Space - Time as a Complete Measure Manifold of Dimension-4

S. C. P. Halakatti*
Department of Mathematics,
Karnatak University Dharwad,
Karnataka India

H. G. Haloli, Soubhagya Baddi
Research scholar,
Department of Mathematics,
Karnatak University Dharwad,
Karnataka India

Abstract: - In this paper we introduce the structure of space-time as a complete measure manifold of dimension-4 endowed with a partial order relation \( \prec \) defined by causal connections. This new approach to space-time structure gives us insight into the dark reason of a space-time that is still measurable.

Key words: Causal connectedness, complete measure manifold, measure manifold, sequential connectedness.

Subject Classification: 57N13, 58C35.

1. INTRODUCTION

Space-time is the 4-dimensional manifold in which all physical events take place. An event is a point in space-time specified by its space and time co-ordinates. In physics space-time is a mathematical model that combines space and time into a single manifold called Minkowski space-time. In cosmology the concept of space-time is by selecting Lorentz metric or Minkowski metric that combines space and time into a single abstract universe. The common practice to study Minkowski space-time is by defining Lorentz metric or Minkowski metric metric that measures the interval between two events in space-time that is, \( ds^2=dx^2+dy^2+dz^2+(ict)^2 \). The interval \( ds^2 \) may be classified into three different types, time-like (\( ds^2<0 \)), light-like (\( ds^2=0 \)) and space-like (\( ds^2>0 \)).

For some physical applications, a space-time continuum is mathematically a 4-dimensional smooth connected Lorentzian manifold \((M, g)\) where \( g \) is a Lorentz metric with signature \((3, 1)\). The causal structure of Lorentzian manifold describes the causal relation between the events in the manifold. Hawking, S. W. Ellis, G.F.R(1973)[4], Penrose R. (1972)[7] have studied causal structure in terms of causal future, causal past, chronological future, chronological past.

The major distinction in the study of physics is the difference between local and global structures, measurement in physics are done in the local neighborhood of the events in the space-time, leading to the study of local structure of space-time in general relativity. The study of global space-time structure is vital in cosmological problems.

Generally, over lapping coordinate charts covers a manifold. The intersection of two charts represents a region of space-time in which two observers can measure physical quantities and compare their results. The concepts of coordinate charts as local observer who perform their measurements to collect and compare their results in the non empty intersection of charts locally, is vital to our mathematical model. The concept of connectedness serves this purpose. Without connectedness this would not be possible.

In this paper we study the concept of causal connected \( \mu_1-\text{a.e.} \), sequentially connected \( \mu_1-\text{a.e.} \) and maximally connected \( \mu_1-\text{a.e.} \), on complete measure manifold of dimension-4 introduced by S. C. P. Halakatti ([9], [10], [11], [12], [13],[14]). We have proved some results on space-time manifold \((M^4, \mathcal{T}_1, \Sigma_1, \mu_1)\). The significance of these results are, sequentially connected \( \mu_1-\text{a.e.} \) is an invariant property under measurable homeomorphism and measure invariant function \( F \). Also, we have developed the concept of maximal connectedness on \((M^4, \mathcal{T}_1, \Sigma_1, \mu_1)\) using sequential connected \( \mu_1-\text{a.e.} \) property. It is interesting to see that if \( \mathcal{A}_0 \) is sequentially connected \( \mu_1-\text{a.e.} \) to \( \mathcal{A}_0 \) and \( \mathcal{A}_0 \) is sequentially connected \( \mu_1-\text{a.e.} \) to \( \mathcal{A}_0 \), then \((M^4, \mathcal{T}_1, \Sigma_1, \mu_1)\) is maximally connected.

The following concepts are introduced and developed by S. C. P. Halakatti to generate a causally connected network manifold, whose applications are in the field of neural network, brain structure, in the study of large scale structures and engineering science.

