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Some New Contractive Mapping Theorem in Partially Ordered Metric Spaces

Rita Shukla
Department of Mathematics
RSR-RCET, Kohka, Bhilai
Chhattisgarh, India

Abstract : In this paper ,we establish some coincidence , common fixed point theorems for monotone f-non-decreasing self-mappings satisfying certain rational type contraction in the context of a metric spaces with partial order . these results generalize and extend well known existing results in the literature .

Keyword: Compatible mappings Partially ordered metric spaces ,Weakly compatible mappings .

Introduction : In fixed point theory ,the classical Banach contraction principle plays a valid role to obtain an unique solution of the result ,lot of variety of generalizations of this Banach contraction principle[1] have been taken place in a metric fixed point theory by improving the underlying contraction condition .some contractive conditions in a partially ordered set which guarantee the existence of fixed points have been recently established in [5] and [6]

Preliminaries:

The following definitions are frequently used in results given in upcoming sections .

1) The triple (X.d,≤) is called a partially ordered metric space ,if (X , ≤) is a partially ordered set together with (X , d) is a metric space .
2) The triple (X , d , ≤) is called a partially ordered complete metric space if (X,d) is a complete metric space .
3) Let (X , ≤) be a partially ordered set . A self mapping $f: X \to X$ said to be strictly increasing if f(x) ≼ $f(y)$ for all x, $y \in X$ with x $< y$ and is also said to be strictly decreasing if f(x) > f(y) for all x, $y \in X$ with x $< y$.
4) Let (X , d) is a metric space and A ⊆ X. Then a point x ∈ A is called a common fixed point (coincidence) point of two self mappings f and T if fx = Tx=x (fx=Tx)
5) The two self mapping f and T defined over a subset A of a metric space (X, d) are called commuting iff fTx = Tfx for x ∈ A.
6) Two self mappings f and T defined over A ⊂ X are compatible ,if for any sequence $\{x_n\}$
7) Two self mapping f and T defined over A⊂ X are said to be weakly compatible ,if they commute at their coincidence point i.e. if fx=Tx then fTx =Tfx
8) Let f and T be two self mappings defined over a partially ordered set (X,≼). A mapping T is called a monotone f non-deceasing if fx ≼ fy implies Tx ≼ Ty ,∀ x,y ∈ X
9) Let A be a non-empty subset of a partially ordered set (X, ≼). If very two elements of A are comparable then it is called well ordered set.
10) A partially ordered metric space (X,d,≼) is called an ordered complete ,if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$,one of the following condition holds.
    1. If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \preccurlyeq x$,for all n ∈N that is x=sup$\{x_n\}$
    2. If $\{x_n\}$ is a non increasing sequence in X such that $x_n \to x$ implies $x \preccurlyeq x_n$,for all n ∈N that is x=inf$\{x_n\}$.

MAIN RESULTS :

We prove coincidence ,common fixed point theorem in the context of ordered metric space

Theorem (1) Let (X,d,≼) be a complete partially ordered metric space .suppose that the self mappings f and T on X are continuous T is a monotone f-nondecreasing T(x) =f(X) and satisfying the following condition

$$d(Tx,Ty) \leq \alpha \cdot \frac{d(fx,Tx).d(fy,Ty)}{d(fxfy)} + \beta . d(fx,fy)$$ ---------------------(1)

for all x, y in X with $f(x) \neq f(y)$ are comparable ,where $\alpha, \beta \in [0,1)$ with $0 \leq \alpha + \beta < 1$.

If there exists a point $x_0 \in X$ such that $f(x_0) \preccurlyeq T(x_0)$ and the mapping T and f are compatible then T and f have a coincidence point in X .

Proof : Let $x_0 \in X$ such that $f(x_0) \preccurlyeq T(x_0)$ since from hypotheses , we have $T(X) \subseteq f(X)$ then we can choose a point $x_1 \in X$ such that $fx_1=Tx_0$ but $Tx_1 \in f(X)$ then again there exists another point $x_2 \in X$ such that $fx_2= Tx_1$ by continuing the same way, we can construct a sequence $\{x_n\}$ in X such that $fx_{n+1}= Tx_n$ for all n.

