

Some Results on $S^\#$ -Continuous Functions in Topological Spaces

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Abstract

The main objective of this paper is to further investigate $S^\#$ -closed sets and introduce the concept of $S^\#$ -continuous functions in topological spaces along with their properties. We also define $S^\#$ -irresolute functions and study their fundamental properties. Several theorems related to $S^\#$ -continuous functions are established. Counter examples are discussed to clarify the limitations of these concepts. We define some stronger forms of continuous functions namely, strongly $S^\#$ -continuous, perfectly $S^\#$ -continuous and completely $S^\#$ -continuous maps in topological spaces and discuss some of their properties.

Keywords : $S^\#$ -closed set, $S^\#$ -irresolute, $S^\#$ -continuous, strongly $S^\#$ -continuous, perfectly $S^\#$ -continuous and completely $S^\#$ -continuous function.

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1. INTRODUCTION

The class of continuous functions plays an crucial role in general topological spaces. The stronger as well as weaker forms of continuity have been introduced and studied by several topologists. In 1960, Levine [6] introduced strong continuous functions. Some stronger forms of continuous functions were introduced and studied by Noiri [13] in 1980. After that, several mathematicians like Levine [6], Arya and Gupta [1], Reilly and Vamanmurthy [16] and Munshi and Bassan [10] have respectively introduced and studied strong continuous functions, perfectly continuous functions, completely continuous functions and super continuous functions which are strong forms of continuous functions. Crossley and Hildebrand [4] introduced the concepts of irresolute functions, which are independent of continuous functions and stronger than semi-continuous functions.

In this paper, the symbols (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces, unless otherwise specified with additional separation axioms. For any subset $D \subseteq X$, the notations $\text{cl}(D)$ and $\text{int}(D)$ represent the closure and interior of D in X , respectively.

2. PRELIMINARIES

In this section we recall definitions of semi-open sets, semi-closed sets, generalized closed sets and related concepts which are used throughout the paper.

Definition 2.1: A member K of a TS X is termed as

- (i) regular open set [17] whenever $K = \text{int}(\text{cl}(K))$ also a regular closed set whenever $k = \text{cl}(\text{int}(K))$.
- (ii) α -open set [12] wherever $K \subseteq \text{int}(\text{cl}(\text{int}(K)))$ also a α -closed set [9] wherever $\text{cl}(\text{int}(\text{cl}(K))) \subseteq K$.
- (iii) pre-open set [8] whenever $K \subseteq \text{int}(\text{cl}((K))$ also a pre-closed set whenever $\text{cl}(\text{int}((K)) \subseteq K$.
- (iv) semi-open set [7] whenever $K \subseteq \text{cl}(\text{int}(K))$ along with a semi-closed set [7] whenever $\text{int}(\text{cl}(K) \subseteq K$.

Definition 2.2: A map $k: X \rightarrow Y$ is termed as

- (i) semi-continuous [7] whenever $k^{-1}(M)$ is SF(X) whereas each M as a $C(Y)$.
- (ii) pre-continuous [8] whenever $k^{-1}(J)$ is PF(X) for each J as a $C(Y)$.
- (iii) α -continuous [9] whenever $k^{-1}(N)$ is $\alpha F(X)$ being each N is $C(Y)$.
- (iv) g -continuous [2] whenever $k^{-1}(D)$ is GF(X) for each D is $C(Y)$.
- (v) αg -continuous [5] whenever $k^{-1}(D)$ is $\alpha GF(X)$ being each D is $C(Y)$.
- (vi) αgs -continuous [15] whenever $k^{-1}(B)$ is $\alpha GSF(X)$ whereas each B is $C(Y)$.
- (vii) g^* -continuous (= strongly g -continuous [14]) [19] whenever $k^{-1}(J)$ is $G^*F(X)$ (= strongly g -closed) whereas each J as a $C(Y)$.

Definition 2.3: A map $k: X \rightarrow Y$ is termed as irresolute [3] if $k^{-1}(G)$ is SO(X) considering each G as SO(Y).

3. $S^\#$ - CONTINUOUS FUNCTIONS

Definition 3.1: A mapping $f: X \rightarrow Y$ is termed as $S^\#$ -continuous whenever the preimage of each $C(Y)$ is $S^\#F(X)$.

Theorem 3.2: A function $f: X \rightarrow Y$ is $S^\#$ -continuous iff the preimage of each $O(Y)$ is $S^\#O(X)$.

Proof: Allow J as an $O(Y)$. Thereupon J^c is $C(Y)$. As f is $S^\#$ -continuous, $f^{-1}(J^c)$ is $S^\#F(X)$. Though $f^{-1}(J^c) = X - f^{-1}(J)$ which is $S^\#F(X)$. Accordingly, $f^{-1}(J)$ is $S^\#O(X)$.

Conversely, assume that the preimage of each $O(Y)$ is $S^\#O(X)$. Taking V be a $C(Y)$. Thereupon V^c is $O(Y)$. Due to hypothesis, $f^{-1}(V^c) = X - f^{-1}(V)$ is $S^\#F(X)$. So $f^{-1}(V)$ is $S^\#F(X)$. Thus f is $S^\#$ -continuous function.

