eigen-values of the transition matrix are less than or equal to 1. The roots of the eigen-value equation $\lambda^2-g\lambda+1=0$ will be less than or equal to 1 if $|g|\le 2$.

Since the error generally increases with the increase of the wave number, let's consider the maximum wave number(Nyquist frequency) as

$$k_x=k_z=\frac{\pi}{h}. \quad (17)$$

Using the equation (17) into the equation (16), we have

$$g=2-\frac{4r^2(1+\gamma)}{\gamma}\sum_{m=1}^{M_1}a_{2m-1} \quad (18)$$

Where $M_1=\text{int}[(M+1)/2]$, int[] is a function to get the integer part of a value.

Therefore the stability condition is

$$\left|2-\frac{4r^2(1+\gamma)}{\gamma}\sum_{m=1}^{M_1}a_{2m-1}\right|\le 2, \quad (19)$$

i.e., $$r\le\left(\frac{(1+\gamma)}{\gamma}\sum_{m=1}^{M_1}a_{2m-1}\right)^{-\frac{1}{2}} \quad (20)$$

As a particular case if we take $\gamma=1$ (the case of isotropic medium)), the stability condition is reduced to

$$r\le\left(2\sum_{m=1}^{M_1}a_{2m-1}\right)^{-\frac{1}{2}} \quad (21)$$

which is discussed by Liu and Sen [12].

To calculate and analyze the stability of the finite difference scheme, we define the stability factor s according to the equation (20) as follows:

$$s=\left(\frac{(1+\gamma)}{\gamma}\sum_{m=1}^{M_1}a_{2m-1}\right)^{-\frac{1}{2}} \quad (22)$$

We calculate the variation of s with r and M.

V. DISPERSION ANALYSIS

Let us define a parameter δ to describe the dispersion of the Finite difference scheme by using equation (9) as follows:

$$\delta=\frac{v_{FD}}{\beta_a}$$
$$=\frac{2}{rkh\sqrt{d}}\sin^{-1}\sqrt{\frac{r^2}{\gamma}\sum_{m=1}^{M}a_m\left[\gamma\sin^2\left(\frac{mkh\cos\theta}{2}\right)+\sin^2\left(\frac{mkh\sin\theta}{2}\right)\right]} \quad (23)$$

If δ is equal to 1, then there is no dispersion. However, if δ is far from 1, a large dispersion will occur. Also kh is equal to π at the Nyquist frequency, so in calculating δ, kh only ranges from 0 to π and the variation of θ is from 0 to $\pi/4$.

VI. NUMERICAL CALCULATION AND DISCUSSIONS WITH RESPECT TO GRAPHICAL REPRESENTATIONS

The numerical calculation of the equation (23) has been done for different values of the parameters γ, d and by taking, $\alpha = 3000$. The phase velocity v_{FD}/β_a of Love wave from the equation (23) verses kh has been computed for different values of $d = 1,\ 0.9,\ 0.8,\ 0.7$, $\gamma = 1, 2, 3$, propagation angle $\theta = 0,\ \pi/16,\ \pi/8,\ 3\pi/16,\ \pi/4$ and $t = 0.0005,\ 0.001,\ 0.0015,\ 0.002,\ 0.0025.$.

To analyze the stability of the finite difference scheme, we calculate the variation of stability factor s from the equation (22) for different values of $r = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ and $M = 6, 8, 10, 12.$

Figs. 2-4 display the dispersion curves of Love waves at different values of M in a homogeneous non porous elastic solid, porous isotropic and anisotropic layer respectively. It is found that dispersion is more for the lower values of M and decreases for higher values of M. Here it is also observed that the increase in porosity leads to the decrease in the magnitude of the phase velocity of Love waves and an increase in anisotropy leads to increase in the phase velocity of Love waves

Fig. 5 displays that the area for stable recursion decreases with the increase of M.

Figs. 6 and 7 displays the variation of stability factor s for different values of r and M. It has been shown that the method is always stable and more stable for lower values of r and higher values of M.