2. PRELIMINARIES

Some basic definitions referred are as follows

Definition 2.1: Convergence point wise almost everywhere on \((M, \mathcal{T}_1, \Sigma_1, \mu_1)\)

Let \( f_n \to f \) point wise almost everywhere in \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) and if any measure manifold \((M, \mathcal{T}, \Sigma, \mu)\) is measurable homeomorphic to \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) then for every \( x \in \mathcal{A}_n \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu) \) \( \exists \phi^{-1}(x) = p \in \phi^{-1}(A_n) \) denoted by

\[ S = \phi^{-1}(A_n) \in (U, \phi) \in (M, \mathcal{T}, \Sigma_1, \mu_1) \text{ and } (f_n \circ \phi) \to f \circ \phi \text{ point wise a.e. in } S \in (U, \phi) \in (M, \mathcal{T}_1, \Sigma_1, \mu_1) \text{ such that, } \]

\[ S = \phi^{-1}(A_n) = \{ p = \phi^{-1}(x) \in (M, \mathcal{T}, \Sigma, \mu_1) : |(f_n \circ \phi)(p)| - (f \circ \phi)(p) | < \varepsilon, \forall n \in N \} \text{ on the chart } \phi \text{ satisfying the following conditions: } \]

\[ (i) \mu_1(S) > 0, \text{ if } |(f_n \circ \phi)(p)| - (f \circ \phi)(p) | < \varepsilon, \forall n \in N, \]

\[ (ii) \mu_1(S) = 0, \text{ if } |(f_n \circ \phi)(p)| - (f \circ \phi)(p) | \geq \varepsilon, \forall n \geq N, \text{ that is, } \mu_1(S) = 0 \text{ as } n \to \infty. \]
Definition 2.2: Convergence $\mu_{\xi}$-a.e. on $(M, \mathcal{T}, \Sigma, \mu)$

Let $f_{n} \xrightarrow{\mu_{\xi}} f$ in $A_{n} \in (\mathbb{R}, \mathcal{T}, \Sigma, \mu)$ and if any measure manifold $(M, \mathcal{T}, \Sigma, \mu)$ is measurable homeomorphic to $(\mathbb{R}, \mathcal{T}, \Sigma, \mu)$ then for every $A \in (\mathbb{R}, \mathcal{T}, \Sigma, \mu)$, $\mu_{\xi}(f_{n} - f) = 0$ if $|g_{\xi}(f_{n})| = 0$, if $|h_{\xi}(f_{n})| > 0$, if $|g_{\xi}(f_{n}) - h_{\xi}(f_{n})| = 0$, if $|h_{\xi}(f_{n}) - h_{\xi}(f_{n})| = 0$, and $\forall x \in S \subseteq (\mathbb{R}, \mathcal{T}, \Sigma, \mu)$. We define $Borel$ subsets $\mu_{\xi}^{-1}(A_{n}) = S \subseteq (\mathbb{R}, \mathcal{T}, \Sigma, \mu)$ and $\mu_{\xi}(f_{n}) = 0$, if $|f_{n} - f| = 0$, if $|f_{n} - f| = 0$, if $|f_{n} - f| = 0$, if $|f_{n} - f| = 0$, and $\forall x \in S \subseteq (\mathbb{R}, \mathcal{T}, \Sigma, \mu)$.

Note: We denote the $Borel$ subsets $\phi^{-1}(A_{n}) = S \subseteq (\mathbb{R}, \mathcal{T}, \Sigma, \mu)$ and $\phi^{-1}(A_{n}) = S \subseteq (\mathbb{R}, \mathcal{T}, \Sigma, \mu)$.

Definition 2.3: Complete Measure Manifold

If $(M, \mathcal{T}, \Sigma, \mu)$ be a measure manifold of dimension $n$ and suppose that for every measure chart $(U, \mathcal{T}, \Sigma, \mu)$, $\mu_{\xi}(f_{n} - f) = 0$ and every $V \subseteq (U, \mathcal{T}, \Sigma, \mu)$, if $\mu_{\xi}(V) = 0$, then $(M, \mathcal{T}, \Sigma, \mu)$ is called a complete measure manifold.

Let $(M, \mathcal{T}, \Sigma, \mu)$ be a complete measure manifold of dimension $n$ which is measurable homeomorphic to a measure space $(\mathbb{R}, \mathcal{T}, \Sigma, \mu)$. Let $\{f_{n}, g_{n}\}$ be measurable real valued functions converging to $f$ and $g$ respectively in $(\mathbb{R}, \mathcal{T}, \Sigma, \mu)$.

