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Some Studies on Simple Semiring

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Abstract: **Authors determine different additive and multiplicative structures of simple semiring which was introduced by Golan [1]. We also proved some results based on the paper P. Sreenivasulu Reddy and Guesh Yfter tela [4].**

1. INTRODUCTION

This paper reveals the properties of simple semiring. Through out this paper simple semiring (S, +, •) means simple semiring (S, +, •) with multiplicative identity 1.

1.1. Definition: A triple (S, +, •) is said to be a semiring if S is a non - empty set and "+, •" are binary operations on S satisfying that
(i) (S, +) is a semigroup
(ii) (S, •) is a semigroup
(iii) a(b + c) = ab + ac and (b + c)a = ba + ca, for all a, b, c in S.

Examples: (i) The set of natural numbers under the usual addition, multiplication
(ii) Every distributive lattice (L, ∨, ∧).
(iii) Any ring (R, +, •).
(iv) If (M, +) is a commutative monoid with identity element zero then the set End(M) of all endomorphism of M is a semiring under the operations of point wise addition and composition of functions.
(vi) Let S = {a, b} with the operations given by the following tables:

+	a	b
a	a	b
b	b	b

•	a	b
a	b	b
b	b	b

Then (S, +, •) is a semiring.

1.2.Definition: An element x in a semigroup (S, •) is said to be multiplicative idempotent if $x^2 = x$.

1.3.Definition: An element 'x' in a semigroup (S, +) is said to be an additive idempotent if x + x = x.

1.4.Definition: A semigroup (S, •) with all of its elements are left (right) cancellable is said to be left (right) cancellative semigroup.

1.5. Definition: A semigroup (S, •) is said to satisfy quasi separative if $x^2 = xy = yx = y^2 \Rightarrow x = y$, for all x, y in S.

1.6. Definition: A semigroup (S, +) is said to satisfy weakly separative if x + x = x + y = y + y ⇒ x = y, for all x, y in S.

1.7. Definition: A semigroup (S, •) is said to be left (right) regular if it satisfies the identity aba = ab (aba = ba) for all a, b in S.

1.8. Definition: A semigroup (S, +) is said to be left (right) singular if it satisfies the identity a + b = a (a + b = b) for all a, b in S.

1.9. Definition: A semigroup (S, .) is said to be left(right) singular if it satisfies the identity ab = a (ab = b) for all a,b in S
1.10. Definition: [3] A semiring S is called simple if a + 1 = 1 + a = 1 for any a∈S.

1.11. Definition: A semiring (S, +, .) with additive identity zero is said to be zero sum free semiring if x + x = 0 for all x in S.

1.12. Definition: . A semiring (S, +, .) is said to be zero square semiring if $x^2 = 0$ for all x in S, where 0 is multiplicative zero.

1.13. Definition: A viterbi semiring is a semiring in which S is additively idempotent and multiplicatively subidempotent.i.e., a + a = a and a + a^2 = a, for all 'a' in S.

1.14. Theorem: A simple semiring is additive idempotent semiring.
Proof: Let (S, +, .) be a simple semiring. Since (S, +, .) is simple, for any a$\in$S, a + 1 = 1. (Where 1 is the multiplicative identity element of S. S^1 = SU {1}.)
Now a = a.1 = a(1 + 1) = a + a $\Rightarrow$ a = a + a $\Rightarrow$ S is additive idempotent semiring.

1.15. Theorem: If (S, +, •) be a simple semiring and (S, +) be a right cancellative then (S, •) be a band.
Proof: From hypothesis, (S, +, •) be a simple semiring $\Rightarrow$ a + 1 = 1 $\Rightarrow$ a(a + 1) = a.1 $\Rightarrow$ a^2 + a = a $\Rightarrow$ a^2 + a = a + a (Since Theorem1.14) $\Rightarrow$ a^2 = a (Since (S, +) be a right cancellative) $\Rightarrow$ (S, •) be a band.

1.16. Theorem: If (S, +, •) be a simple semiring and (S, •) be a rectangular band then (S, •) be a singular.
Proof: From hypothesis, (S, +, •) be a simple semiring $\Rightarrow$ a + 1 = 1 $\Rightarrow$ b(a + 1) = b.1 $\Rightarrow$ba + b = b $\Rightarrow$ a(ba + b) = ab $\Rightarrow$ aba + ab = ab $\Rightarrow$ a + ab = ab (Since (S, •) be a rectangular band) $\Rightarrow$ a(1 + b) = ab $\Rightarrow$ a = ab $\Rightarrow$ ab= a $\Rightarrow$ (S, •) be a left singular. $\rightarrow$(1)
Again, a + 1 = 1 $\Rightarrow$ (a+1)b = 1.b $\Rightarrow$ ab + b = b $\Rightarrow$ (ab + b)a = ba $\Rightarrow$ aba + ba = ba $\Rightarrow$ a + ba = ba (Since (S, •) be a rectangular band)$\Rightarrow$ a(1+b)=ba $\Rightarrow$ a = ba $\Rightarrow$ ba = a $\Rightarrow$ (S, •) be a right singular. $\rightarrow$(2)
From (1) and (2), (S, •) be a singular.