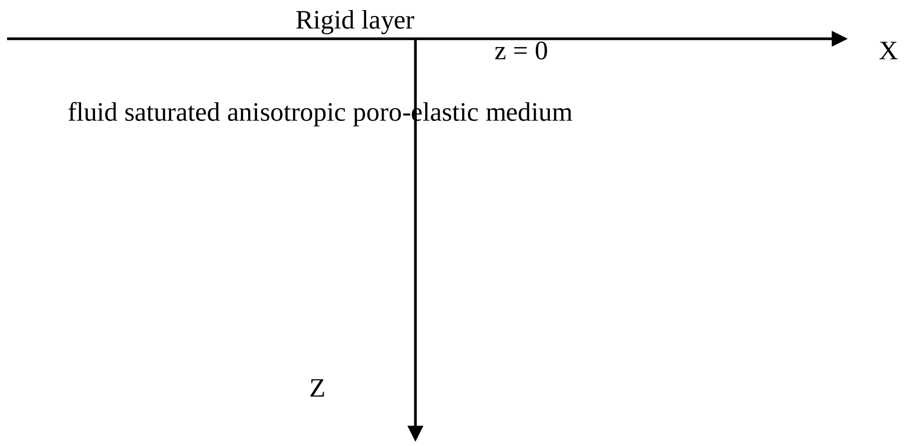

Fig. 1: Geometry of the problem.

VII. CONCLUSION

It is observed that the higher order time dependent Finite Difference Method plays an important role in the propagation of Love wave in a porous layer. Graphically, we have shown that the dispersion curves of Love waves are less dispersed for higher order Finite Difference Method. The results are compared for different order finite difference scheme (M= 6, 8, 10, 12). Here it is also observed that the increase in porosity leads to the decrease in the magnitude of the phase velocity of Love waves and an increase in anisotropy leads to increase in the phase velocity of Love waves

Stability analysis has been done following conventional Eigen- value method. The variation of stability factor has been derived for r and M (for different order finite difference schemes). It has been shown that the method is always stable as we have considered $r \leq 1$. It is also shown graphically that it is more stable for lower values of r and for higher order finite difference schemes.

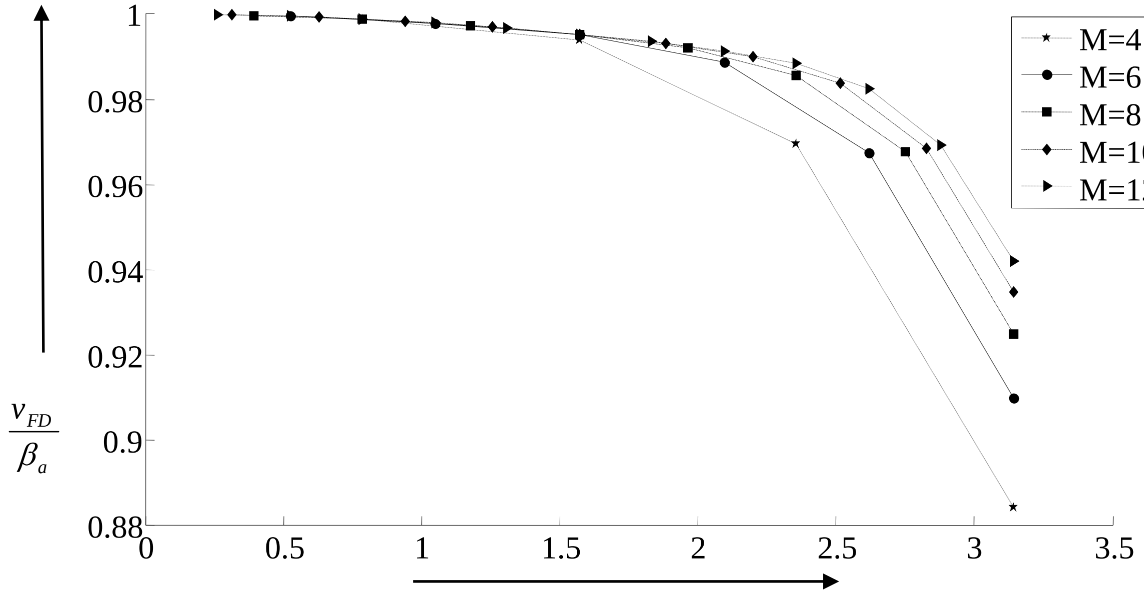

Fig.2 Dispersion curves for Love waves for different values of M when d = 1 and γ = 1.