Again ,by hypotheses we have $f(x_0) \preccurlyeq T(x_0) = f(x_1)$ and T is a monotone f-nondecreasing mapping then ,we get $T(x_0) \preccurlyeq T(x_1)$. Similarly ,we obtain $T(x_1) \preccurlyeq T(x_2)$ since $f(x_1) \preccurlyeq f(x_2)$ and then by continuing the same procedure ,we obtain that

$$T(x_0) \preccurlyeq T(x_1) \preccurlyeq T(x_2) \preccurlyeq \text{---------------} \preccurlyeq T(x_n) \preccurlyeq T(x_{n+1})$$

The equality $T(x_{n+1}) = T(x_n)$ is impossible because $f(x_{n+2}) \neq f(x_{n+1})$ for all $n \in N$. Thus

$d(T(x_n) , T(x_{n+1})) > 0$ for all $n \geq 0$.therefore from contraction condition (1) ,we have

$$d(Tx_{n+1},Tx_n) \leq \alpha \frac{d(fx_{n+1},Tx_{n+1}).d(fx_n,Tx_n).}{d(fx_{n+1},fx_n)} + \beta[d(fx_{n+1},fx_n)]$$

$$d(Tx_{n+1},Tx_n) \leq \alpha \frac{d(Tx_n,Tx_{n+1}).d(Tx_{n-1},Tx)}{d(Tx_n,Tx_{n-1})} + \beta d(Tx_n,Tx_{n-1})$$

$$d(Tx_{n+1},Tx_n) \leq \alpha d(Tx_n,Tx_{n+1}) + \beta d(Tx_n,Tx_{n-1})$$

$$(1-\alpha)\, d(Tx_{n+1},Tx_n) \leq \beta d(Tx_n,Tx_{n-1})$$

$$d(Tx_{n+1},Tx_n) \leq \left(\frac{\beta}{1-\alpha}\right) d(Tx_n,Tx_{n-1})$$

continuing the same process up to (n-1) times,

we get

$$d(Tx_{n+1},Tx_n) \leq \left(\frac{\beta}{1-\alpha}\right)^n d(Tx_1, Tx_0)$$

Let $k = \left(\frac{\beta}{1-\alpha}\right) \in [0,1]$ then from triangular

inequality for $m \geq n$ .we have

$$d(Tx_m ,Tx_n) \leq d(Tx_m ,Tx_{m-1}) + d(Tx_{m-1} ,Tx_{m-2}) + d(Tx_{m-1} ,Tx_{m-2}) + \text{--------+}$$

$$d(Tx_{n+1},Tx_n) \leq (k^{m-1} + k^{m-2} + \text{-----------------+} k^n)\, d(Tx_1,Tx_0)$$

$$\leq \left(\frac{k^n}{1-k}\right) d(Tx_1 , Tx_0)$$

as $m, n \to \infty$, $d(Tx_m ,Tx_n) \to 0$ which shows that the sequence $\{Tx_n\}$ is a Cauchy sequence in X . so by the completeness of X , there exist appoint $\mu \in X$ the continuity of T ,we have

$$\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} Tx_n) = T\mu$$

But $fx_{n+1} = T x_n$, then $fx_{n+1} \to \mu$ as $n \to \infty$ and from the compatibility for T and f

$$\lim_{n\to\infty} d(T(fx_n), f(Tx_n)) = 0$$

$$d(T\mu, f\mu) = d(T\mu, Tfx_n) + d(T(fx_n),f(Tx_n)) + d(f(Tx_n),f\mu)$$

on taking limit as $n \to \infty$ in both sides of above equation and using the fact that T and f are continuous then we get $d(T\mu, f\mu) = 0$ thus

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$T\mu = f\mu$ .Hence μ is coincidence point of T and f in X.

Theorem 2 : Let (X, d ,≼) be a complete partially ordered metric space .suppose that f and T are self-mapping on X ,T is a monotone f-non-decreasing , T(X) ⊆ $f(X)$ and satisfying

d(Tx, Ty) ≤ $\alpha .\frac{d(fx,Tx).d(fy,Ty)}{d(fx,fy)} + \beta.d(fx,fy)$ .for all x, y in X with f(x)≠f(y) are compatible and for some $\alpha, \beta \in [0,1]$ with 0≤ $\alpha + \beta < 1$.If there exist a point $x_0 \in X$ such that f(x_0) ≼ $T(x_0)$ and $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$

then $x_n \preccurlyeq x$ for all n∈ N .If f(X) is a complete subset of X Then T and f have a coincidence point in X .Further, if T and f are weakly compatible ,then T and f have a common fixed point in X . Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X .