Theorem 3.3: When $f: X \rightarrow Y$ is continuous, consequently f is $S^\#$ -continuous.

Proof: Permit V as an $O(Y)$. As f is continuous, $f^{-1}(V)$ is $O(X)$. And therefore $f^{-1}(V)$ is $S^\#O(X)$. Henceforth f is $S^\#$ -continuous.

Remark 3.4: The example below makes clear that reverse of the theorem 3.3 is untrue.

Example 3.5: Taking $X = Y = \{6, 7, 8\}$, $\tau = \{X, \phi, \{6\}, \{6, 7\}\}$ and $\sigma = \{Y, \phi, \{7\}, \{7, 8\}\}$. For an Id. Function, f is $S^\#$ -continuous however not a continuous, as $\{7, 8\} O(Y)$ in Y , $f^{-1}(\{7, 8\}) = \{7, 8\}$ isn't $O(X)$ yet it is $S^\#O(X)$.

Theorem 3.6: Each g -continuous is $S^\#$ -continuous however converse is untrue.

Proof: Allow $f: X \rightarrow Y$ be an g -continuous function. Make F as a $C(Y)$. Thereupon $f^{-1}(V)$ is $GF(X)$. Accordingly, $f^{-1}(V)$ is $S^\#F(X)$. Henceforth, f is $S^\#$ -continuous function.

Example 3.7: Taking $X = Y = \{7, 8, 9\}$, $\tau = \{X, \phi, \{7\}, \{7, 8\}\}$ and $\sigma = \{Y, \phi, \{7\}, \{8, 9\}\}$. For an Id. Function, f is $S^\#$ -continuous however not a g -continuous, as $\{8, 9\} O(Y)$ in Y , $f^{-1}(\{8, 9\}) = \{8, 9\}$ is not $GO(X)$ yet it is $S^\#O(X)$.

Theorem 3.8: Each g^* -continuous is $S^\#$ -continuous though converse is impossible.

Proof: Enable $k: X \rightarrow Y$ be a g^* -continuous function. Allow V be a $C(Y)$. At that time $k^{-1}(V)$ is $G^*F(X)$ as f is g^* -continuous. Accordingly, $k^{-1}(V)$ is $S^\#F(X)$. Henceforth, k is $S^\#$ -continuous function.

Example 3.9: Taking $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{b, c\}\}$. At that time Id. function $\beta: X \rightarrow Y$ is $S^\#$ -continuous though not a g^* -continuous, as $\{a\}$ in $C(Y)$,

$\beta^{-1}(\{a\}) = \{a\}$ is not a $G^*F(X)$ yet it is $S^\#F(X)$.

Theorem 3.10: Each α -continuous function is $S^\#$ -continuous yet the converse is untrue.

Proof: Permit $\beta: X \rightarrow Y$ be an α -continuous function. Allow W be a $C(Y)$. Thereupon $\beta^{-1}(W)$ is $\alpha F(X)$. Accordingly, $\beta^{-1}(W)$ is $S^\#F(X)$. Henceforth, β is $S^\#$ -continuous function.

Example 3.11: Enable $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ along with $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Accordingly, Id. function $f: X \rightarrow Y$ is $S^\#$ -continuous however not an α -continuous, as for $\{a, c\} \in C(Y)$, $f^{-1}(\{a, c\}) = \{a, c\}$ is not an $\alpha F(X)$ yet it is $S^\#F(X)$.

Theorem 3.12: Each α gs-continuous function is $S^\#$ -continuous however reverse is impossible.

Proof: Enable $\beta: X \rightarrow Y$ be a α gs-continuous function. Make D be a $C(Y)$. Thereupon $\beta^{-1}(D)$ is $\alpha GSF(X)$. So, $\beta^{-1}(D)$ is $S^\#F(X)$. Consequently, β is $S^\#$ -continuous function.

Example 3.13: Consider $X = Y = \{a, b, c\}$ equipped with topologies $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ as well as $\sigma = \{Y, \phi, \{b\}, \{b, c\}\}$. Id. function f is $S^\#$ -continuous yet not α gs-continuous, due to $\{a, c\} \in C(Y)$, $f^{-1}(\{a, c\}) = \{a, c\}$ is $S^\#F(X)$ however not $\alpha GSF(X)$.

Theorem 3.14: Each α g-continuous function is $S^\#$ -continuous yet converse is untrue.

Proof: Allow $j: X \rightarrow Y$ be a α g-continuous function. Enable K be a $C(Y)$. Subsequently $j^{-1}(K)$ is $\alpha GF(X)$. And therefore $j^{-1}(K)$ is $S^\#F(X)$. Therefore, j is $S^\#$ -continuous

Example 3.15: Referring to an example 3.7 along with $Y = \{a, b, c, d\}$, $\sigma = \{Y, \phi, \{b\}\}$. Id. function f is $S^\#$ -continuous however not α g-continuous, as for $\{b\} \in C(Y)$, $f^{-1}(\{b\}) = \{b\}$ is not $\alpha GF(X)$ though it is $S^\#F(X)$.

