Some New Separation Axioms In Ideal Topological Spaces

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ABSTRACT. In this paper, b-$\mathcal{I}$-open sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.

1. Introduction

The notion of R_0 topological spaces is introduced by Shanin [13] in 1943. Later, Davis [7] rediscovered it and studied some properties of this weak separation axiom. Several topologists (e.g. [8], [9], [11]) further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker than T_2. The subject of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathasamy [14]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [14] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every open neighbourhood U of $x\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $\star$-topology, which is finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$, When there is no chance of confusion, $A^\star(\mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ will denote the closure and interior of A in (X, τ), respectively. A subset A of a topological space (X, τ) is said to be b-open [1] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$. The notion of b-open sets has been studied extensively in recent years by many topologists [see [4, 5, 12]] because b-open sets are only natural generalization of open sets. More importantly, they also suggest several new properties

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of topological spaces. In this paper, b-$\mathcal{I}$-open sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.

2. Preliminaries

A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-open [6] if $S \subset \mathrm{Int}(\mathrm{Cl}^\star(S)) \cup \mathrm{Cl}^\star(\mathrm{Int}(S))$. The complement of a b-$\mathcal{I}$-open set is called a b-$\mathcal{I}$-closed set [6]. The intersection of all b-$\mathcal{I}$-closed sets containing S is called the b-$\mathcal{I}$-closure of S and is denoted by $b\mathcal{I}\,\mathrm{Cl}(S)$. The b-$\mathcal{I}$-Interior of S is defined by the union of all b-$\mathcal{I}$-open sets contained in S and is denoted by $b\mathcal{I}\,\mathrm{Int}(S)$. The family of all b-$\mathcal{I}$-open (resp. b-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $B\mathcal{I}O(X, x)$ (resp. $B\mathcal{I}C(X, x)$). A subset U of X is called a b-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists a b-$\mathcal{I}$-open set V of $(X, \tau, \mathcal{I})$ such that $x \in V \subset U$. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma,)$ is said to be b-$\mathcal{I}$-continuous [6] if $f^{-1}(V) \in B\mathcal{I}O(X)$ for every open set V of Y.

Definition 2.1. *A topological space (X, τ) is said to be:*

(1) *b-R_0 [4] if every b-open set contains the b-closure of each of its singletons.*
(2) *b-R_1 [4] if for x, y in X with $b\,\mathrm{Cl}(\{x\}) \neq b\,\mathrm{Cl}(\{y\})$, there exist disjoint b-open sets U and V such that $b\,\mathrm{Cl}(\{x\}) \subset U$ and $b\,\mathrm{Cl}(\{y\}) \subset V$.*

R_0 and R_1 spaces are similarly defined in (see, [7], [13]*).*

Definition 2.2. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be:*

(1) *b-$\mathcal{I}$-T_1 [3] if for each pair of distinct points x and y of X, there exist b-$\mathcal{I}$-open sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.*
(2) *b-$\mathcal{I}$-T_2 [3] if for each pair of distinct points x and y in X, there exist disjoint b-$\mathcal{I}$-open sets U and V in X such that $x \in U$ and $y \in V$.*

3. On b-$\mathcal{I}$-R_0 spaces

Definition 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then the b-$\mathcal{I}$-kernel of A, denoted by $b\mathcal{I}\,\mathrm{Ker}(A)$ is defined to be the set $b\mathcal{I}\,\mathrm{Ker}(A) = \cap\,\{G \in B\mathcal{I}O(X) \mid A \subset G\}$.*

Lemma 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $x \in X$. Then, $y \in b\mathcal{I}\,\mathrm{Ker}(\{x\})$ if and only if $x \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$.*

Proof. Suppose that $y \notin b\mathcal{I}\,\mathrm{Ker}(\{x\})$. Then there exists $U \in B\mathcal{I}O(X, x)$ such that $y \notin U$. Therefore, we have $x \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$. The proof of the converse case can be done similarly. □

Lemma 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A a subset of X. Then, $b\mathcal{I}\operatorname{Ker}(A) = \{x \in X \mid b\mathcal{I}\operatorname{Cl}(\{x\}) \cap A \neq \varnothing\}$.*

Proof. Let $x \in b\mathcal{I}\operatorname{Ker}(A)$ and $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap A = \varnothing$. Hence $x \notin X \setminus b\mathcal{I}\operatorname{Cl}(\{x\})$ which is a b-$\mathcal{I}$-open set containing A. This is impossible, since $x \in b\mathcal{I}\operatorname{Ker}(A)$. Consequently, $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap A \neq \varnothing$. Next, let $x \in X$ such that $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap A \neq \varnothing$ and suppose that $x \notin b\mathcal{I}\operatorname{Ker}(A)$. Then, there exists a b-$\mathcal{I}$-open set U containing A and $x \notin U$. Let $y \in b\mathcal{I}\operatorname{Cl}(\{x\}) \cap A$. Hence, U is a b-$\mathcal{I}$-neighbourhood of y which does not contains x. By this contradiction $x \in b\mathcal{I}\operatorname{Ker}(A)$ and hence the claim. □

Definition 3.4. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be a b-$\mathcal{I}$-R_0 space if every b-$\mathcal{I}$-open set contains the b-$\mathcal{I}$-closure of each of its singletons.*

Remark 3.5. *Since an ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1 if and only if the singletones are b-$\mathcal{I}$-closed* [3]*, it is clear that every b-$\mathcal{I}$-T_1 space b-$\mathcal{I}$-R_0. But the converse is not true in general.*