Proof : Suppose f(X) is a complete subset of X the sequence {Tx_n} is a Cauchy sequence and hence {fx_n} is also a Cauchy sequence in (f(X),d) as fx_{n+1}= Tx_n and T(X) ⊆ $f(X)$.since f(X) is complete then there exists some fu∈ $f(X)$ such that $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} f(x_n)$ = f(u) .The sequences $\{Tx_n\}$ and {fx_n} are non-decreasing and from hypotheses, we have T(x_n)≼ $f(u)$ and f(x_n)≼ $f(u)$ for all n∈ N. But T is a monotone f-nondecreasing that ,we get T(x_n)≼ $T(u)$ for all n. Letting n→ ∞ we obtain that f(u) ≼T(u).

Suppose that f(u) ≺ $T(u)$ then define a sequence $\{u_n\}$ by u_0 = u and fu_{n+1}=Tu_n for all n∈ N.

An argument similar to that in the proof of theorem (1) yields that {fx_n} is a non-decreasing sequence and $\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} T(u_n)$=f(v) for some v∈ X so from hypotheses it is clear that

Sup f(u_n) ≼f(v) and sup T(u_n) ≼ $f(v)$ for all n∈ N.

f(x_n)≼ f(u) ≼ f(u_1)≼----------≼f(u_n) ≼ f(v)

Case I st –Suppose if there exists some $n_0 \geq 1$ such that f(x_{n_0}) = f(u_{n_0}) then, we have

f(x_{n_0}) =f(u) =f(u_{n_0})= f(u_1) = T(u) .

Hence u is a coincidence point of T and f in X .

Case IInd – suppose that f(x_{n_0}) ≠ f(u_{n_0}) for all n then we have

d(fx_{n+1},fu_{n+1}) = d(Tx_n ,Tu_n) ≤ $\alpha.\frac{d(fx_n,Tu_n).d(fu_n,Tu_n)}{d(fx_n,fu_n)} + \beta\, d(fx_n, fu_n)$

taking limit as $n \to \infty$ on both sides of the above inequality ,we get

d(fu, fv) $\leq \alpha.\frac{d(f(u),f(u)).d(f(v),f(v))}{d(f(u),f(v))}$ +β d(f(u),f(v))

$$\leq 0 + \beta d(fu, fv)$$

$$< d(fu, fv) \quad \text{since } \beta < 1$$

Thus we have f(u)=f(v)=f(u_1)= T(u)

Hence ,we conclude that u is a coincidence point of T and f in X .Now suppose that T and f are weakly compatible .Let w be a coincidence point then

T(w) =T(f(z)) = f(T(z)) = f(w)

Since w =T(z) = f(z) for some z∈ X

Now by contraction condition ,we have

d(T(z),T(w)) ≤ $\alpha.\frac{d(fz,Tz).d(fw,Tw)}{d(fz,fw)}$ +$\beta.d(fz,fw)$

$$\leq \beta\, d(T(z), T(w))$$

As < 1 , d(T(z),T(w)) =0

Therefore , T(z) =T(w) =f(w)=w .Hence w is a common fixed point of T and f in X . Now

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suppose that the set of common fixed points of T and f is well ordered ,we have to show that common fixed point of T and f is unique .let u and v be two common fixed points of T and f such that u $\neq v$ then

$$d(u,v) \leq \alpha.\frac{d(fu,Tu).d(fv,Tv)}{d(fu,fv)} + \beta d(fu,fv)$$

$$\leq \beta d(u,v)$$

$$< d(u,v) \quad \text{since } \beta < 1$$

Which is a contradiction , Thus u = v .

Conversely , suppose T and f have only one common fixed point then the set of common fixed points of T and f being a singleton I well ordered .This completes the proof .

Theorem 3 : Let (X, d, ≼) be a complete partially ordered metric space . suppose that f and T are self- mappigs on X ,T is a monotone f-non-decreasing T(X) $\subseteq f(X)$ and satisfying

$$d(Tx,Ty) \leq \alpha\, d(fx,fy) + \beta[d(fx,Tx) + d(fy,Ty)] + \gamma[d(fx,Ty) + d(fy,Tx)]$$

for all x, y in X with f(x)$\neq f(y)$ are compatible ,where $\alpha, \beta, \gamma \in [0,1]$ with if there exists point $0 \leq \alpha + 2\beta + 2\gamma < 1$,$x_0 \in X$ such that f(x_0)≼T(x_0) and the mappings T and f are compatible ,then T and f have a coincidence point in X .

Proof : Let $x_0 \in X$ such that f(x_0)≼T(x_0) since from hypothesis ,we have T(X) $\subseteq f(X)$ then ,we can choose a point $x_1 \in X$ such that fx_1 =T x_0 but T$x_1 \in f(X)$ then again there exists another point $x_2 \in X$ such that fx_2=Tx_1. By continuing ,the same way , we can construct a sequence $\{x_n\}$ in X such that fx_{n+1}=Tx_n for all n.