1.17. Theorem: If (S, +, •) be a zero sum free and simple semiring with additive identity 0 then ab = 0 for every a, b in (S, +, •).
Proof: Since (S, +, •) be a simple semiring, b + 1 = 1 $\Rightarrow$ a(b + 1)=a.1 $\Rightarrow$ ab + a = a $\Rightarrow$ ab + a + a = a + a (Since theorem1.14) $\Rightarrow$ ab + 0 = 0 ((S, +, •) be a zero sum free semiring) $\Rightarrow$ ab = 0.

1.18. Theorem: If (S, +, •) be a zero square and simple semiring with additive identity 0 then aba = 0 and bab = 0 for every a, b in (S, +, •).
Proof: Since (S, +, •) be a simple semiring, b + 1 = 1 $\Rightarrow$ a(b + 1)=a.1 $\Rightarrow$ ab + a = a $\Rightarrow$ (ab + a)a = a.a $\Rightarrow$ aba + a^2 = a^2 $\Rightarrow$ aba + 0 = 0 (Since (S, +, •) be a zero square semiring) $\Rightarrow$ aba = 0.
Again, a + 1 = 1 $\Rightarrow$ b(a + 1)=b.1 $\Rightarrow$ ba + b = b $\Rightarrow$ (ba + b)b = b.b $\Rightarrow$ bab + b^2 = b^2 $\Rightarrow$ bab + 0 = 0 (Since (S, +, •) be a zero square semiring) $\Rightarrow$ bab = 0.

1.19. Theorem: Let (S, +, •) be a simple semiring. If (S, •) is a singular then (S, +) is a singular.
Proof: Let (S, +, •) be a simple semiring in which (S, •) is a singular that is ab = a $\Rightarrow$ ab + b = a + b $\Rightarrow$ (a + 1)b = a + b $\Rightarrow$ b = a + b $\Rightarrow$ a + b = b $\Rightarrow$ (S, +) is a right singular. $\rightarrow$(1)
Again, ab = b $\Rightarrow$ a + ab = a + b $\Rightarrow$ a(1 + b) = a + b $\Rightarrow$ a.1 = a + b $\Rightarrow$ a = a + b $\Rightarrow$ a + b = a $\Rightarrow$ (S, +) is a left singular. $\rightarrow$(2)
From (1) and (2), (S, +) is a singular.
Example: The following example satisfies the conditions of theorem

+	1	a	b
1	1	a	b
a	1	a	b
b	1	a	b

.	1	a	b
1	1	a	b
a	a	a	a
b	b	a	b

1.20. Theorem: Let (S, +, •) be a simple semiring. If (S, •) be a left regular semigroup then (S, +) is an E-inversive semigroup E(+).
Proof: By hypothesis (S, •) be a left regular semigroup then aba = ab for every a, b in (S, •)
b+ 1 = 1 $\Rightarrow$ a(b + 1) = a.1 $\Rightarrow$ ab + a = a $\Rightarrow$ b(ab + a) = ba $\Rightarrow$ bab + ba = ba $\Rightarrow$ ba + ba = ba, $\forall$a,b$\in$E(+). Where E(+) is the set of all idempotent elements in (S, +). This means that there exists a in S such that ba + ba = ba implies ba is an E-inversive element. Hence (S, +) is an E-inversive semigroup.

1.21. Theorem: If (S, +, •) be a simple semiring with multiplicative identity which is also additive identity then (S, •) is a quasi-seperative semigroup.
Proof: If (S, +, •) be a simple semiring with multiplicative identity which is also additive identity then ab + a = a.
Let a^2 = ab $\Rightarrow$ a^2 = a(b + e) $\Rightarrow$ a^2 = ab + a.e $\Rightarrow$ a^2 = ab +a $\Rightarrow$ a^2 = a.
Similarly, b^2 = ba $\Rightarrow$ b^2 = b(a + e) $\Rightarrow$ b^2 = ba + b.e $\Rightarrow$ b^2 = ba + b $\Rightarrow$ b^2 = b.
If a^2 = ab = ba = b^2 then a = b. Hence (S, •) is a quasi-seperative semigroup.

1.22. Theorem: If (S, +, •) be a simple semiring with multiplicative identity which is also additive identity then (S, •) is a (i) seperative semigroup.
(ii) weakly seperative semigroup.

Proof: Proof is similar to above theorem1.22.

1.23. Theorem: Every simple semiring (S, +, •) is a viterbi semiring.
Proof: By hypothesis (S, +, •) be a simple semiring
From the theorem1.14 (S, +, •) be an additive idempotent semiring that is $a + a = a \rightarrow (1)$
And $1 + a = 1 \Rightarrow a(1 + a) = a \Rightarrow a + a^2 = a \rightarrow (2)$
From (1) & (2), (S, +, •) is a viterbi semiring.
Remark: Converse of theorem1.15, is true if (S, •) is left cancellative and (S, +) is commutative.
Proof: Consider $a + a^2 = a$, for all 'a' in S
$\Rightarrow a.1 + a^2 = a .1$
$\Rightarrow a (1 + a) = a .1$
$\Rightarrow 1 + a = 1$ (Since (S, •) is left cancellative)
$\Rightarrow 1 + a = a + 1 = 1$ (Since (S, +) is commutative)
$\Rightarrow$ (S, +, •) be a simple semiring.
Example: This is an example for theorem 1.23

+	1	a
1	1	1
a	1	a

•	1	a
1	1	a
a	a	a

1.24. Theorem: Every simple semiring (S, +, •) is a multiplicative sub idempotent semiring.
Proof: Proof is similar to above theorem1.23.

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