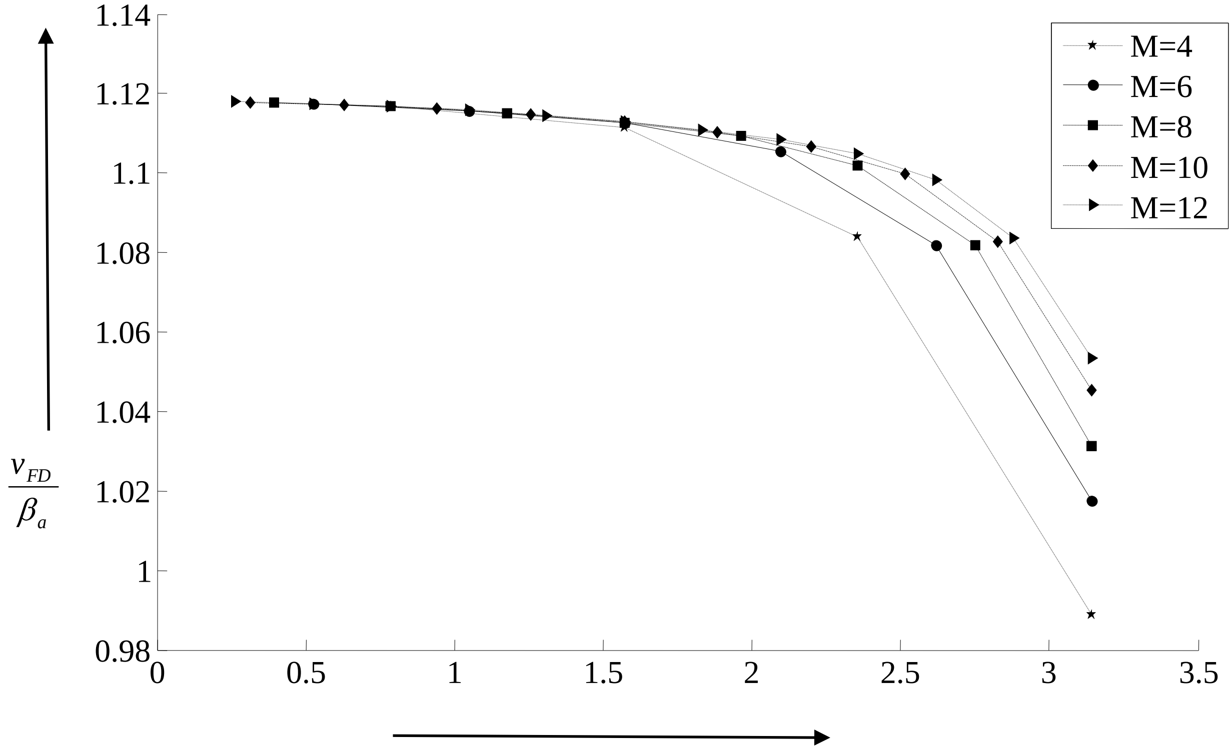

Fig.3 Dispersion curves for Love waves at different values of M when d = 0.8 and γ = 1.