Since $f$ is measurable homeomorphic from $(M, \mathcal{T}, \Sigma, \mu)$ to $(\mathbb{R}, \mathcal{T}, \Sigma, \mu)$ for every $\{f_{n}\}$ and $\{g_{n}\}$ on $(\mathbb{R}, \mathcal{T}, \Sigma, \mu)$ there exist corresponding measurable real valued functions $\{f_{n} \rightarrow f\}$ and $\{g_{n} \rightarrow g\}$ converging to $f \circ f$ and $g \circ g$ on $(M, \mathcal{T}, \Sigma, \mu)$.

The ordered pairs $\{f_{n} \rightarrow f\}$, $\{f \circ f\}$ induces a Borel subset $S \subseteq (U, \mathcal{T}, \Sigma, \mu)$ in $(M, \mathcal{T}, \Sigma, \mu)$ satisfying the following condition:

$$S = \{ p \in (M, \mathcal{T}, \Sigma, \mu), (f_{n} \rightarrow f)(p) - (f \circ f)(p) | < \epsilon, \forall n \in N \}$$

Definition 2.4: Locally path connected $\mu_{\xi}$-a.e. on complete measure manifold

The Borel subset $S$ is locally path connected $\mu_{\xi}$-a.e. if $\exists \gamma \in C^{\infty}$- map $\gamma : [0, 1] \rightarrow S \subseteq (U, \mathcal{T}, \Sigma, \mu)$ such that

$$\gamma(0) = p \in S, \gamma(1) = q \in S,$$

That is, $p$ is locally path connected $\mu_{\xi}$-a.e. to $q$ in $S \subseteq (U, \mathcal{T}, \Sigma, \mu)$. Let $A_{n}$ be a sequence of $Borel$ subsets $\mathbb{A}$.

Definition 2.5: If $\mu_{\xi}(S) = 0$ where $\{f_{\xi} \rightarrow f\}$ and $S \subseteq (U, \mathcal{T}, \Sigma, \mu)$ is called a dark region in the chart $(U, \mathcal{T})$.

Let $(M, \mathcal{T}, \Sigma, \mu)$ be a complete measure manifold on which $\{f_{\xi} \rightarrow f\}$ and $\{g_{\xi} \rightarrow g\}$ are sequence of real valued measurable functions converging to $f \circ f$ and $g \circ g$ respectively $\mu_{\xi}$-a.e. on $(U, \mathcal{T}, \Sigma, \mu)$ and $(V, \psi)$. We define $Borel$ subsets $A_{\mu, \psi}$ be a measurable subset of $A$.

Definition 2.6: Interconnected $\mu_{\xi}$-a.e on complete measure manifold

The Borel subset $S \subseteq (U, \mathcal{T}, \Sigma, \mu)$ is interconnect to the Borel subset $R \subseteq (V, \mu)$ in $(M, \mathcal{T}, \Sigma, \mu)$. Let $f, g$ and $h$ be measurable functions converging to $f \circ f$ and $g \circ g$ respectively $\mu_{\xi}$-a.e. on $(U, \mathcal{T}, \Sigma, \mu)$ and $(V, \psi)$. We define $Borel$ subsets $A_{\mu, \psi}$ be a measurable subset of $A$.

Let $(\mathbb{R}, \mathcal{T}, \Sigma, \mu)$ be a measure space and $\{f_{\xi}, g_{\xi}\}$ and $\{h_{\xi}\}$ be sequence of measurable functions on $(\mathbb{R}, \mathcal{T}, \Sigma, \mu)$. We define $Borel$ subsets $A_{\mu, \psi}$ be a measurable subset of $A$. We define $Borel$ subsets $A_{n}$ and $C_{n}$.
If \((M, \mathcal{J}, \Sigma, \mu_1)\) is a complete measure manifold that is measurable homeomorphic and measure invariant to \((\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\). Then \(\exists\) a measurable homeomorphism and measure invariant transformation \(\varphi: (M, \mathcal{J}_1, \Sigma_1, \mu_1) \rightarrow (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\), such that, \(\{f_m \circ \varphi\}, \{g_n \circ \varphi\}\) and \(\{h_n \circ \varphi\}\) are sequences of real valued measurable functions converging to \(f \circ \varphi, g \circ \varphi\) and \(h \circ \varphi\) point wise \(\mu_1\)-a.e. on \((U, \varphi), (V, \psi)\) and \((W, \chi)\) belonging to the atlases \(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\) respectively. Also, for every induced Borel subsets \(B_n \in \mathcal{B}_{\mathbb{R}}\) and \(C_n \in (\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\), \(\exists\) the corresponding induced Borel subsets namely
\[
S \in (U, \varphi) \in \mathcal{A}_1 \in \mathcal{A}(\mathbb{R}^n), R \in (V, \psi) \in \mathcal{A}_2 \in \mathcal{A}(\mathbb{R}^n) \text{ and } Q \in (W, \chi) \in \mathcal{A}_3 \in \mathcal{A}(\mathbb{R}^n) \text{ and for each } \mu_1(S) > 0.
\]
Let \(R_n \in \mathcal{A}(\mathbb{R}^n)\) and \(Q_n \in \mathcal{A}(\mathbb{R}^n)\) be Borel subsets of \(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\), then we say that \(\mathcal{A}(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) is maximally connected if \(\forall\) a path \(\gamma(0) = p \in \mathcal{A}(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) and \(\gamma(1) \in \mathcal{A}(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) such that \(\gamma(0) = p \in S \in (U, \varphi) \in \mathcal{A}(\mathbb{R}^n)\) for which \(\mu_1(S) > 0\).