Remark 3.16: The composition of two $S^\#$ -continuous functions isn't $S^\#$ -continuous.

Example 3.17: Considering $X = Y = Z = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$ and $\eta = \{Z, \phi, \{c\}\}$. Id. functions $\beta: X \rightarrow Y$ as well as $\mu: Y \rightarrow Z$ are $S^\#$ -continuous functions however $\mu \circ \beta: X \rightarrow Z$ is not an $S^\#$ -continuous, as for $\{a, b\} \in C(Z)$, $(\mu \circ \beta)^{-1}(\{a, b\}) = \beta^{-1}(\mu^{-1}(\{a, b\})) = \beta^{-1}(\{a, b\}) = \{a, b\}$ is not $S^\#F(X)$

4. $S^\#$ -IRRESOLUTE FUNCTIONS

Definition 4.1: A function $f: X \rightarrow Y$ is named as $S^\#$ -irresolute function whenever preimage of each $S^\#F(Y)$ is $S^\#F(X)$.

Theorem 4.2: A map $k: X \rightarrow Y$ is $S^\#$ -irresolute iff preimage of each $S^\#O(Y)$ is $S^\#O(X)$.

Proof: Speculate $k: X \rightarrow Y$ is $S^\#$ -irresolute function. Allow B as a $S^\#O(Y)$. Accordingly B^c is $S^\#F(Y)$. As k is $S^\#$ -irresolute, $k^{-1}(B^c)$ is $S^\#F(X)$. Accordingly, $k^{-1}(B)$ is $S^\#O(X)$.

Reversely, predict the preimage of each $S^\#O(Y)$ is $S^\#O(X)$. Allow W be a $S^\#F(Y)$. Consequently W^c is $S^\#O(Y)$. Due to hypothesis, $k^{-1}(W^c)$ is $S^\#O(Y)$. That is $k^{-1}(W)$ is $S^\#F(X)$. Hence k is $S^\#$ -irresolute.

Theorem 4.3: When $h: X \rightarrow Y$ is $S^\#$ -irresolute, thereupon f is $S^\#$ -continuous, however reverse is untrue.

Proof: Enable Q be any $C(Y)$. Consequently, Q is $S^\#F(Y)$. As h is $S^\#$ -irresolute, $h^{-1}(Q)$ is $S^\#F(X)$. Accordingly, h is $S^\#$ -continuous function.

Example 4.4: Entitle $X=Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Id. function $f: X \rightarrow Y$ is $S^\#$ -continuous though it isn't $S^\#$ -irresolute, due to $\{b\}$ a $S^\#F(Y)$, $f^{-1}(\{b\}) = \{b\}$ is not a $S^\#F(X)$.

Theorem 4.5: Whenever $\mu: X \rightarrow Y$ is $S^\#$ -irresolute and $\beta: Y \rightarrow Z$ is $S^\#$ -continuous, subsequently $\beta \circ \mu: X \rightarrow Z$ is $S^\#$ -continuous function.

Proof: Enable T be an $O(Z)$. As β is $S^\#$ -continuous, $\beta^{-1}(T)$ is $S^\#O(Y)$. Again μ is $S^\#$ -irresolute, $\mu^{-1}(\beta^{-1}(T))$ is $S^\#O(X)$. However, $\mu^{-1}(\beta^{-1}(T)) = (\beta \circ \mu)^{-1}(T)$ is $S^\#O(X)$. Thus $(\beta \circ \mu)$ is $S^\#$ -continuous function.

Theorem 4.6: Whenever $\lambda: X \rightarrow Y$, $\mu: Y \rightarrow Z$ be two $S^\#$ -irresolute functions, consequently $\mu \circ \lambda: X \rightarrow Z$ is a $S^\#$ -irresolute function.

Proof: Allow V be a $S^\#O(Z)$. Thereupon $\lambda^{-1}(V)$ is $S^\#O(Y)$ as μ is $S^\#$ -irresolute. Again λ is $S^\#$ -irresolute, $\lambda^{-1}(\mu^{-1}(V))$ is $S^\#O(X)$. Though, $\lambda^{-1}(\mu^{-1}(V)) = (\mu \circ \lambda)^{-1}(V)$ is $S^\#O(X)$. Henceforth $(\mu \circ \lambda)$ is $S^\#$ -irresolute.

Remark 4.7: The notions of irresolute functions and $S^\#$ -irresolute functions are not related to each other as seen.

Example 4.8: Taking $X=Y=\{p_1, p_2, p_3\}$, $\tau = \{X, \phi, \{p_1\}\}$ and $\sigma = \{Y, \phi, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}$. Id. function $f: X \rightarrow Y$ is $S^\#$ -irresolute however not irresolute as for $\{p_2, p_3\}$ a $SO(Y)$, $f^{-1}(\{p_2, p_3\}) = \{p_2, p_3\}$ isn't $SO(X)$.

Example 4.9: Consider $X=Y=\{p_1, p_2, p_3\}$, $\tau = \{X, \phi, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}$ and $\sigma = \{Y, \phi, \{p_1\}, \{p_2, p_3\}\}$. Id. function $f: X \rightarrow Y$ is irresolute however not $S^\#$ -irresolute as for $\{p_1, p_2\}$ $S^\#F(Y)$, $f^{-1}(\{p_1, p_2\}) = \{p_1, p_2\}$ is not $S^\#F(X)$.

Theorem 4.10: Whenever X is T_s -space, accordingly Y is also T_s -space wherein $h: X \rightarrow Y$ be onto $S^\#$ -irresolute as well as closed function.

Proof: Permit Q be a $S^\#F(Y)$. Consequently $f^{-1}(Q)$ is $S^\#F(X)$ as h is $S^\#$ -irresolute function. As X is T_s -space, $h^{-1}(Q)$ is $C(X)$. Again h is closed function, $h(h^{-1}(Q)) = Q$ is $C(Y)$. Thus Y is T_s -space.

Theorem 4.11: When f is $S^\#$ -continuous and g is $S^\#$ -irresolute along with Y is T_s -space, thereupon $g \circ f: X \rightarrow Z$ is $S^\#$ -irresolute function.

Proof: Consider D be a $S^\#F(Z)$. At that time $g^{-1}(D)$ is $S^\#F(Y)$ as g is $S^\#$ -irresolute. As Y is T_s -space, $g^{-1}(D)$ is $C(Y)$. Again f is $S^\#$ -continuous, $f^{-1}(g^{-1}(D))$ is $S^\#F(X)$. However, $f^{-1}(g^{-1}(D)) = (g \circ f)^{-1}(D)$ is $S^\#F(X)$. Hence $(g \circ f)$ is $S^\#$ -irresolute.

Theorem 4.12: Whenever $j: X \rightarrow Y$ is injection $S^\#$ -irresolute function along with Y is $S^\#$ - T_0 -space, thereupon X is $S^\#$ - T_0 -space.

Proof: Speculate Y is $S^\#$ - T_0 -space. Consider a and b be two unlike points in X . As j is an injection, $j(a)$ and $j(b)$ are separate points in Y . In view of Y is $S^\#$ - T_0 -space, there arises G as a $S^\#O(Y)$ so as $j(a) \in G$ and $j(b) \notin G$. Again as j is $S^\#$ -irresolute, $j^{-1}(G)$ is $S^\#O(X)$ so as $a \in j^{-1}(G)$ as well as $b \notin j^{-1}(G)$. Henceforth, X is $S^\#$ - T_0 -space.

Theorem 4.13: Whenever $f: X \rightarrow Y$ is injective $S^\#$ -irresolute function from a T_S X into $S^\#$ - T_1 -space Y , subsequently, X is $S^\#$ - T_1 -space.

Proof: Allow a and b be dissimilar points of X . In view of f is injective, $f(a)$ and $f(b)$ are unlike points of Y . In view of Y is $S^\#$ - T_1 -space, there prevails $S^\#$ -open sets G and H so as $f(a) \in G$, $f(b) \notin G$ and $f(a) \notin H$, $f(b) \in H$. Again f is $S^\#$ -irresolute, $f^{-1}(G)$ and $f^{-1}(H)$ are $S^\#$ -open sets in X thereby $a \in f^{-1}(G)$, $b \notin f^{-1}(G)$ and $a \notin f^{-1}(H)$, $b \in f^{-1}(H)$. Henceforth, X is $S^\#$ - T_1 -space

5. STRONGER FORMS OF $S^\#$ -CONTINUOUS FUNCTIONS

Definition 5.1: A function $f: X \rightarrow Y$ is termed as strongly $S^\#$ -continuous whenever the preimage of each $S^\#O(Y)$ is $O(X)$.

Theorem 5.2: A function $f: X \rightarrow Y$ is strongly $S^\#$ -continuous iff the inverse image of each $S^\#F(Y)$ is $C(X)$.

Proof: Speculate $f: X \rightarrow Y$ is strongly $S^\#$ -continuous. Allow W be a $S^\#F(Y)$. Then $Y - W$ is $S^\#O(Y)$. As f is strongly $S^\#$ -continuous, $f^{-1}(Y - W)$ is $O(X)$. Though $f^{-1}(Y - W) = X - f^{-1}(W)$. Accordingly, $f^{-1}(W)$ is $C(X)$.

Conversely, speculate that the preimage of each $S^\#F(Y)$ is $C(X)$. Make U as a $S^\#O(Y)$ at that time $Y - U$ is $S^\#F(Y)$. Referring to hypothesis, $f^{-1}(Y - U)$ is $C(X)$. However, $f^{-1}(Y - U) = X - f^{-1}(U)$ is $C(X)$. Accordingly, $f^{-1}(U)$ is $O(X)$. Hence f is strongly $S^\#$ -continuous function.

Theorem 5.3: Each strongly $S^\#$ -continuous function is a continuous however conversely not.

Proof: Allow $f: X \rightarrow Y$ be a strongly $S^\#$ -continuous function. Make W be an $O(Y)$. Consequently, W is $S^\#O(Y)$. In view of f is strongly $S^\#$ -continuous, $f^{-1}(W)$ is $O(X)$. Consequently, f is continuous function.

Example 5.4: Taking $X=Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Entitle $f: X \rightarrow Y$ an Id. function. Thereupon, f is continuous however not strongly $S^\#$ -continuous, due to $\{a, c\} S^\#O(Y)$, $f^{-1}(\{a, c\}) = \{a, c\}$ isn't $O(X)$.

Theorem 5.5: Each strongly continuous function is strongly $S^\#$ -continuous, though reverse is untrue.

Proof: Consider $f: X \rightarrow Y$ be a strongly continuous function. Enable W be a $S^\#O(Y)$. At that time $f^{-1}(W)$ is both $O(X)$ as well as $C(X)$. Thereupon, $f^{-1}(W)$ is $O(X)$. Hence f is strongly $S^\#$ -continuous function.

Example 5.6: Taking $X=Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. At that time Id. function $f: X \rightarrow Y$ is strongly $S^\#$ -continuous however not strongly continuous, as for $\{a, b\}$ in Y , $f^{-1}(\{a, b\}) = \{a, b\}$ is $O(X)$ however not a $C(X)$.

Theorem 5.7: When $f: X \rightarrow Y$ is continuous and Y is T_s -space. Thereupon f is strongly $S^\#$ -continuous.

Proof: Enable $f: X \rightarrow Y$ be a continuous function. Consider R as a $S^\#O(Y)$. Thereupon, R is $O(Y)$ as Y is T_s -space. As f is continuous, $f^{-1}(R)$ is $O(X)$. Henceforth, f is strongly $S^\#$ -continuous function.

Theorem 5.8: When $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is also strongly $S^\#$ -continuous functions, thereupon $gof: X \rightarrow Z$ is strongly $S^\#$ -continuous.

Proof: Make J as a $S^\#O(Z)$. In view of g is strongly $S^\#$ -continuous, $g^{-1}(J)$ is $O(Y)$. Therefore, $g^{-1}(J)$ is $S^\#O(Y)$. Again f is strongly $S^\#$ -continuous, $f^{-1}(g^{-1}(J))$ is $O(X)$. Subsequently, $f^{-1}(g^{-1}(J)) = (gof)^{-1}(J)$ which is $O(X)$. Henceforth (gof) is strongly $S^\#$ -continuous.

Theorem 5.9: Whenever $f: X \rightarrow Y$ is continuous as well as $g: Y \rightarrow Z$ are strongly $S^\#$ -continuous, thereupon $gof: X \rightarrow Z$ is strongly $S^\#$ -continuous.

Proof: Entitle H as a $S^\#O(Z)$. In view of g is strongly $S^\#$ -continuous, $g^{-1}(H)$ is $O(Y)$. At that time $f^{-1}(g^{-1}(H))$ is $O(X)$ as f is continuous. Consequently, $f^{-1}(g^{-1}(H)) = (gof)^{-1}(H)$ is $O(X)$. So (gof) is strongly $S^\#$ -continuous function.

Theorem 5.10: Whenever $\beta: X \rightarrow Y$ is $S^\#$ -continuous and $\mu: Y \rightarrow Z$ is strongly $S^\#$ -continuous, thereupon $\mu\beta: X \rightarrow Z$ is $S^\#$ -irresolute.

Proof: Enable R be a $S^\#O(Z)$. Accordingly, $\mu^{-1}(R)$ is $O(Y)$ as μ is strongly $S^\#$ -continuous. As β is $S^\#$ -continuous, $\beta^{-1}(\mu^{-1}(R))$ is $S^\#O(X)$. But $\beta^{-1}(\mu^{-1}(R)) = (\mu\beta)^{-1}(R)$ is $S^\#O(X)$. Hence $(\mu\beta)$ is $S^\#$ -irresolute.

Theorem 5.11: Whenever $p: X \rightarrow Y$ is strongly $S^\#$ -continuous and $q: Y \rightarrow Z$ is continuous, subsequently $qop: X \rightarrow Z$ is continuous function.

Proof: Make R as an $O(Z)$. Subsequently $q^{-1}(R)$ is $O(Y)$ as q is continuous. So $q^{-1}(R)$ is $S^\#O(Y)$. Again p is strongly $S^\#$ -continuous, $p^{-1}(q^{-1}(R))$ is $O(X)$. Accordingly $p^{-1}(q^{-1}(R)) = (qop)^{-1}(R)$ is $O(X)$. Henceforth (qop) is continuous function.

Definition 5.12: A function $f: X \rightarrow Y$ is named as perfectly $S^\#$ -continuous whenever the preimage of each $S^\#O(Y)$ is both $O(X)$ as well as $C(X)$.

Lemma 5.13: A mapping $f: X \rightarrow Y$ is perfectly $S^\#$ -continuous iff preimage of each $S^\#F(X)$ is $O(X)$ as well as $C(X)$.

Theorem 5.14: A mapping $f: X \rightarrow Y$ is perfectly $S^\#$ -continuous consequently, f is strongly $S^\#$ -continuous, however reverse is impossible.

Proof: Enable $f: X \rightarrow Y$ be a perfectly $S^\#$ -continuous function. Allow W be a $S^\#O(Y)$. Thereupon $f^{-1}(W)$ is $O(X)$ besides $C(X)$. Therefore, $f^{-1}(W)$ is $O(X)$. Henceforth f is strongly $S^\#$ -continuous.

Example 5.15: In Example 5.6, the function f is strongly $S^\#$ -continuous however not perfectly $S^\#$ -continuous, due to the $\{a, b\}$ a $S^\#O(Y)$, $f^{-1}(\{a, b\}) = \{a, b\}$ is $O(X)$ though not $C(X)$.

Theorem 5.16: Every perfectly $S^\#$ -continuous function is continuous function, however reverse is impossible.

Proof: Entitle $f: X \rightarrow Y$ be a perfectly $S^\#$ -continuous function. Let G be an $O(Y)$. Subsequently G is $S^\#O(Y)$. As f is perfectly $S^\#$ -continuous, $f^{-1}(G)$ is both $O(X)$ and $C(X)$. Therefore, $f^{-1}(G)$ is $O(X)$. Consequently f is continuous function.

Example 5.17: In Example 3.4.4, the function f is continuous yet it isn't perfectly $S^\#$ -continuous, as for $\{a, c\}$ a $S^\#O(Y)$, $f^{-1}(\{a, c\}) = \{a, c\}$ is not both $O(X)$ as well as $C(X)$.

Theorem 5.18: Each perfectly $S^\#$ -continuous function is perfectly continuous, reverse is impossible.

Proof: Consider $f: X \rightarrow Y$ be a perfectly $S^\#$ -continuous function. Make W be an $O(Y)$. So G is $S^\#O(Y)$. As f is perfectly $S^\#$ -continuous, $f^{-1}(W)$ is both $O(X)$ and $C(X)$. Henceforth, f is perfectly continuous.

Example 5.19: Allow $X=Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. At that time Id. function $f: X \rightarrow Y$ is perfectly continuous though not perfectly $S^\#$ -continuous, as for $\{a, c\}$ a $S^\#O(Y)$, $f^{-1}(\{a, c\}) = \{a, c\}$ is neither $O(X)$ nor $C(X)$.

Theorem 5.20: When $f: X \rightarrow Y$ is perfectly continuous along with Y is T_s -space, subsequently f is perfectly $S^\#$ -continuous.

Proof: Enable G be a $S^\#O(Y)$. Consequently, G is $O(Y)$ as Y is T_s -space. As f is perfectly continuous, $f^{-1}(G)$ is both $O(X)$ and $C(X)$. Accordingly, f is perfectly $S^\#$ -continuous function.

Theorem 5.21: Assuming $k: X \rightarrow Y$ be a function with X as a discrete TS , Y be any TS . Thereupon the succeeding are comparable.

- i) k is perfectly $S^\#$ -continuous
- ii) k is strongly $S^\#$ -continuous.

Proof: (i) $\Rightarrow$ (ii): Follows in view of Theorem 5.14.

(ii) $\Rightarrow$ (i): Permit T be a $S^\#O(Y)$. With reference to hypothesis, $f^{-1}(T)$ is $O(X)$. As X is discrete TS , $f^{-1}(T)$ is $C(X)$. Accordingly, $f^{-1}(T)$ is $O(X)$ as well as $C(X)$. Thence f is perfectly $S^\#$ -continuous function.

Theorem 5.22: Whenever $h: X \rightarrow Y$ as well as $k: Y \rightarrow Z$ are perfectly $S^\#$ -continuous functions, at that time $k \circ h: X \rightarrow Z$ is perfectly $S^\#$ -continuous.

Proof: Make G be a $S^\#O(Z)$. Thereupon $k^{-1}(G)$ is $O(Y)$ together with $C(Y)$ as k is perfectly $S^\#$ -continuous. So $k^{-1}(G)$ is $S^\#O(Y)$. In view of h is perfectly $S^\#$ -continuous, $h^{-1}(k^{-1}(G))$ is $O(X)$ as well as $C(X)$. That is $h^{-1}(k^{-1}(G)) = (k \circ h)^{-1}(G)$ is $O(X)$ along with $C(X)$. Like so $(k \circ h)$ is perfectly $S^\#$ -continuous.

Definition 5.23: A function $f: X \rightarrow Y$ is referred as completely $S^\#$ -continuous if the inverse image of each $S^\#O(Y)$ is $RO(X)$.

Theorem 5.24: A function $f: X \rightarrow Y$ is completely $S^\#$ -continuous iff preimage of each $S^\#F(Y)$ is $RF(X)$.

Proof: Imagine a function $f: X \rightarrow Y$ is completely $S^\#$ -continuous. Allow Q be a $S^\#F(Y)$. Accordingly $Y - Q$ is $S^\#O(Y)$. As f is completely $S^\#$ -continuous, $f^{-1}(Y - Q)$ is $RO(X)$. That is

$$f^{-1}(Y - Q) = X - f^{-1}(Q) \text{ is } RO(X). \text{ Consequently } f^{-1}(Q) \text{ is } RF(X).$$

Conversely, guess that preimage of each $S^\#F(Y)$ is $RF(X)$. Taking K as a $S^\#O(Y)$. At that time $Y - K$ is $S^\#F(Y)$. Referring to hypothesis, $f^{-1}(Y - K)$ is $RF(X)$. That is $f^{-1}(Y - K) = X - f^{-1}(K)$ is $RF(X)$. Consequently, $f^{-1}(K)$ is $RO(X)$. Henceforth, f is completely $S^\#$ -continuous function.

Theorem 5.25: Whenever $f: X \rightarrow Y$ is completely $S^\#$ -continuous thereupon f is continuous, however reverse is untrue.

Proof: Enable G be an $O(Y)$. Thereupon G is $S^\#O(Y)$. As f is completely $S^\#$ -continuous, $f^{-1}(G)$ is $RO(X)$. So $f^{-1}(G)$ is $O(X)$. Henceforth, f is continuous.

Example 5.26: In Example 5.6, f is continuous yet it isn't completely $S^\#$ -continuous, as for $\{a, b\}$ a $S^\#O(Y)$, $f^{-1}(\{a, b\}) = \{a, b\}$ is not $RO(X)$.

Theorem 5.27: Each completely $S^\#$ -continuous function completely continuous, however converse is untrue.

Proof: Allow $f: X \rightarrow Y$ be a completely $S^\#$ -continuous function. Consider W be an $O(Y)$. Consequently W is $S^\#O(Y)$. In view of f is completely $S^\#$ -continuous, $f^{-1}(W)$ is $RO(X)$. Consequently f is completely continuous.

Example 5.28: Taking $X=Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Enable an Id. function $f: X \rightarrow Y$. At that time f is completely continuous yet it isn't completely $S^\#$ -continuous, as for the $\{a, b\}$ a $S^\#O(Y)$, $f^{-1}(\{a, b\}) = \{a, b\}$ isn't $RO(X)$.

Theorem 5.29: Whenever $f: X \rightarrow Y$ is completely $S^\#$ -continuous at that time f is strongly $S^\#$ -continuous, though the reverse is not possible.

Proof: Consider $f: X \rightarrow Y$ be completely $S^\#$ -continuous. Entitle D be $S^\#O(Y)$. As f is completely $S^\#$ -continuous, $f^{-1}(D)$ is $RO(X)$. Therefore, $f^{-1}(D)$ is $O(X)$. Henceforth, f is strongly $S^\#$ -continuous.

Example 5.30: In Example 5.6, the function f is strongly $S^\#$ -continuous though not completely $S^\#$ -continuous, as for the $\{a, b\}$ a $S^\#O(Y)$, $f^{-1}(\{a, b\}) = \{a, b\}$ is not $RO(X)$.

Example 5.31: Taking $X=Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. At that time Id. function $f: X \rightarrow Y$ is strongly $S^\#$ -continuous however not strongly continuous, as for the subset $\{a, b\}$ in Y , $f^{-1}(\{a, b\}) = \{a, b\}$ is $O(X)$ but not a $C(X)$.

Theorem 5.32: Whenever $f: X \rightarrow Y$ is completely continuous and Y is T_s -space, consequently is completely $S^\#$ -continuous.

Proof: Make G be a $S^\#O(Y)$. At that time G is an $O(Y)$ as Y is T_s -space. As f is completely continuous, $f^{-1}(G)$ is $RO(X)$. Accordingly, f is completely $S^\#$ -continuous function.

Theorem 5.33: Whenever $p: X \rightarrow Y$ is completely continuous and $q: Y \rightarrow Z$ completely $S^\#$ -continuous, consequently $(q \circ p): X \rightarrow Z$ is completely $S^\#$ -continuous.

Proof: Enable K be a $S^\#O(Z)$. Then $q^{-1}(K)$ is $RO(Y)$ as q is completely $S^\#$ -continuous. So $q^{-1}(K)$ is $O(Y)$. In view of p is completely continuous, $p^{-1}(q^{-1}(K))$ is $RO(X)$. That is $p^{-1}(q^{-1}(K)) = (q \circ p)^{-1}(K)$ is $RO(X)$. Subsequently, $(q \circ p)$ is completely $S^\#$ -continuous.

Theorem 5.34: Whenever $f: X \rightarrow Y$ is completely $S^\#$ -continuous together with $g: Y \rightarrow Z$ is $S^\#$ -irresolute, subsequently $g \circ f: X \rightarrow Z$ is completely $S^\#$ -continuous.

Proof: Enable G be a $S^\#O(Z)$. As g is $S^\#$ -irresolute, $g^{-1}(G)$ is $S^\#O(Y)$. As f is completely $S^\#$ -continuous, $f^{-1}(g^{-1}(G))$ is $RO(X)$. Particularly $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is $RO(X)$. Hence $(g \circ f)$ is completely $S^\#$ -continuous.

6. CONCLUDING REMARKS

The concluding remarks of the paper are as follows

- **$S^\#$ -Continuous Functions and Standard Continuity:** The research demonstrates that $S^\#$ -continuous functions form a broader category than standard continuous functions. Every continuous function, g -continuous function, and αg -continuous function is automatically an $S^\#$ -continuous function, though the reverse relationships do not hold true.
- **$S^\#$ -Irresolute Functions and Their Behavior:** The authors show that while every $S^\#$ -irresolute function qualifies as an $S^\#$ -continuous function, an $S^\#$ -continuous function is not guaranteed to be $S^\#$ -irresolute. Additionally, the concepts of irresolute functions and $S^\#$ -irresolute functions operate independently of each other.
- **Strongly $S^\#$ -Continuous Functions:** The paper establishes that every strongly continuous function and every completely $S^\#$ -continuous function acts as a strongly $S^\#$ -continuous function. Furthermore, every strongly $S^\#$ -continuous function is continuous, though continuous functions are not always strongly $S^\#$ -continuous.
- **Perfectly $S^\#$ -Continuous Functions:** The findings reveal that perfectly $S^\#$ -continuous functions represent a very restrictive category. Any function that is perfectly $S^\#$ -continuous is also strongly $S^\#$ -continuous, continuous, and perfectly continuous, but none of these individual conditions naturally imply perfect $S^\#$ -continuity on their own.
- **Completely $S^\#$ -Continuous Functions:** The paper proves that completely $S^\#$ -continuous functions are inherently continuous, completely continuous, and strongly $S^\#$ -continuous, though the reverse implications fail in general topological settings.
- **Preservation of Topological Properties:** The research demonstrates that when an onto, closed, and $S^\#$ -irresolute function maps from a $S^\#$ -space, the destination space is also preserved as a T_s -space. Similarly, one-to-one $S^\#$ -irresolute mappings preserve $S^\#$ - T_0 and $S^\#$ - T_1 space properties from the target space back to the domain space.
- **Composition Properties Across Mappings:** The study confirms that combining two strongly $S^\#$ -continuous functions, two perfectly $S^\#$ -continuous functions, or a completely $S^\#$ -continuous function

with an $S^\#$ -irresolute function preserves their respective strong continuity traits through function composition.

- **Behavior in Specialized Topological Spaces:** The authors conclude that in discrete topological spaces, a function is perfectly $S^\#$ -continuous if and only if it is strongly $S^\#$ -continuous. Moreover, when the destination space is a T_s -space, standard continuous functions become strongly $S^\#$ -continuous, perfectly continuous functions become perfectly $S^\#$ -continuous, and completely continuous functions become completely $S^\#$ -continuous.

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REFERENCES

- [1] S. P. Arya and R Gupta, On strongly continuous mappings, Kyungpook Math., JI.14 (1974)131-143.
- [2] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Kochi Univ. Ser. A. Math., 12(1991), 5-13.
- [3] P. Bhattacharya and B. K. Lahiri, Semi-generalized closed sets in topology, Indian. J. Math., 29(3) (1987), 375-382.
- [4] S. G. Crossley and S. K. Hildebrand, Semi-topological properties, Fund. Math., 74(1974), 233-254.
- [5] R. Devi, K. Balachandran and H. Maki, On generalized α -continuous maps and α -generalized continuous maps, Far East J. Math., Sci., Special Volume, Part I(1997),1-15.
- [6] N. Levine, Strong continuity in topological spaces, Amer. Math. Monthly, 67 (1960), 269.
- [7] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.
- [8] A. S. Mashhour, M. E. Abd El-Monsef and S. N. EL-Deeb, On pre-continuous and weak pre-continuous mappings, Proc. Math and Phys. Soc. Egypt, 53 (1982), 47-53.
- [9] A. S. Mashhour, I. A. Hasanein and S. N. EL-Deeb, α -continuous and α -open mappings, Acta Math. Hung., 41(3-4) (1983), 213-218.
- [10] B. M. Munshi and D. S. Bassan, Super continuous mappings, Indian. J. Pure appl. Math., 13 (1982), 229-236.
- [11] G.B. Navalagi, Sujata Mookanagoudar and Sujata Tallur, on A study of new generalized closed set in topological space, HSSS, 12(2), No-3, July-Dec 2023, 102-105.
- [12] O. Njastad, on some classes of nearly open sets, Pacific. J. Math., 15(1965), 961-970.
- [13] T. Noiri, Super continuity and some strong forms of continuity, Indian JI. Pure Appl.Math., 15(1984), 241-250.
- [14] A. Pushpalatha, Studies on generalizations of mappings in topological spaces, Ph.D., thesis, Bharathiar University, Coimbatore (2000).
- [15] M.Rajamani and K.Viswanathan, On α gs-continuous maps in topological spaces. Acta Ciencia Indica, vol.31 No. 1(2005), 293-303.
- [16] I. L. Reilly and M. K. Vamanmurthy, On super continuous mappings, Indian JI. Pure Appl. Math., 14(1983), 767-772.

- [17] **M. Stone**, Application of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41 (1937), 374-481.
- [18] **Sujata Mookanagoudar, Sujata Tallur, M.B.Page and G.B. Navalagi**, Properties On $S^\#$ -Neighborhoods in topological spaces”, IOSR journal of Mathematics, Vol.22(2), (Mar.-Apr. 2026), 80-84
- [19] **J. Tong**, A decomposition of continuity, Acta Math. Hungar., 48 (1986), 11-15.