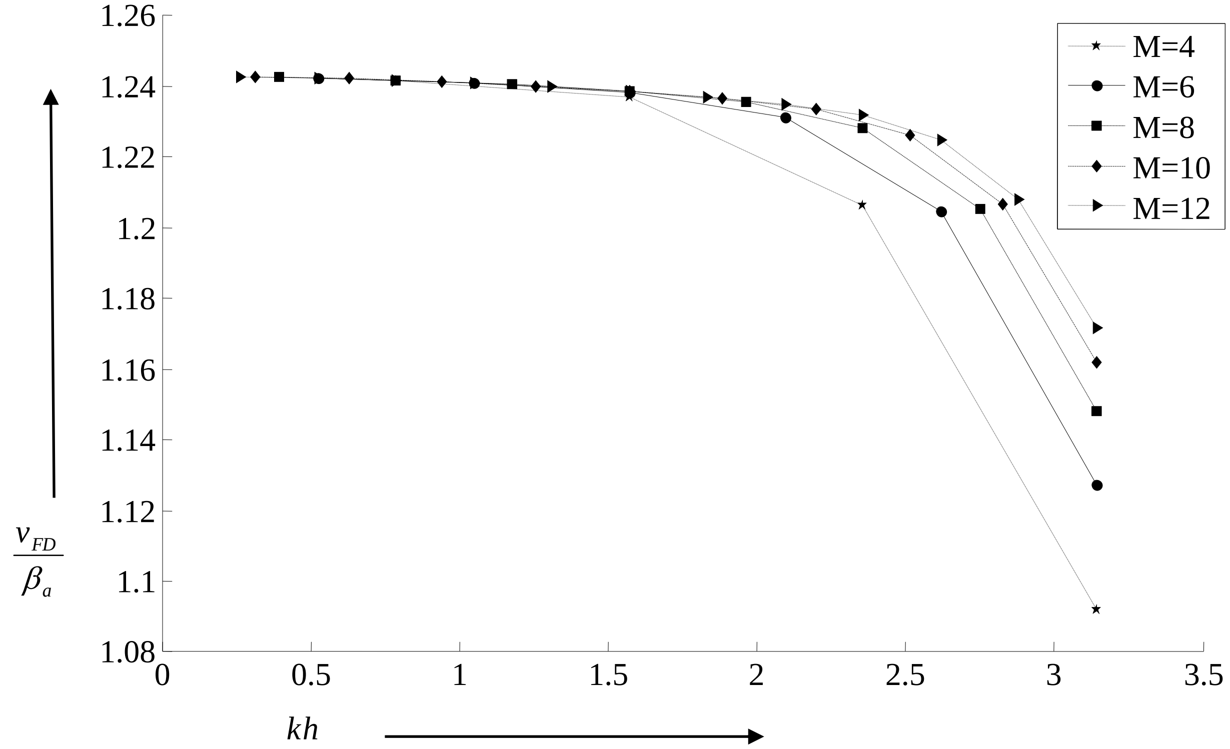

Fig.4 Dispersion curves for Love waves for different values of M when d = 0.8 and γ = 2.