**Definition 2.8: Maximal connected \(\mu_1\)-a.e on complete measure manifold**

Let \((M, \mathcal{J}, \Sigma, \mu)\) be a complete measure manifold and let \(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in \mathcal{A}(\mathbb{R}^n)\) be atlases on \((M, \mathcal{J}, \Sigma, \mu_1)\). Let \(S, R, Q\) be Borel subsets of \(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\) respectively. Then we say that \(\mathcal{A}(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu_1)\) is maximally connected if \(\exists\) a map \(\gamma: [0,1] \rightarrow \mathbb{R}^n \cup U \cup V \cup W\) such that \(\gamma(0) = p \in S \in (U, \varphi) \in \mathcal{A}(\mathbb{R}^n, \mathcal{T}, \Sigma, \mu)\) for which \(\mu_1(S) > 0\).

**Definition 3.1: Causally connected \(\mu_1\)-a.e.**

Let \((U, \varphi) \in (M, \mathcal{J}_1, \Sigma_1, \mu_1)\) and let \(p_i\) and \(p_j\), \(\forall\ i < j, i = 1, \ldots, n\) be events in \((U, \varphi)\) then we say that \((U, \varphi)\) is causally connected \(\mu_1\)-a.e. if there exists a \(C^0\) map \(\gamma : [0,1] \rightarrow \mathbb{R}^n \cup U \cup V \cup W\) such that \(\gamma(0) = p_i \in S \in (U, \varphi) \in (M, \mathcal{J}_1, \Sigma_1, \mu_1)\) and \(\gamma(1) = p_j \in S \in (U, \varphi) \in (M, \mathcal{J}_1, \Sigma_1, \mu_1)\) for which \(\mu_1(S) > 0\).

**Definition 3.2: Sequentially connected \(\mu_1\)-a.e.**

Let \((M, \mathcal{J}_1, \Sigma_1, \mu_1)\) be a complete measure manifold and let \(A\) be a atlas in \(\mathcal{A}(\mathbb{R}^n)\) where \((U, \varphi) \in A\) then we say \(\mathcal{A}\) is sequentially connected \(\mu_1\)-a.e. if \(\exists\) a \(C^0\) map \(\gamma : [0,1] \rightarrow \mathbb{R}^n \cup U \cup V \cup W\) such that \(\gamma(0) = p_i \in S \in (U, \varphi) \in A \in (M, \mathcal{J}_1, \Sigma_1, \mu_1)\) and \(\gamma(1) = p_j \in S \in (U, \varphi) \in (M, \mathcal{J}_1, \Sigma_1, \mu_1)\) for which \(\mu_1(S) > 0\) satisfying a causality relation \(\prec\) such that \(p_i < \ldots < p_i < \ldots < p_j\) for \(i < j\).