Again by hypothese ,we have f(x_0)≼ T(x_0)= fx_1 and T is a monotone f-nondecreasing mapping ,then we get T(x_0)≼ $T(x_1)$.

Similarly ,we obtain T(x_1)≼ $T(x_2)$, since f(x_1)≼ $f(x_2)$ and then by continuing the same process we obtain that

$$T(x_0) \preccurlyeq T(x_1) \preccurlyeq T(x_2) \preccurlyeq \text{----} \preccurlyeq T(x_n) \preccurlyeq T(x_{n+1}) \preccurlyeq \text{-----------------------}$$

The equality T(x_{n+1}) = T(x_n) is impossible because f(x_{n+2})≠f(x_{n+1}) for all n $\in N$. Thus

d(Tx_n, Tx_{n+1}) > 0 for all n≥ 0

therefore, from contraction condition ,we have

$$d(Tx_{n+1},Tx_n) \leq \alpha d(fx_{n+1}, fx_n) + \beta[d(fx_{n+1}, Tx_{n+1}) + d(fx_n, Tx_n)] + \gamma[d(fx_{n+1},Tx_n) + d(fx_n, Tx_{n+1})]$$

$$d(Tx_{n+1},Tx_n) \leq \alpha d(Tx_n, Tx_{n-1}) + \beta[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)] + \gamma[d(Tx_n,Tx_n) + d(Tx_{n-1}, Tx_{n+1})]$$

$$d(Tx_{n+1},Tx_n) \leq \alpha d(Tx_n, Tx_{n-1}) + \beta[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)] + \gamma[d(Tx_{n-1}, Tx_{n+1})]$$

$$d(Tx_{n+1},Tx_n) \leq \alpha d(Tx_n, Tx_{n-1}) + \beta[d(Tx_n, Tx_{n+1}) + d(Tx_{n-1}, Tx_n)] + \gamma[d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})]$$

$$d(Tx_{n+1},Tx_n) \leq \alpha d(Tx_n, Tx_{n-1}) + \beta d(Tx_n, Tx_{n+1}) + \beta\, d(Tx_{n-1}, Tx_n) + \gamma d(Tx_{n-1}, Tx_n) + \gamma d(Tx_n, Tx_{n+1})$$

$$(1- \beta - \gamma) \quad d(Tx_{n+1},Tx_n) = (\alpha + \beta + \gamma)\, d(Tx_n,Tx_{n-1})$$

$$d(Tx_{n+1},Tx_n) \leq \frac{(\alpha + \beta + \gamma)}{(1- \beta - \gamma)} . d(Tx_n,Tx_{n-1})$$

continuing the same process up to (n-1) times we get $d(Tx_{n+1},Tx_n) \leq \left(\frac{(\alpha + \beta + \gamma)}{(1- \beta - \gamma)}\right)^n d(Tx_1,Tx_0)$

Let $k = \frac{(\alpha + \beta + \gamma)}{(1- \beta - \gamma)} \in [0,1)$ then from triangular inequality for m $\geq n$,we have d(Tx_m,Tx_n)≤ d(Tx_m, Tx_{m-1})+ d(Tx_{m-1}, Tx_{m-2}) +--------------+d(Tx_{n+1},Tx_n)

$\leq (k^{m-1}+ k^{m-2} +$ ------------- $+ k^n)$ d(Tx_1 ,Tx_0)

$\leq \left(\frac{k^n}{1-k}\right)$ d(Tx_1,Tx_0)

as m, n → ∞ d(Tx_m ,Tx_n) → 0 which shows that the sequence $\{Tx_n\}$ is a Cauchy sequence in X so

by the completeness of X there exists a point $\mu \in X$ such that $Tx_n \to \mu$ as $n \to \infty$. Again by the

continuity of T , we have $\lim_{n\to\infty} T(Tx_n) =$ T($\lim_{n\to\infty} Tx_n$) =T μ

But fx_{n+1} = Tx_n then fx_{n+1} → μ as n→ ∞ and from the compatibility for T and f we have

$\lim_{n\to\infty} d(T(fx_n), f(Tx_n)) = 0$

Further by triangular inequality , we have

d(Tμ, f μ) = d(Tμ, T(f x_n)) + d[T(fx_n) ,f(Tx_n)] +d[f(Tx_n) ,fμ]

on taking limit as n→ ∞ in both sides of the above equation and using the fact that T and f are continuous then ,we get d(Tμ, $f\mu$)= 0 thus Tμ = fμ. Hence μ is a coincidence point of T and f in X.

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