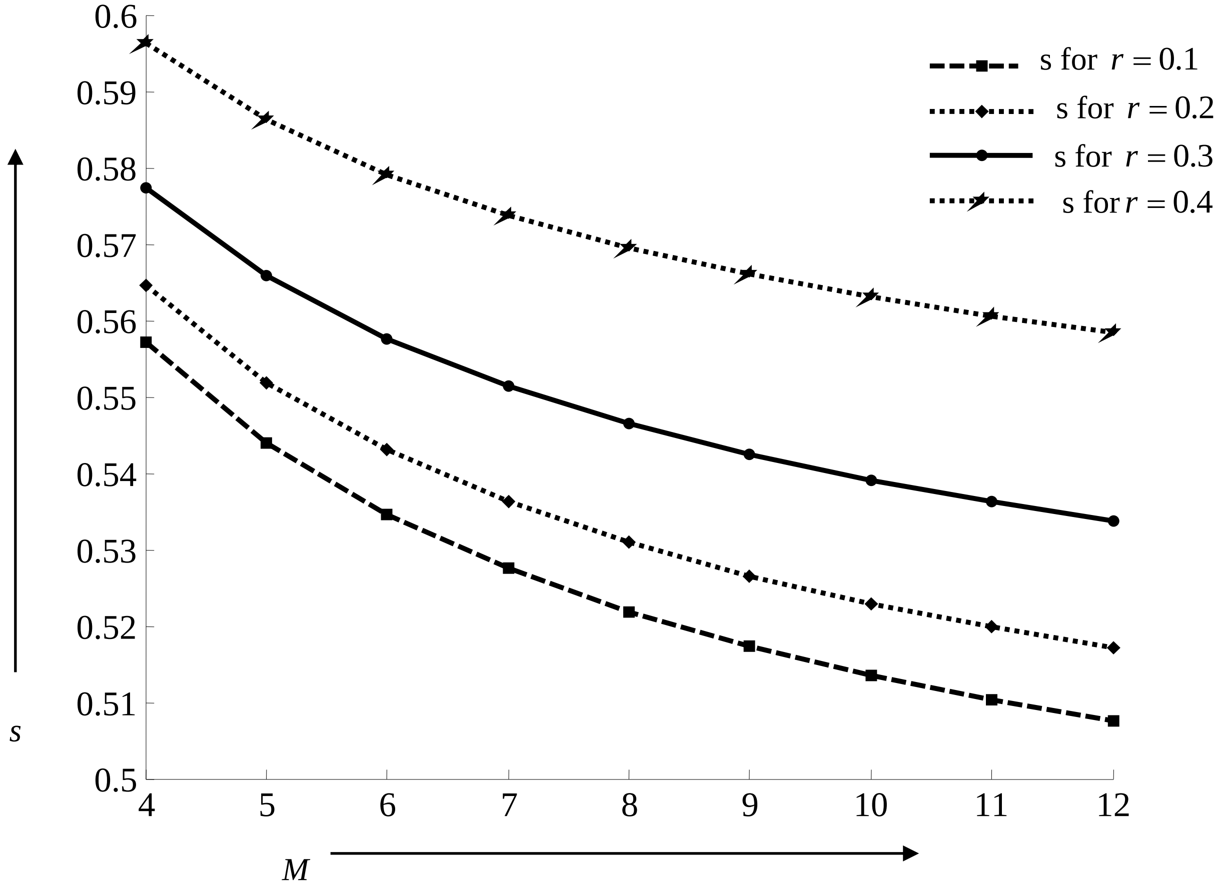

Fig. 5 The variation of stability factor s verses M for different values of r .