Then the relation \(\prec\) is called a sequentially connected \(\mu_1\)-a.e. on \((U, \varphi) \in (M, \mathcal{J}_1, \Sigma_1, \mu_1)\), if \(\mu_1(S) > 0, \mu_i(U) > 0\), where \(U = U_{i=1}^n (U_i, \varphi_i)\).
Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure manifold and let \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \in \mathcal{A}^i(M^4)\) and \(\{f_i\}, \{g_i\}, \{h_i\}\) be measurable functions converging to real valued functions \(f, g, h\) on \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k\). Then the ordered pairs \((f_i, g), (g_i, h)\) induces the following sets \(S_1, S_2, S_3\) on \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k\) respectively, where \(\mathcal{A}_i\) is sequentially connected to \(\mathcal{A}_j\) and \(\mathcal{A}_j\) is sequentially connected to \(\mathcal{A}_k\) by the functions \(F, G\) and \(F\) respectively.

\[
S_1 = \{ p \in (U_i, \phi_i) \in \mathcal{A}_i \in \mathcal{A}^i(M^4) : |f_i(p) - f(p)| < \epsilon \ \forall \ n \in \mathbb{N}, i = 1, \ldots, n, \]
\[
S_2 = \{ q \in (V_j, \psi_j) \in \mathcal{A}_j \in \mathcal{A}^j(M^4) : |g_i(q) - g(q)| < \epsilon \ \forall \ n \in \mathbb{N}, i = 1, \ldots, n, \]
\[
S_3 = \{ r \in (W_k, \chi_k) \in \mathcal{A}_k \in \mathcal{A}^k(M^4) : |h_i(r) - h(r)| < \epsilon \ \forall \ n \in \mathbb{N}, i = 1, \ldots, n, \}
\]

where \(g_i = f_i \circ F^{-1}\) on \(\mathcal{A}_j\) and \(h_i = g_i \circ F^{-1}\) on \(\mathcal{A}_k\).

Since \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k\) are sequentially connected by measurable homeomorphism and measure invariant map \(F\) and \(G\), \(\mathcal{A}_i\) and \(\mathcal{A}_j\) are sequentially connected by measurable homeomorphism and measure invariant map \(G\). If \(\mathcal{A}_k\) is sequentially connected to \(\mathcal{A}_j\) by measurable homeomorphism and measure invariant map \(G \circ F\) then \((M^4, \mathcal{T}_1, \Sigma, \mu)\) is maximally connected. Therefore, one can develop the following definition.

**Definition 3.3: Maximal Connectedness \(\mu\)-a.e. on \((M^4, \mathcal{T}_1, \Sigma, \mu)\)**

Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure manifold and \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \in \mathcal{A}^i(M^4)\) be mutually sequentially connected by measurable homeomorphism and measure invariant maps \(F, G\) and \(G \circ F\) respectively, then \((M^4, \mathcal{T}_1, \Sigma, \mu)\) is maximally connected \(\mu\)-a.e.

**4. SOME RESULTS ON SPACE-TIME MANIFOLD**

It is necessary to note that if any atlas \(\mathcal{A}_i\) is sequentially connected \(\mu\)-a.e. then we show that any other atlas \(\mathcal{A}_j\), \(i \neq j \in \mathbb{N}\) is also sequentially connected \(\mu\)-a.e. that is, sequentially connected \(\mu\)-a.e. is an invariant property under measurable homeomorphism and measure invariant function \(F\).

**Theorem 4.1:**

Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure-manifold, where \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \in \mathcal{A}^i(M^4)\). Let \(F: \mathcal{A}_i \rightarrow \mathcal{A}_j\) be a \(C^\infty\) measurable homeomorphism and measure invariant map. If \(\mathcal{A}_k\) is sequentially connected \(\mu\)-a.e., then \(\mathcal{A}_j\) is also sequentially connected \(\mu\)-a.e. in \(\mathcal{A}^i(M^4)\).

**Proof:** Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure manifold and \(\mathcal{A}_i, \mathcal{A}_j \in \mathcal{A}^i(M^4)\). Let \(F: \mathcal{A}_i \rightarrow \mathcal{A}_j\) be a \(C^\infty\) measurable homeomorphism and measure invariant map.

Let \(\mathcal{A}_i\) be sequentially connected \(\mu\)-a.e.

Let \(\mathcal{A}_j\) be sequentially connected \(\mu\)-a.e. on \(S \in (U_i, \phi_i) \in \mathcal{A}_i \in \mathcal{A}^i(M^4)\).