Example 3.6. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0 but none of b-$\mathcal{I}$-T_0 and b-$\mathcal{I}$-T_1.*

Remark 3.7. *Th following example and Examlpe 3.6 shows that the notions b-$\mathcal{I}$-T_0-ness b-$\mathcal{I}$-R_0-ness are independent.*

Example 3.8. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0 but not b-$\mathcal{I}$-R_0.*

Proposition 3.9. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0 space;*
(2) *For any $F \in B\mathcal{I}C(X)$, $x \notin F$ implies $F \subset U$ and $x \notin U$ for some $U \in B\mathcal{I}O(X)$;*
(3) *For any $F \in B\mathcal{I}C(X)$, $x \notin F$ implies $F \cap b\mathcal{I}\operatorname{Cl}(\{x\}) = \varnothing$;*
(4) *For any distinct points x and y of X, either $b\mathcal{I}\operatorname{Cl}(\{x\}) = b\mathcal{I}\operatorname{Cl}(\{y\})$ or $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap b\mathcal{I}\operatorname{Cl}(\{y\}) = \varnothing$.*

Proof. (1)⇒(2): Let $F \in B\mathcal{I}C(X)$ and $x \notin F$. Then by (1) $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset X \setminus F$. Set $U = X \setminus b\mathcal{I}\operatorname{Cl}(\{x\})$, then $U \in B\mathcal{I}O(X)$, $F \subset U$ and $x \notin U$. (2)⇒(3): Let $F \in B\mathcal{I}C(X)$ and $x \notin F$. There exists $U \in B\mathcal{I}O(X)$ such that $F \subset U$ and $x \notin U$. Since $U \in B\mathcal{I}O(X)$, $U \cap b\mathcal{I}\operatorname{Cl}(\{x\}) = \varnothing$ and $F \cap b\mathcal{I}\operatorname{Cl}(\{x\}) = \varnothing$. (3)⇒(4): Suppose that $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$ for distinct points $x, y \in X$. There exists $z \in b\mathcal{I}\operatorname{Cl}(\{x\})$ such that $z \notin b\mathcal{I}\operatorname{Cl}(\{y\})$ (or $z \in b\mathcal{I}\operatorname{Cl}(\{y\})$ such that $z \notin b\mathcal{I}\operatorname{Cl}(\{x\})$). There exists $V \in B\mathcal{I}O(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin b\mathcal{I}\operatorname{Cl}(\{y\})$. By (3), we obtain $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap b\mathcal{I}\operatorname{Cl}(\{y\}) = \varnothing$. The proof for otherwise is similar. (4)⇒(1): Let $V \in$

$B\mathcal{I}O(X,x)$. For each $y \notin V$, $x \neq y$ and $x \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$. This shows that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. By (4), $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \cap b\mathcal{I}\,\mathrm{Cl}(\{y\}) = \varnothing$ for each $y \in X \setminus V$ and hence $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \cap (\cup_{y\in X\setminus V}\, b\mathcal{I}\,\mathrm{Cl}(\{y\})) = \varnothing$. On other hand, since $V \in B\mathcal{I}O(X)$ and $y \in X \setminus V$, we have $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset X \setminus V$ and hence $X \setminus V = \cup_{\,y\in X\setminus V}\, b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Therefore, we obtain $(X\setminus V) \cap b\mathcal{I}\,\mathrm{Cl}(\{x\}) = \varnothing$ and $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset V$. This shows that $(X,\tau,\mathcal{I})$ is a b-$\mathcal{I}$-R_0 space. □

Theorem 3.10. *An ideal topological space $(X,\tau,\mathcal{I})$ is a b-$\mathcal{I}$-R_0 space if and only if for any x and y in X, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$ implies $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \cap b\mathcal{I}\,\mathrm{Cl}(\{y\}) = \varnothing$.*

Proof. Suppose that $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-R_0 and x, $y \in X$ such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Then, there exists $z \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$ such that $z \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$ (or $z \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$) such that $z \notin b\mathcal{I}\,\mathrm{Cl}(\{x\})$. There exists $V \in B\mathcal{I}O(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Thus $x \in X \setminus b\mathcal{I}\,\mathrm{Cl}(\{y\}) \in B\mathcal{I}O(X)$, which implies $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset X \setminus b\mathcal{I}\,\mathrm{Cl}(\{y\})$ and $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \cap b\mathcal{I}\,\mathrm{Cl}(\{y\}) = \varnothing$. The proof for otherwise is similar. Conversely, let $V \in B\mathcal{I}O(X,x)$. We will show that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset V$. Let $y \in X \setminus V$. Then $x \neq y$ and $x \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$. This shows that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. By assumption, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \cap b\mathcal{I}\,\mathrm{Cl}(\{y\}) = \varnothing$. Hence $y \notin b\mathcal{I}\,\mathrm{Cl}(\{x\})$ and therefore $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset V$. □

Lemma 3.11. *The following statements are equivalent for any points x and y in an ideal topological space $(X,\tau,\mathcal{I})$:*

(1) $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \neq b\mathcal{I}\,\mathrm{Ker}(\{y\})$;
(2) $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$.

Proof. (1)⇒(2): Suppose that $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \neq b\mathcal{I}\,\mathrm{Ker}(\{y\})$, then there exists a point z in X such that $z \in b\mathcal{I}\,\mathrm{Ker}(\{x\})$ and $z \notin b\mathcal{I}\,\mathrm{Ker}(\{y\})$. It follows from $z \in b\mathcal{I}\,\mathrm{Ker}(\{x\})$ that $\{x\} \cap b\mathcal{I}\,\mathrm{Cl}(\{z\}) \neq \varnothing$. This implies that $x \in b\mathcal{I}\,\mathrm{Cl}(\{z\})$. By $z \notin b\mathcal{I}\,\mathrm{Ker}(\{y\})$, we have $\{y\} \cap b\mathcal{I}\,\mathrm{Cl}(\{z\}) = \varnothing$. Since $x \in b\mathcal{I}\,\mathrm{Cl}(\{z\})$, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset b\mathcal{I}\,\mathrm{Cl}(\{z\})$ and $\{y\} \cap b\mathcal{I}\,\mathrm{Cl}(\{x\}) = \varnothing$. Therefore, it follows that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Now $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \neq b\mathcal{I}\,\mathrm{Ker}(\{y\})$ implies that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$.
(2)⇒(1): Suppose that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Then there exists a point z in X such that $z \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$ and $z \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Then, there exists a b-$\mathcal{I}$-open set containing z and therefore x but not y, namely, $y \notin b\mathcal{I}\,\mathrm{Ker}(\{x\})$ and thus $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \neq b\mathcal{I}\,\mathrm{Ker}(\{y\})$. □

Theorem 3.12. *An ideal topological space $(X,\tau,\mathcal{I})$ is a b-$\mathcal{I}$-R_0 space if and only if for any pair of points x and y in X, $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \neq b\mathcal{I}\,\mathrm{Ker}(\{y\})$ implies $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \cap b\mathcal{I}\,\mathrm{Ker}(\{y\}) = \varnothing$.*

Proof. Suppose that $(X,\tau,\mathcal{I})$ is a b-$\mathcal{I}$-R_0 space. Thus by Lemma 3.11, for any points x and y in X if $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \neq b\mathcal{I}\,\mathrm{Ker}(\{y\})$, then $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Now we prove that $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \cap b\mathcal{I}\,\mathrm{Ker}(\{y\})$

$= \varnothing$. Assume that $z \in b\mathcal{I}\operatorname{Ker}(\{x\}) \cap b\mathcal{I}\operatorname{Ker}(\{y\})$. By $z \in b\mathcal{I}\operatorname{Ker}(\{x\})$ and Lemma 3.11, it follows that $x \in b\mathcal{I}\operatorname{Cl}(\{z\})$. Since $x \in b\mathcal{I}\operatorname{Cl}(\{x\})$, by Theorem 3.10 $b\mathcal{I}\operatorname{Cl}(\{x\}) = b\mathcal{I}\operatorname{Cl}(\{z\})$. Similarly, we have $b\mathcal{I}\operatorname{Cl}(\{y\}) = b\mathcal{I}\operatorname{Cl}(\{z\}) = b\mathcal{I}\operatorname{Cl}(\{x\})$. This is a contradiction. Therefore, we have $b\mathcal{I}\operatorname{Ker}(\{x\}) \cap b\mathcal{I}\operatorname{Ker}(\{y\}) = \varnothing$. Conversely, let $(X, \tau, \mathcal{I})$ be an ideal topological space such that for any points x and y in X, $b\mathcal{I}\operatorname{Ker}(\{x\}) \neq b\mathcal{I}\operatorname{Ker}(\{y\})$ implies $b\mathcal{I}\operatorname{Ker}(\{x\}) \cap b\mathcal{I}\operatorname{Ker}(\{y\}) = \varnothing$. If $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$, then by Lemma 3.2, $b\mathcal{I}\operatorname{Ker}(\{x\}) \neq b\mathcal{I}\operatorname{Ker}(\{y\})$. Hence, $b\mathcal{I}\operatorname{Ker}(\{x\}) \cap b\mathcal{I}\operatorname{Ker}(\{y\}) = \varnothing$ which implies $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap b\mathcal{I}\operatorname{Cl}(\{y\}) = \varnothing$. Because $z \in b\mathcal{I}\operatorname{Cl}(\{x\})$ implies that $x \in b\mathcal{I}\operatorname{Ker}(\{z\})$ and therefore $b\mathcal{I}\operatorname{Ker}(\{x\}) \cap b\mathcal{I}\operatorname{Ker}(\{y\}) \neq \varnothing$. By hypothesis, we have $b\mathcal{I}\operatorname{Ker}(\{x\}) = b\mathcal{I}\operatorname{Ker}(\{z\})$. Then $z \in b\mathcal{I}\operatorname{Cl}(\{x\}) \cap b\mathcal{I}\operatorname{Cl}(\{y\})$ implies that $b\mathcal{I}\operatorname{Ker}(\{x\}) = b\mathcal{I}\operatorname{Ker}(\{z\}) = b\mathcal{I}\operatorname{Ker}(\{y\})$. This is a contradiction. Therefore, $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap b\mathcal{I}\operatorname{Cl}(\{y\}) = \varnothing$ and by Theorem 3.10 $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-R_0 space. □

Theorem 3.13. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-R_0 space;*
(2) *For any nonempty sets A, $G \in B\mathcal{I}O(X)$ such that $A \cap G \neq \varnothing$, there exists $F \in B\mathcal{I}C(X)$ such that $A \cap F \neq \varnothing$ and $F \subset G$;*
(3) *Any $G \in B\mathcal{I}O(X)$, $G = \cup\{F \in B\mathcal{I}C(X) \mid F \subset G\}$;*
(4) *Any $F \in B\mathcal{I}C(X)$, $F = \cap\{G \in B\mathcal{I}O(X) \mid F \subset G\}$;*
(5) *For any $x \in X$, $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset b\mathcal{I}\operatorname{Ker}(\{x\})$.*

Proof. (1)⇒(2): Let A be a nonempty set of X and $G \in B\mathcal{I}O(X)$ such that $A \cap G \neq \varnothing$. There exists $x \in A \cap G$. Since $x \in G \in B\mathcal{I}O(X)$, $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset G$. Set $F = b\mathcal{I}\operatorname{Cl}(\{x\})$, then $F \in B\mathcal{I}C(X)$, $F \subset G$ and $A \cap F \neq \varnothing$. (2)⇒(3): Let $G \in B\mathcal{I}O(X)$, then $G \supset \cup\{F \in B\mathcal{I}C(X) \mid F \subset G\}$. Let x be any point of G. There exists $F \in B\mathcal{I}C(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in B\mathcal{I}C(X) \mid F \subset G\}$ and hence $G = \cup\{F \in B\mathcal{I}C(X) \mid F \subset G\}$. (3)⇒(4): This is obvious. (4)⇒(5): Let x be any point of X and $y \notin b\mathcal{I}\operatorname{Ker}(\{x\})$. There exists $V \in B\mathcal{I}O(X, x)$ $y \notin V$; hence $b\mathcal{I}\operatorname{Cl}(\{y\}) \cap V = \varnothing$. By (4) $(\cap\{G \in B\mathcal{I}O(X) \mid b\mathcal{I}\operatorname{Cl}(\{y\}) \subset G\}) \cap V = \varnothing$ and there exists $G \in B\mathcal{I}O(X)$ such that $x \notin G$ and $b\mathcal{I}\operatorname{Cl}(\{y\}) \subset G$. Therefore, $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap G = \varnothing$ and $y \notin b\mathcal{I}\operatorname{Cl}(\{x\})$. Consequently, we obtain $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset b\mathcal{I}\operatorname{Ker}(\{x\})$. (5)⇒(1): Let $G \in B\mathcal{I}O(X, x)$. Let $y \in b\mathcal{I}\operatorname{Ker}(\{x\})$, then $x \in b\mathcal{I}\operatorname{Cl}(\{y\})$ and $y \in G$. This implies that $b\mathcal{I}\operatorname{Ker}(\{x\}) \subset G$. Therefore, we obtain $x \in b\mathcal{I}\operatorname{Cl}(\{x\}) \subset b\mathcal{I}\operatorname{Ker}(\{x\}) \subset G$. This shows that $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-R_0 space. □

Corollary 3.14. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-R_0 space;*

(2) $b\mathcal{I}\operatorname{Cl}(\{x\}) = b\mathcal{I}\operatorname{Ker}(\{x\})$ *for all* $x \in X$.

Proof. (1)⇒(2): Suppose that $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-R_0 space. By Theorem 3.13, $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset b\mathcal{I}\operatorname{Ker}(\{x\})$ for each $x \in X$. Let $y \in b\mathcal{I}\operatorname{Ker}(\{x\})$, then $x \in b\mathcal{I}\operatorname{Cl}(\{y\})$ and by Theorem 3.10 $b\mathcal{I}\operatorname{Cl}(\{x\}) = b\mathcal{I}\operatorname{Cl}(\{y\})$. Therefore, $y \in b\mathcal{I}\operatorname{Cl}(\{x\})$ and hence $b\mathcal{I}\operatorname{Ker}(\{x\}) \subset b\mathcal{I}\operatorname{Cl}(\{x\})$. This shows that $b\mathcal{I}\operatorname{Cl}(\{x\}) = b\mathcal{I}\operatorname{Ker}(\{x\})$. (2)⇒(1): This is obvious by Theorem 3.13. □

Corollary 3.15. *If for any point x of a b-$\mathcal{I}$-R_0 space $(X, \tau, \mathcal{I})$, $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap b\mathcal{I}\operatorname{Ker}(\{x\}) = \{x\}$, then $b\mathcal{I}\operatorname{Ker}(\{x\}) = \{x\}$.*

Proof. The proof follows from Theorem 3.13(v). □

Theorem 3.16. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-R_0 space;*
(2) *$x \in b\mathcal{I}\operatorname{Cl}(\{y\})$ if and only if $y \in b\mathcal{I}\operatorname{Cl}(\{x\})$ for any points x and y in X.*

Proof. (1)⇒(2): Assume that $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0. Let $x \in b\mathcal{I}\operatorname{Cl}(\{y\})$ and $A \in B\mathcal{I}O(X, y)$. Now by hypothesis, $x \in A$. Therefore, every b-$\mathcal{I}$-open set containing y contains x. Hence $y \in b\mathcal{I}\operatorname{Cl}(\{x\})$. (2)⇒(1): Let $U \in B\mathcal{I}O(X, x)$. If $y \notin U$, then $x \notin b\mathcal{I}\operatorname{Cl}(\{y\})$ and hence $y \notin b\mathcal{I}\operatorname{Cl}(\{x\})$. This implies that $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset U$. Hence $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0. □

Theorem 3.17. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) *$(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-R_0 space;*
(2) *If F is a b-$\mathcal{I}$-closed subset of X, then $F = b\mathcal{I}\operatorname{Ker}(F)$;*
(3) *If F is a b-$\mathcal{I}$-closed subset of X and $x \in F$, then $b\mathcal{I}\operatorname{Ker}(\{x\}) \subset F$;*
(4) *If $x \in X$, then $b\mathcal{I}\operatorname{Ker}(\{x\}) \subset b\mathcal{I}\operatorname{Cl}(\{x\})$.*

Proof. (1)⇒(2): Let F be b-$\mathcal{I}$-closed subset of X and $x \notin F$. Thus $X \backslash F \in B\mathcal{I}O(X, x)$. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0, $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset X \setminus F$. Thus $b\mathcal{I}\operatorname{Cl}(\{x\}) \cap F = \varnothing$ and Lemma 3.3 $x \notin b\mathcal{I}\operatorname{Ker}(F)$. Therefore, $b\mathcal{I}\operatorname{Ker}(F) = F$. (2)⇒(3): In general, $A \subset B$ implies $b\mathcal{I}\operatorname{Ker}(A) \subset b\mathcal{I}\operatorname{Ker}(B)$. Therefore, it follows from (2) that$b\mathcal{I}\operatorname{Ker}(\{x\}) \subset b\mathcal{I}\operatorname{Ker}(F) = F$. (3)⇒(4): Since $x \in b\mathcal{I}\operatorname{Cl}(\{x\})$ and $b\mathcal{I}\operatorname{Cl}(\{x\})$ is b-$\mathcal{I}$-closed, by (3) $b\mathcal{I}\operatorname{Ker}(\{x\}) \subset b\mathcal{I}\operatorname{Cl}(\{x\})$. (4)⇒(1): We show the implication by using Theorem 3.16. Let $x \in b\mathcal{I}\operatorname{Cl}(\{y\})$. Then by Lemma 3.2 $y \in b\mathcal{I}\operatorname{Ker}(\{x\})$. Since $x \in b\mathcal{I}\operatorname{Cl}(\{x\})$ and $b\mathcal{I}\operatorname{Cl}(\{x\})$ is b-$\mathcal{I}$-closed, by (4) we obtain $y \in b\mathcal{I}\operatorname{Ker}(\{x\}) \subset b\mathcal{I}\operatorname{Cl}(\{x\})$. Therefore, $x \in b\mathcal{I}\operatorname{Cl}(\{x\})$ implies $y \in b\mathcal{I}\operatorname{Cl}(\{x\})$. The converse is obvious and $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0. □

Definition 3.18. *A filterbase $\mathcal{F}$ is called b-$\mathcal{I}$-convergent to a point x in X, if for any $U \in B\mathcal{I}O(X,x)$, there exists $B \in \mathcal{F}$ such that B is a subset of U.*

Lemma 3.19. *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and let x and y be any two points in X such that every net in X b-$\mathcal{I}$-converging to y b-$\mathcal{I}$-converges to x. Then $x \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n\in N}$ is a net in $b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Since $\{x_n\}_{n\in N}$ b-$\mathcal{I}$-converges to y, then $\{x_n\}_{n\in N}$ b-$\mathcal{I}$-converges to x and this implies that $x \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$. □

Theorem 3.20. *For an ideal topological space $(X,\tau,\mathcal{I})$, the following statements are equivalent:*

(1) *$(X,\tau,\mathcal{I})$ is a b-$\mathcal{I}$-R_0 space;*
(2) *If x, $y \in X$, then $y \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$ if and only if every net in X b-$\mathcal{I}$-converging to y b-$\mathcal{I}$-converges to x.*

Proof. (1)⇒(2): Let x, $y \in X$ such that $y \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha\in N}$ be a net in X such that $\{x_\alpha\}_{\alpha\in N}$ b-$\mathcal{I}$-converges to y. Since $y \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$, by Theorem 3.10 we have $b\mathcal{I}\,\mathrm{Cl}(\{x\}) = b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Therefore $x \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in N}$ b-$\mathcal{I}$-converges to x. Conversely, let x, $y \in X$ such that every net in X b-$\mathcal{I}$-converging to y b-$\mathcal{I}$-converges to x. Then $x \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$ by Lemma 3.3. By Theorem 3.10, we have $b\mathcal{I}\,\mathrm{Cl}(\{x\}) = b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Therefore $y \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$. (2)⇒(1): Assume that x and y are any two points of X such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \cap b\mathcal{I}\,\mathrm{Cl}(\{y\}) \neq \varnothing$. Let $z \in b\mathcal{I}\,\mathrm{Cl}(\{x\}) \cap b\mathcal{I}\,\mathrm{Cl}(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in N}$ in $b\mathcal{I}\,\mathrm{Cl}(\{x\})$ such that $\{x_\alpha\}_{\alpha\in N}$ b-$\mathcal{I}$-converges to z. Since $z \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$, then $\{x_\alpha\}_{\alpha\in N}$ b-$\mathcal{I}$-converges to y. It follows that $y \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$. By the same token we obtain $x \in b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Therefore $b\mathcal{I}\,\mathrm{Cl}(\{x\}) = b\mathcal{I}\,\mathrm{Cl}(\{y\})$ and by Theorem 3.10 $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-R_0. □

4. On b-$\mathcal{I}$-R_1 spaces

Definition 4.1. *An ideal topological space $(X,\tau,\mathcal{I})$ is said to be b-$\mathcal{I}$-R_1 if for x, y in X with $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$, there exist disjoint b-$\mathcal{I}$-open sets U and V such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset U$ and $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset V$.*

Proposition 4.2. *If $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-R_1, then it is b-$\mathcal{I}$-R_0.*

Proof. Let $U \in B\mathcal{I}O(X,x)$. If $y \notin U$, then since $x \notin b\mathcal{I}\,\mathrm{Cl}(\{y\})$, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Hence there exists a b-$\mathcal{I}$-open V_y such that $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin b\mathcal{I}\,\mathrm{Cl}(\{x\})$. Thus $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset U$. Therefore $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-R_0. □

Theorem 4.3. *An ideal topological space $(X,\tau,\mathcal{I})$ is b-$\mathcal{I}$-R_1 if and only if for x, $y \in X$, $b\mathcal{I}\,\mathrm{Ker}(\{x\}) \neq b\mathcal{I}\,\mathrm{Ker}(\{y\})$, there exist disjoint b-$\mathcal{I}$-open sets U and V such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset U$ and $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset V$.*

Proof. It follows from Lemma 3.11 □

Theorem 4.4. *The following proprties are equivalent:*

(1) $(X, \tau, \mathcal{I})$ *is* b-$\mathcal{I}$-T_2,
(2) $(X, \tau, \mathcal{I})$ *is* b-$\mathcal{I}$-R_1 *and* b-$\mathcal{I}$-T_1, *and*
(3) $(X, \tau, \mathcal{I})$ *is* b-$\mathcal{I}$-R_1 *and* b-$\mathcal{I}$-T_0.

Proof. (1)⇒(2): Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2, then it is b-$\mathcal{I}$-T_1. If $x, y \in X$ such that $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$, then $x \neq y$ and there exist disjoint b-$\mathcal{I}$-open sets U and V such that $x \in U$ and $y \in V$ and $b\mathcal{I}\operatorname{Cl}(\{x\}) = \{x\} \subset U$ and $b\mathcal{I}\operatorname{Cl}(\{y\}) = \{y\} \subset V$. Hence $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_1. (2)⇒(3): Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_0. (3)⇒(1): Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_1, and b-$\mathcal{I}$-T_1, then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0 and b-$\mathcal{I}$-T_0, which implies $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_1. Let x, $y \in X$ such that $x \neq y$. Since $b\mathcal{I}\operatorname{Cl}(\{x\}) = \{x\} \neq \{y\} = b\mathcal{I}\operatorname{Cl}(\{y\})$, then there exist disjoint b-$\mathcal{I}$-open sets U and V such that $x \in U$ and $y \in V$. Hence, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2. □

Theorem 4.5. *The following properties are eqiuvalent:*

(1) $(X, \tau, \mathcal{I})$ *is* b-$\mathcal{I}$-R_1,
(2) *for each* $x, y \in X$ *one of the following holds:*
  (a) *If* U *is* b-$\mathcal{I}$-*open, then* $x \in U$ *if and only if* $y \in U$.
  (b) *there exist disjoint* b-$\mathcal{I}$-*open sets* U *and* V *such that* $x \in U$ *and* $y \in V$, *and*
(3) *If* $x, y \in X$ *such that* $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$, *then there exist* b-$\mathcal{I}$-*closed sets* F_1 *and* F_2 *such that* $x \in F_1$, $y \notin F_2$, $y \in F_1$, $x \notin F_2$, *and* $X = F_1 \cup F_2$.

Proof. (1)⇒(2): Let $x, y \in X$. Then $b\mathcal{I}\operatorname{Cl}(\{x\}) = b\mathcal{I}\operatorname{Cl}(\{y\})$ or $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$. If $b\mathcal{I}\operatorname{Cl}(\{x\}) = b\mathcal{I}\operatorname{Cl}(\{y\})$ and U is b-$\mathcal{I}$-open, then $x \in U$ implies $y \in b\mathcal{I}\operatorname{Cl}(\{x\}) \subset U$ and $y \in U$ implies $x \in b\mathcal{I}\operatorname{Cl}(\{y\}) \subset U$. Thus consider the case that $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$. Then there exist disjoint b-$\mathcal{I}$-open sets U and V such that $x \in b\mathcal{I}\operatorname{Cl}(\{x\}) \subset U$ and $y \in b\mathcal{I}\operatorname{Cl}(\{y\}) \subset V$. (2)⇒(3): Let $x, y \in X$ such that $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$. Then $x \notin b\mathcal{I}\operatorname{Cl}(\{y\})$ or $y \notin b\mathcal{I}\operatorname{Cl}(\{x\})$, say $x \notin b\mathcal{I}\operatorname{Cl}(\{y\})$. Then there exist a b-$\mathcal{I}$-open set A such that $x \in$ A and $y \notin A$, which implies there exist disjoint b-$\mathcal{I}$-open sets U and V such that $x \in U$ and $y \in V$. Then $F_1 = X \setminus V$ and $F_2 = X \setminus U$ are b-$\mathcal{I}$-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$. (3)⇒(1): Let U be b-$\mathcal{I}$-open and let $x \in U$. Then $b\mathcal{I}\operatorname{Cl}(\{x\}) \subset U$, for suppose not. Let $y \in b\mathcal{I}\operatorname{Cl}(\{x\}) \cap (X \setminus U)$. Then $b\mathcal{I}\operatorname{Cl}(\{x\}) \neq b\mathcal{I}\operatorname{Cl}(\{y\})$ and there exist b-$\mathcal{I}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$. Then $y \in F_2 \setminus F_1 = X \setminus F_1$, which is b-$\mathcal{I}$-open, and $x \notin X \setminus F_1$, which is a contradiction. Hence, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0. Let $a, b \in X$ such that $b\mathcal{I}\operatorname{Cl}(\{a\}) \neq b\mathcal{I}\operatorname{Cl}(\{b\})$. Then there exist b-$\mathcal{I}$-closed sets A_1 and A_2 such that $a \in A_1$, $b \notin A_1$, $a \notin A_2$, and $X = A_1 \cup A_2$. Thus $a \in A_1 \setminus A_2$ and $b \in A_2 \setminus A_1$, which are b-$\mathcal{I}$-open, which implies $b\mathcal{I}\operatorname{Cl}(\{a\}) \subset A_1 \setminus A_2$ and $b\mathcal{I}\operatorname{Cl}(\{b\}) \subset A_2 \setminus A_1$. Hence, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_1. □

Theorem 4.6. *An ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2 if and only if for x, $y \in X$ such that $x \neq y$, there exist b-$\mathcal{I}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \notin F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.*

Proof. The straightforward proof is omitted. □

Remark 4.7. *If $\{x_\lambda\}_{\lambda \in A}$ is a net in $(X, \tau, \mathcal{I})$, $b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A}) = \{x \in X : \{x_\lambda\}_{\lambda\in A}$b-$\mathcal{I}$-converges to $x\}$.*

Theorem 4.8. *The following properties are eqiuvalent:*

(1) *$(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_1;*
(2) *for $x, y \in X$, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) = b\mathcal{I}\,\mathrm{Cl}(\{y\})$, whenever there exists a net $\{x_\lambda\}_{\lambda\in A}$ such that $x, y \in b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A})$;*
(3) *$(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0, and for every b-$\mathcal{I}$-convergent net $\{x_\lambda\}_{\lambda\in A}$ in X, $b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A}) = b\mathcal{I}\,\mathrm{Cl}(\{x\})$ for some $x \in X$.*

Proof. (1)⇒(2): Let $x, y \in X$ such that there exists a net $\{x_\lambda\}_{\lambda\in A}$ in X such that $x, y \in b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A})$. Then (a) if U is b-$\mathcal{I}$-open, then $x \in U$ if and only if $y \in U$ or (b) there exist disjoint b-$\mathcal{I}$-open sets U and V such that $x \in U$ and $y \in V$. Since $x, y \in b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A})$, then (1) is satisfied, which implies $b\mathcal{I}\,\mathrm{Cl}(\{x\}) = b\mathcal{I}\,\mathrm{Cl}(\{y\})$. (2)⇒(3): Let $U \in BIO(X, x)$. Let $y \notin U$. For each $n \in N$ let $x_n = x$. Then $\{x_n\}_{n\in N}$ b-$\mathcal{I}$-converges to x and since $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$, that $y \in A$ and $x \notin A$. Thus, $y \notin b\mathcal{I}\,\mathrm{Cl}(\{x\})$ and $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset U$. Hence $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0. Let $\{x_\lambda\}_{\lambda\in A}$ be a b-$\mathcal{I}$-convergent net in X. Let $x \in X$ such that $\{x_\lambda\}_{\lambda\in A}$ b-$\mathcal{I}$-converges to x. If $y \in b\mathcal{I}\,\mathrm{Cl}(\{x\})$, then $\{x_\lambda\}_{\lambda\in A}$ b-$\mathcal{I}$-converges to y, which implies $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A})$ and if $y \in b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A})$, then $x, y \in b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A})$, which implies $y \in b\mathcal{I}\,\mathrm{Cl}(\{y\}) = b\mathcal{I}\,\mathrm{Cl}(\{x\})$. Hence $b\mathcal{I}\lim(\{x_\lambda\}_{\lambda\in A}) = b\mathcal{I}\,\mathrm{Cl}(\{x\})$. (3)⇒(1): Assume that $(X, \tau, \mathcal{I})$ is not b-$\mathcal{I}$-R_1. Then there exists $x, y \in X$ such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$ and every b-$\mathcal{I}$-open set containing $b\mathcal{I}\,\mathrm{Cl}(\{x\})$ intersects every b-$\mathcal{I}$-open set containing $b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_0, then every b-$\mathcal{I}$-open set containing x contains $b\mathcal{I}\,\mathrm{Cl}(\{x\})$ and every b-$\mathcal{I}$-open set containing y contains $b\mathcal{I}\,\mathrm{Cl}(\{y\})$, which implies that every b-$\mathcal{I}$-open set containing x intersects every b-$\mathcal{I}$-open set containing y. Let $D_x = \{U \subset X : U \in BIO(X, x)\}$. Let $\geq_x$ be the binary relation on D_x defined by $U_1 \geq_x U_2$ if and only if $U_1 \subset U_2$. Then, clearly $(D_x, \geq_x)$ is a directed set. Let $D_y = \{U \subset X : U \in BIO(X, y)\}$ and let $\geq_y$ be the binary relation on D_y defined by $U_1 \geq_y U_2$ if and only if $U_1 \subset U_2$. Then, $(D_y, \geq_y)$ is also a directed set. Let $D = \{(U_1, U_2) : U_1 \in D_x$ and $U_2 \in D_y\}$ and let $\geq$ be the binary relation on D defined by $(U_1, U_2) \geq (V_1, V_2)$ if and only if $U_1 \geq_x V_1$ and $U_2 \geq_y V_2$. Then, $(D, \geq)$ is a directed set. For each $(U_1, U_2) \in D$, let $x_{(U_1,U_2)} \in (U_1, U_2)$. Then $\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}$ is a net in X that b-$\mathcal{I}$-converges to both x and y. Thus, there exists $z \in X$ such that $b\mathcal{I}\lim(\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}) = b\mathcal{I}\,\mathrm{Cl}(\{z\})$, which implies $x, y \in b\mathcal{I}\,\mathrm{Cl}(\{z\})$. Since $\{b\mathcal{I}\,\mathrm{Cl}(\{w\}) : w \in X\}$ is a decomposition of X, then $b\mathcal{I}\,\mathrm{Cl}(\{x\}) = b\mathcal{I}\,\mathrm{Cl}(\{z\}) = b\mathcal{I}\,\mathrm{Cl}(\{y\})$, which is a contradiction. Hence $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_1. □

Theorem 4.9. *An ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2 if and only if every b-$\mathcal{I}$-convergent net in X b-$\mathcal{I}$-converges to a unique point.*

Proof. The proof follows from Theorems 4.8 4.4. □

Theorem 4.10. *The following properties are eqiuvalent:*

(1) *$(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_1,*
(2) *for each pair $x, y \in X$, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$, there exists a b-$\mathcal{I}$-open, b-$\mathcal{I}$-closed set V such and $y \notin V$;*
(3) *for each pair $x, y \in X$, $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$, there exists a b-$\mathcal{I}$-continuous function $f : (X, \tau, \mathcal{I}) \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.*

Proof. (1)⇒(2): Let $x, y \in X$ such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Then there exist disjoint b-$\mathcal{I}$-open sets U and W such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset U$ and $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset W$ and $V = b\mathcal{I}\,\mathrm{Cl}(U)$ is b-$\mathcal{I}$-open, b-$\mathcal{I}$-closed such that $x \in V$ and $y \notin V$. (2)⇒(3): Let $x, y \in X$ such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Let V be b-$\mathcal{I}$-open, b-$\mathcal{I}$-closed set of X such that $x \in V$ and $y \notin V$. Thus, the function $f : (X, \tau, \mathcal{I}) \to [0, 1]$ defined by $f(z) = 0$ if $z \in V$ and $f(z) = 1$ if $z \notin V$ satisfies the desired properties. (3)⇒(1): Let $x, y \in X$ such that $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \neq b\mathcal{I}\,\mathrm{Cl}(\{y\})$. Let $f : (X, \tau, \mathcal{I}) \to [0, 1]$ such that f is b-$\mathcal{I}$-continuous, $f(x) = 0$ and $f(y) = 1$. Then $U = f^{-1}([0, 0.5))$ and $V = f^{-1}((0.5, 1])$ are disjoint such that b-$\mathcal{I}$-open, b-$\mathcal{I}$-closed set of X and $b\mathcal{I}\,\mathrm{Cl}(\{x\}) \subset U$ and $b\mathcal{I}\,\mathrm{Cl}(\{y\}) \subset V$. □

Theorem 4.11. *The following properties are eqiuvalent:*

(1) *$(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-R_1,*
(2) *for each pair $x, y \in X$, $x \neq y$, there exists a b-$\mathcal{I}$-open, b-$\mathcal{I}$-closed set V such and $y \notin V$;*
(3) *for each pair $x, y \in X$, $x \neq y$, there exists a b-$\mathcal{I}$-continuous function $f : (X, \tau, \mathcal{I}) \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.*

Proof. The proof is similar to that of Theorem 4.10. □

Theorem 4.12. *(1). An ideal topological space $(X, \tau, \{\varnothing\})$ is b-$\mathcal{I}$-R_0 (resp. b-$\mathcal{I}$-R_1) if and only if it is b-R_0 (resp. b-R_1).*
(2). An ideal topological space $(X, \tau, \mathcal{N})$ is b-$\mathcal{I}$-R_0 (resp. b-$\mathcal{I}$-R_1) if and only if it is b-$\mathcal{I}$-R_0 (resp. b-R_1) ($\mathcal{N}$ is the ideal of all nowhere dense sets).
(3). An ideal topological space $(X, \tau, \mathcal{P}(X))$ is b-$\mathcal{I}$-R_0 (resp. b-$\mathcal{I}$-R_1) if and only if it is R_0 (resp. R_1).

Proof. It follows from Proposition 2 of [2]. □

Remark 4.13. *In the following diagram we denote by arrows the implications between the seperation axioms which we have introduced and* *discussed in this paper and examples show that no other implications hold between them:*

$$\begin{array}{ccc} R_0 & \leftarrow & R_1 \\ \downarrow & & \downarrow \\ b\text{-}\mathcal{I}\text{-}R_0 & \leftarrow & b\text{-}\mathcal{I}\text{-}R_1 \\ \downarrow & & \downarrow \\ b\text{-}R_0 & \leftarrow & b\text{-}R_1 \end{array}$$

Example 4.14. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}\}$. *Then* $(X, \tau, \mathcal{I})$ *is* b-$\mathcal{I}$-R_0 *but none of* R_0 *and* b-$\mathcal{I}$-R_1.

Example 4.15. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{c\}\}$. *Then* $(X, \tau, \mathcal{I})$ *is* b-$\mathcal{I}$-R_1 *but not* R_1.

Example 4.16. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{b\}\}$. *Then* $(X, \tau, \mathcal{I})$ *is* b-R_0 *but not* b-$\mathcal{I}$-R_0.

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