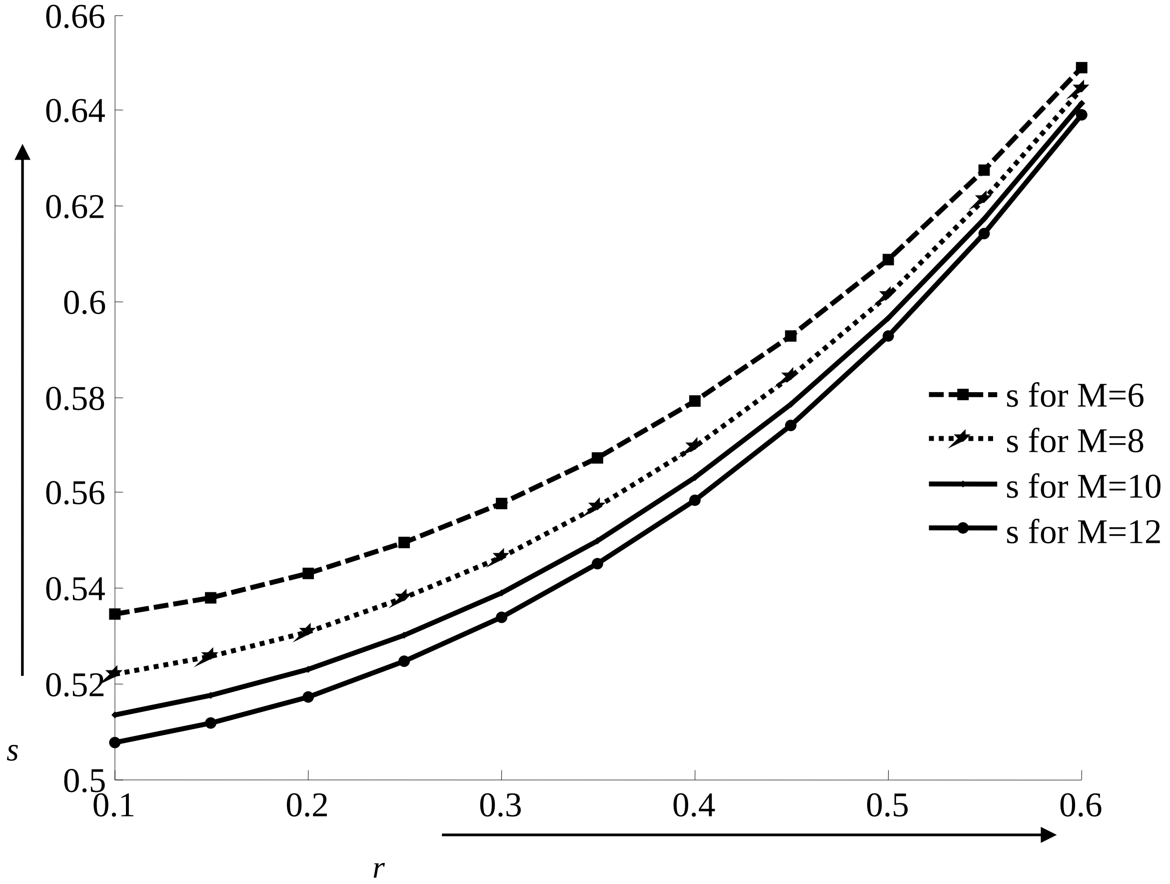

Fig 6. The variation of stability factor s verses r for different values of M .

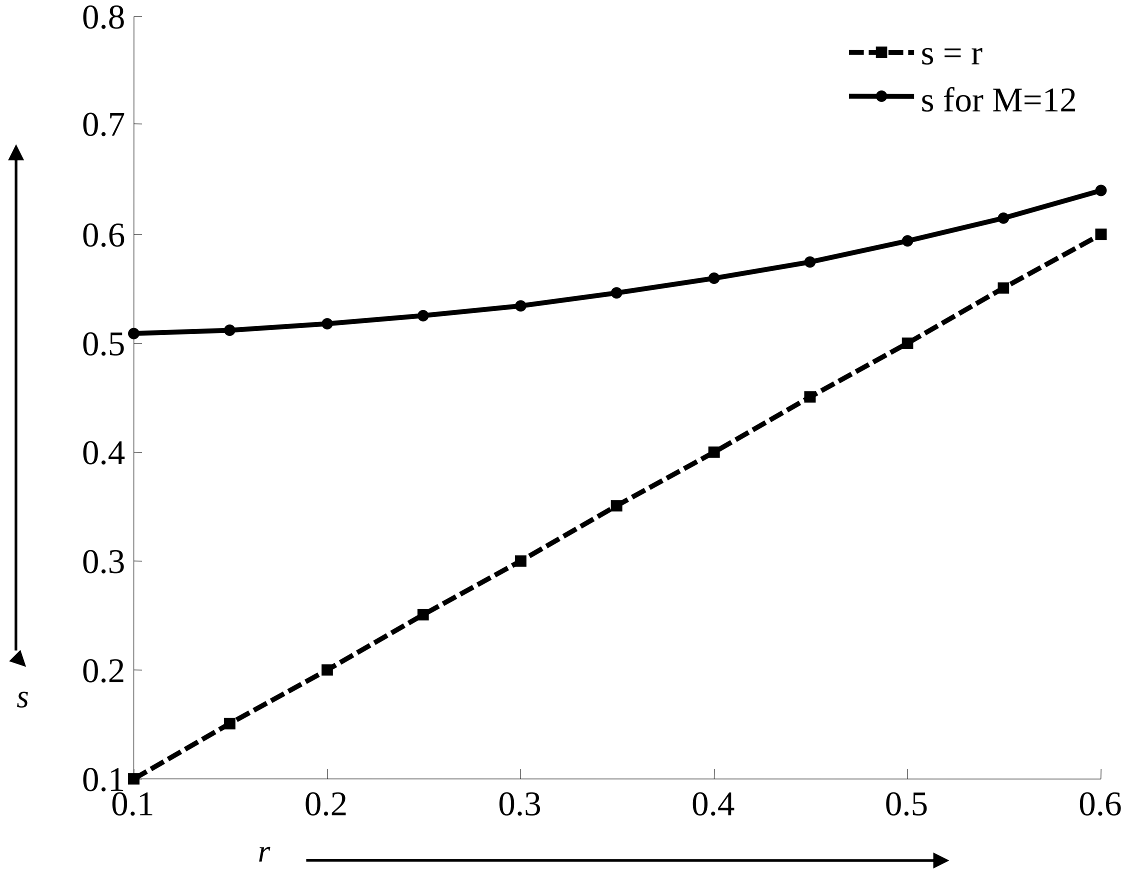

Fig 7 The variation of stability factor s verses r for different values of M .

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