By definition of sequentially connected, the induced set \(S\) is defined as, \(S = \{ p_i \in (U_i, \phi_i) \in \mathcal{A}_i \in \mathcal{A}^i(M^4) : |f_i(p) - f(p)| < \epsilon \ \forall \ i = 1, \ldots, n, \}\)

where \(f_i = f \circ F^{-1}\) on \(\mathcal{A}_j\) and \(\mathcal{A}_k\) respectively.

Since \(\mathcal{A}_j\) and \(\mathcal{A}_k\) are sequentially connected by measurable homeomorphism and measure invariant map \(G \circ F\) then \((M^4, \mathcal{T}_1, \Sigma, \mu)\) is maximally connected. Therefore, one can develop the following definition.

**Definition 3.3: Maximal Connectedness \(\mu\)-a.e. on \((M^4, \mathcal{T}_1, \Sigma, \mu)\)**

Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure manifold and \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \in \mathcal{A}^i(M^4)\) be mutually sequentially connected by measurable homeomorphism and measure invariant maps \(F, G\) and \(G \circ F\) respectively, then \((M^4, \mathcal{T}_1, \Sigma, \mu)\) is maximally connected \(\mu\)-a.e.

**4. SOME RESULTS ON SPACE-TIME MANIFOLD**

It is necessary to note that if any atlas \(\mathcal{A}_i\) is sequentially connected \(\mu\)-a.e. then we show that any other atlas \(\mathcal{A}_j\), \(i \neq j \in \mathbb{N}\) is also sequentially connected \(\mu\)-a.e. that is, sequentially connected \(\mu\)-a.e. is an invariant property under measurable homeomorphism and measure invariant function \(F\).

**Theorem 4.1:**

Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure-manifold, where \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \in \mathcal{A}^i(M^4)\). Let \(F: \mathcal{A}_i \rightarrow \mathcal{A}_j\) be a \(C^\infty\) measurable homeomorphism and measure invariant map. If \(\mathcal{A}_k\) is sequentially connected \(\mu\)-a.e., then \(\mathcal{A}_j\) is also sequentially connected \(\mu\)-a.e. in \(\mathcal{A}^i(M^4)\).

**Proof:** Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure manifold and \(\mathcal{A}_i, \mathcal{A}_j \in \mathcal{A}^i(M^4)\). Let \(F: \mathcal{A}_i \rightarrow \mathcal{A}_j\) be a \(C^\infty\) measurable homeomorphism and measure invariant map.

We show that if \(\mathcal{A}_i\) is sequentially connected \(\mu\)-a.e. then \(\mathcal{A}_j\) is also sequentially connected \(\mu\)-a.e. Since \(\mathcal{A}_i\) and \(\mathcal{A}_j\) are sequentially connected by measurable homeomorphism and measure invariant map \(G \circ F\) then \((M^4, \mathcal{T}_1, \Sigma, \mu)\) is maximally connected. Therefore, one can develop the following definition.

**Definition 3.3: Maximal Connectedness \(\mu\)-a.e. on \((M^4, \mathcal{T}_1, \Sigma, \mu)\)**

Let \((M^4, \mathcal{T}_1, \Sigma, \mu)\) be a complete measure manifold and \(\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k \in \mathcal{A}^i(M^4)\) be mutually sequentially connected by measurable homeomorphism and measure invariant maps \(F, G\) and \(G \circ F\) respectively, then \((M^4, \mathcal{T}_1, \Sigma, \mu)\) is maximally connected \(\mu\)-a.e.
Theorem 4.2: Let $(M^4, T, \Sigma, \mu_1)$ be a complete measure manifold that is maximally path connected $\mu_1$-a.e. where $A_i, A_j, A_k \in \Lambda^4(M^4)$. If $A_i$ is sequentially connected $\mu_1$-a.e to $A_j$ by measurable homeomorphism and measure invariant map $F$ and if $A_j$ is sequentially connected $\mu_1$-a.e to $A_k$ by measurable homeomorphism and measure invariant map $G$ then $A_i$ is sequentially connected $\mu_1$-a.e. to $A_k$ by measurable homeomorphism and measure invariant map $G \circ F$.\[\Box\]

Proof: Let $(M^4, T, \Sigma, \mu_1)$ be a complete measure manifold which is maximally path connected $\mu_1$-a.e. and $A_i, A_j, A_k \in \Lambda^4(M^4)$. Let $F : A_i \rightarrow A_j$ and $G : A_j \rightarrow A_k$ be $C^0$ measurable homeomorphism and measure invariant maps such that $A_i$ is sequentially connected $\mu_1$-a.e. to $A_j$ and $A_j$ is sequentially connected $\mu_1$-a.e. to $A_k$. Now, to show that $A_i$ is sequentially connected $\mu_1$-a.e. to $A_k$ there exists a $C^0$ measurable homeomorphism and measure invariant map $G \circ F : A_i \rightarrow A_k$, since $F$ and $G$ are measurable homeomorphism and measure invariant maps therefore composite function $G \circ F$ is measurable homeomorphism and measure invariant.

According to definition 3.2, $A_i$ is sequentially path connected $\mu_{i\rightarrow j}$ to $A_j$ and $A_j$ is sequentially path connected $\mu_{j\rightarrow k}$ to $A_k$ in $A^4(M^4)$. Then we show that $A_i$ is sequentially path connected $\mu_{i\rightarrow k}$ to $A_k$ in $A^4(M^4)$.

Let $S_i = \{p_1 \in (U_i, \phi_i) \in A_i \in A^4(M^4) : f_i(p_1) - f_i(p_0) | < \in \forall n \in [0,1] \}$, $S_j = \{q_1 \in (U_j, \psi_j) \in A_j \in A^4(M^4) : g_j(q_1) - g_j(q_0) | < \in \forall n \in [0,1] \}$, where $g_j = g_j \circ f_i \circ F$ on $A_j$ and $\forall$. According to definition 3.4 and theorems 3.1, sequentially connectedness is invariant under measurable homeomorphism and measure invariant function. Since, for every $C^0$ map $\gamma : [0,1] \rightarrow A^4(M^4)$ that connects all the events in sequential way $p_1 < \ldots < p_i < \ldots < p_{n-1} < p_n$ for $t_1 < \ldots < t_l < \ldots < t_k$, $\forall i \in [1,n]$ where $\mu_i (S_i) > 0$, $\mu_i (U_i) > 0$, $\exists F \circ \gamma : [0,1] \rightarrow A$ such that, it connects all the events in $A_j$ in sequential way $q_1 < \ldots < q_i < \ldots < q_{n-1} < q_n$ for $t_1 < \ldots < t_i < \ldots < t_k$, $\forall i \in [1,n]$ where $\mu_i (F(S_i)) > 0$, $\mu_i (V_i) > 0$, $\mu_i (A_i) > 0$.

Similarly, if for every $F \circ \gamma : [0,1] \rightarrow A_j \exists G \circ F \circ \gamma : [0,1] \rightarrow A_k$ where $G \circ F \circ \gamma$ connects all the events sequentially in $A_k$, such that $G \circ F \circ \gamma (0) = r_1 \in \beta \in (W_1, \chi_1) \in A_i \in A^4(M^4), \mu_i (S_1) > 0$, $\mu_i (W_1) > 0$.

$G \circ F \circ \gamma (\frac{K}{2}) = r_k \in (W_1, \chi_1) \in A_i \in A^4(M^4), \forall (\frac{K}{2^n}) \in (0,1)$, $k < 2^n, 1 < n \in \mathbb{N}, \mu_i (S_k) > 0$ and $\mu_i (W_1) > 0$.

Hence, $A_i$ is sequentially connected $\mu_{i\rightarrow j}$ to $A_j$, $A_j$ and $A_k$, $\forall i, j, l \in [1, n]$ then $M^4$ is maximally connected.

A maximally connectedness property on $(M^4, T, \Sigma, \mu_1)$ defines a causal structure on space-time of dimension 4.

Definition 4.3: A complete measure manifold $(M^4, T, \Sigma, \mu_1)$ admitting a partial ordered relation `$<`' that generates a network manifold of dimension 4.

5 CONCLUSION

A measure manifold $(M^4, T, \Sigma, \mu_1)$ admitting a partial ordered relation `$<`' on it denoted by $(M^4, T, \Sigma, \mu_1, <)$ generates a network manifold of dimension-4. This approach provides a new vision to the space-time as a 4-dimensional complete measure manifold. The advantage of
such approach is to generate a causally connected network manifold, whose applications are in the field of neural network, brain structure, in the study of large scale structures and engineering science.

REFERENCES: