Some Mapping on $\alpha c^* g$-Open & Closed Maps in Topological Spaces

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Abstract

In this paper we have introduced the concept of Closed maps, Open maps, Irresolute and Homeomorphism on the $\alpha c^* g$-closed set and study some properties on them.

1. Introduction

Malghan [1] introduced and investigated some properties of generalized closed maps in topological spaces. The concept of generalized open map was introduced by Sundaram[2]. In this paper we introduced the concepts of $\alpha c^* g$-closed maps and $\alpha c^* g$-open maps in topological spaces.

2. Premilinaries

Definition 2.1: A subset A of a topological space $(X, r)$ is called
(i) Generalized closed set (g-closed)[3] if cl(A) $\subseteq U$ whenever $A \subseteq U$, and U is open in $X$.
(ii) $\alpha$-generalized closed set (g-closed)[4] if $\alpha$cl(A) $\subseteq U$ whenever $A \subseteq U$, and U is open in $X$.
(iii) $\alpha c^*$g-closed set[5] if $\alpha$cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is C-set. The complement of $\alpha c^*$g-closed set is $\alpha c^*$g-open set[5].
(iv) $\alpha c$g-closed set[5] if $\alpha$cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is C*-set. The complement of $\alpha c$g-closed set is $\alpha c$g-open set[5].
(v) $\alpha$g-closed set[5] if $\alpha$cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is G-set.

Definition 2.2: A map $f: X \rightarrow Y$ is said to be
(i) g-closed[3] in $X$ for each closed set F in $Y$.
(ii) $\alpha$-generalized continuous (g-continuous)[15] if $f^{-1}(F)$ is $\alpha g$-closed in $X$ for each closed set F in $Y$.
(iii) closed map[1] if for each closed set F in $X$, $f(F)$ is closed in $Y$.
(iv) open map[1] if for each open set F in $X$, $f(F)$ is open in $Y$.

3. $\alpha c^* g$-Closed maps & $\alpha c^* g$-Open maps in topological spaces

Definition 3.1: A map $f: X \rightarrow Y$ from a topological space $X$ into a topological space $Y$ is called $\alpha c^* g$-closed map if for each closed set $F$ in $X$, $f(F)$ is a $\alpha c^* g$-closed set in $Y$.

Theorem 3.2: If a map $f: X \rightarrow Y$ is closed map then it is $\alpha c^* g$-closed map but not conversely.

Proof: Since every closed set is $\alpha c^* g$-closed set then it is $\alpha c^* g$-closed map.

The converse of the above theorem need not be true as seen from the following example.

Example 3.3: Let $X = Y = \{a, b, c\}$. Let $f$ be a identity map such that

$f: (X, \tau_1) \rightarrow (Y, \tau_2)$.  
$\tau_1 = \{\varphi, Y, \{b, c\}\}$,  
$\tau_2 = \{\varphi, X, \{a\}, \{a, c\}, \{a, b\}\}$.  

Here

$C(\tau_1, \tau_2) = \{\phi_y, \{a\}\} \cup C(X, \tau_2) = \{\phi, X, \{b, c\}, \{b\}, \{c\}\}$.  

Then $f$ is $\alpha c^* g$-closed map but not closed map.

Since for the closed set $\{a\}$ in $(X, \tau_1)$,  
$f([a]) = \{a\}$ is not closed in $Y$. 

Theorem 3.4: If a map \( f : X \to Y \) is g-closed map then it is \( \alpha g \)-closed map but not conversely.

**Proof:** Let \( f : X \to Y \) be a g-closed map. Then for each closed set \( F \) in \( X \), \( f(F) \) is g-closed in \( Y \). Since every g-closed set is \( \alpha g \)-closed set. Therefore \( \overline{f(F)} \) is \( \alpha g \)-closed set. Hence \( f \) is \( \alpha g \)-closed map.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.5:** Let \( X = Y = \{a, b, c\} \). Let \( f \) be a identity map such that \( f : (X, \tau_1) \to (Y, \tau_2) \).
\[
\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \quad \text{and} \quad \tau_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}.
\]
Then \( f \) is \( \alpha g \)-closed but not g-closed because for the closed set \( \{a, c\} \) in \( X \), \( f(\{a, c\}) = \{a, c\} \) is not g-closed in \( Y \). Therefore \( f \) is not g-closed map.

**Theorem 3.6:** If a map \( f : X \to Y \) is \( \alpha \)-closed map then it is \( \alpha g \)-closed map but not conversely.

**Proof:** Let \( f : X \to Y \) be a \( \alpha \)-closed map. Then for each closed set \( F \) in \( X \), \( f(F) \) is \( \alpha \)-closed set in \( Y \). Since every \( \alpha \)-closed set is \( \alpha g \)-closed set. Therefore \( f(F) \) is \( \alpha g \)-closed set. Hence \( f \) is \( \alpha g \)-closed map.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.7:** Let \( X = Y = \{a, b, c\} \). Let \( f \) be a identity map such that \( f : (X, \tau_1) \to (Y, \tau_2) \).
\[
\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \quad \text{and} \quad \tau_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}.
\]
Then \( f \) is \( \alpha g \)-closed but not \( \alpha \)-closed because for the closed set \( \{a, c\} \) in \( X \), \( f(\{a, c\}) = \{a, c\} \) is not \( \alpha \)-closed in \( Y \). Therefore \( f \) is not \( \alpha \)-closed map.

**Theorem 3.8:** If a map \( f : X \to Y \) is \( \alpha g \)-closed map then it is \( \alpha g \)-closed map but not conversely.

**Proof:** Let \( f : X \to Y \) be a \( \alpha g \)-closed map. Then for each closed set \( F \) in \( X \), \( f(F) \) is \( \alpha g \)-closed set in \( Y \). Since every \( \alpha g \)-closed set is \( \alpha g \)-closed set. Therefore \( f(F) \) is \( \alpha g \)-closed set. Hence \( f \) is \( \alpha g \)-closed map.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.9:** Let \( X = Y = \{a, b, c\} \). Let \( f \) be a identity map such that \( f : (X, \tau_1) \to (Y, \tau_2) \).
\[
\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \quad \text{and} \quad \tau_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}.
\]
Then \( f \) is \( \alpha g \)-closed but not \( \alpha g \)-closed because for the closed set \( \{a, c\} \) in \( X \), \( f(\{a, c\}) = \{a, c\} \) is not \( \alpha g \)-closed in \( Y \). Therefore \( f \) is not \( \alpha g \)-closed map.

**Theorem 3.10:** If a map \( f : X \to Y \) is gs-closed map then it is \( \alpha g \)-closed map but not conversely.

**Proof:** Let \( f : X \to Y \) be a gs-closed map. Then for each closed set \( F \) in \( X \), \( f(F) \) is gs-closed set in \( Y \). Since every gs-closed set is \( \alpha g \)-closed set. Therefore \( f(F) \) is \( \alpha g \)-closed set. Hence \( f \) is \( \alpha g \)-closed map.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.11:** Let \( X = Y = \{a, b, c\} \). Let \( f \) be a identity map such that \( f : (X, \tau_1) \to (Y, \tau_2) \).
\[
\tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \quad \text{and} \quad \tau_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}.
\]
Then \( f \) is \( \alpha g \)-closed but not gs-closed because for the closed set \( \{a, c\} \) in \( X \), \( f(\{a, c\}) = \{a, c\} \) is not gs-closed in \( Y \). Therefore \( f \) is not gs-closed map.

**Definition 3.12:** A map \( f : X \to Y \) from a topological space \( X \) into a topological space \( Y \) is called \( \alpha g \)-open map if \( f(F) \) is a \( \alpha g \)-open set in \( Y \) for every open set \( F \) in \( X \).

**Theorem 3.13:** If a map \( f : X \to Y \) is open map then it is \( \alpha g \)-open map but not conversely.

**Proof:** Let \( f : X \to Y \) be a open map. Let \( F \) be any open set in \( X \), \( f(F) \) is open set in \( Y \). Then \( f(F) \) is \( \alpha g \)-open set. Since every open set is \( \alpha g \)-open set. Hence \( f \) is \( \alpha g \)-open map.
The converse of the above theorem need not be true as seen from the following example.

Example 3.14: Let $X = Y = \{a, b, c\}$. Let $f'$ be a identity map such that $f': (X, \tau_1) \rightarrow (Y, \tau_2)$. 
\[ \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \]
\[ \tau_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \]. Then $f'$ is $\alpha g$-open map but not open map because for the open set $\{b\}$ in $X$, $f'([b]) = \{b\}$ is not open in $Y$. Therefore $f'$ is not open map.

Theorem 3.15: If a map $f: X \rightarrow Y$ is $g$-open map then it is $\alpha^c g$-open map but not conversely.

Proof: Let $f: X \rightarrow Y$ be a $g$-open map. Let $F$ be any open set in $X$, $f(F)$ is $g$-open set in $Y$. Since every $g$-open set is $\alpha^* g$-open set. Then $f(F)$ is $\alpha g$-open set. Hence $f$ is $\alpha^* g$-open map.

The converse of the above theorem need not be true as seen from the following example.

Example 3.16: Let $X = Y = \{a, b, c\}$. Let $f$ be a identity map such that $f: (X, \tau_1) \rightarrow (Y, \tau_2)$. 
\[ \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \]
\[ \tau_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \]. Then $f$ is $\alpha^* g$-open map but not $g$-open map because for the open set $\{b\}$ in $X$, $f([b]) = \{b\}$ is not $g$-open in $Y$. Therefore $f$ is not $g$-open map.

Theorem 3.17: If a map $f: X \rightarrow Y$ is $\alpha g$-open map then it is $\alpha^* g$-open map but not conversely.

Proof: Let $f: X \rightarrow Y$ be a $\alpha g$-open map. Let $F$ be any open set in $X$, $f(F)$ is $\alpha g$-open set in $Y$. Since every $\alpha g$-open set is $\alpha^* g$-open set. Then $f(F)$ is $\alpha^* g$-open set. Hence $f$ is $\alpha^* g$-open map.

The converse of the above theorem need not be true as seen from the following example.

Example 3.18: Let $X = Y = \{a, b, c\}$. Let $f$ be a identity map such that $f: (X, \tau_1) \rightarrow (Y, \tau_2)$. 
\[ \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \]
\[ \tau_2 = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \]. Then $f$ is $\alpha^* g$-open map but not $g$-open map because for the open set $\{b\}$ in $X$, $f([b]) = \{b\}$ is not $g$-open in $Y$. Therefore $f$ is not $g$-open map.

Theorem 3.19: If a map $f: X \rightarrow Y$ is $\alpha$-open map then it is $\alpha^* g$-open map but not conversely.

Proof: Let $f: X \rightarrow Y$ be a $\alpha$-open map. Let $F$ be any open set in $X$, $f(F)$ is $\alpha$-open set in $Y$. Since every $\alpha$-open set is $\alpha^* g$-open set. Then $f(F)$ is $\alpha^* g$-open set. Hence $f$ is $\alpha^* g$-open map.

The converse of the above theorem need not be true as seen from the following example.

Example 3.20: Let $X = Y = \{a, b, c\}$. Let $f$ be a identity map such that $f: (X, \tau_1) \rightarrow (Y, \tau_2)$. 
\[ \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \]
\[ \tau_2 = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}\} \]. Then $f$ is $\alpha^* g$-open map but not $\alpha$-open map because for the open set $\{b\}$ in $X$, $f([b]) = \{b\}$ is not $\alpha$-open in $Y$. Therefore $f$ is not $\alpha$-open map.

Theorem 3.21: If a map $f: X \rightarrow Y$ is $gs$-open map then it is $\alpha^* g$-open map but not conversely.

Proof: Let $f: X \rightarrow Y$ be a $gs$-open map. Let $F$ be any open set in $X$, $f(F)$ is $gs$-open set in $Y$. Since every $gs$-open set is $\alpha^* g$-open set. Then $f(F)$ is $\alpha^* g$-open set. Hence $f$ is $\alpha^* g$-open map.

The converse of the above theorem need not be true as seen from the following example.

Example 3.22: Let $X = Y = \{a, b, c\}$. Let $f$ be a identity map such that $f: (X, \tau_1) \rightarrow (Y, \tau_2)$. 
\[ \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \]
\[ \tau_2 = \{\emptyset, Y, \{a\}, \{a, b\}, \{a, c\}\} \]. Then $f$ is $\alpha^* g$-open map but not $gs$-open map because for the open set $\{b\}$ in $X$, $f([b]) = \{b\}$ is not $gs$-open in $Y$. Therefore $f$ is not $gs$-open map.
Theorem 3.23: If \( f : X \to Y \) is \( \alpha^c \text{-} g \)-continuous and \( \alpha^c g \)-closed and \( A \) is a \( \alpha^c g \)-closed set of \( X \), then \( f(A) \) is \( \alpha^c g \)-closed in \( Y \).

Proof: Let \( f(A) \subseteq O \), where \( O \) is \( \alpha^c \)-set of \( Y \), Since \( f \) is \( \alpha^c \text{-} g \)-continuous, \( f^{-1}(O) \) is \( \alpha \)-set containing \( A \). Hence \( cl(A) \subseteq f^{-1}(O) \) as \( A \) is \( \alpha^c g \)-closed. Since \( f \) is \( \alpha^c g \)-closed, \( f(cl(A)) \) is \( \alpha^c \)-set contained in \( \alpha \)-set \( O \), which implies that \( cl(f(cl(A))) \subseteq O \) and hence \( cl(f(A)) \subseteq O \). So \( f(A) \) is \( \alpha^c g \)-closed in \( Y \).

Corollary 3.24: If \( f : X \to Y \) is continuous and closed map and if \( A \) is \( \alpha^c g \)-closed set in \( X \), then \( f(A) \) is \( \alpha^c g \)-closed in \( Y \).

Proof: Since every continuous map is \( \alpha^c \)-continuous and every closed map is \( \alpha^c \)-closed, by the above theorem the result follows.

Theorem 3.25: If \( f : X \to Y \) is closed and \( h : Y \to Z \) is \( \alpha^c \)-closed then \( h \circ f : X \to Z \) is \( \alpha^c \)-closed.

Proof: Let \( f : X \to Y \) is a closed map and \( h : Y \to Z \) is \( \alpha^c \)-closed map. Let \( V \) be any closed set in \( X \). Since \( f(X) \) is closed in \( Y \) and since \( h(Y) \) is \( \alpha^c \)-closed \( h(f(V)) \) is \( \alpha^c \)-closed set in \( Z \). Therefore \( h \circ f : X \to Z \) is \( \alpha^c \)-closed map.

Theorem 3.26: If \( f : X \to Y \) is \( \alpha^c g \)-closed and \( A \) is closed set in \( X \). Then \( f^{-1}(A) \to Y \) is \( \alpha^c g \)-closed.

Proof: Let \( V \) be closed set in \( A \). Then \( V \) is closed in \( X \). Therefore \( f^{-1}(A) \) is \( \alpha^c g \)-closed set in \( Y \). By theorem 1.24 \( f(V) \) is \( \alpha^c g \)-closed. That is \( f^{-1}(A) \) is \( \alpha^c g \)-closed set in \( Y \). Therefore \( f^{-1}(A) \to Y \) is \( \alpha^c g \)-closed.

4. \( \alpha^c g \) - irresolute map in Topological Spaces

Crossley and Hildebrand[9] introduced and investigated the concept of irresolute function in topological spaces. Sundaram[2], Maheshwari and Prasad[10], Jankovic[11] have defined gc- irresolute maps, \( \alpha \)-irresolute maps and p-open maps in topological spaces.

In this section, we have introduced a new class of map called \( \alpha^c g \) - irresolute map and study some of their properties.

Definition 4.1: A map \( f : X \to Y \) from topological space \( X \) into a topological space \( Y \) is called \( \alpha^c g \) - irresolute map in the inverse of every \( \alpha^c g \)-closed (\( \alpha^c g \)-open) set in \( Y \) is \( \alpha^c g \)-closed (\( \alpha^c g \)-open) in \( X \).

Theorem 4.2: If a map \( f : X \to Y \) is \( \alpha^c g \)-irresolute, then it is \( \alpha^c g \)-continuous, but not conversely.

Proof: Assume that \( f \) is \( \alpha^c g \)-irresolute. Let \( F \) be any closed set in \( Y \). Since every closed set is \( \alpha^c g \)-closed, \( F \) is \( \alpha^c g \)-closed in \( Y \). Since \( f \) is \( \alpha^c g \)-irresolute, irresolute, \( f^{-1}(F) \) is \( \alpha^c g \)-closed in \( X \). Therefore \( f \) is \( \alpha^c g \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 4.3: Consider the topological space \( X = Y = \{a, b, c\} \) with topology \( \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{\emptyset, Y, \{a\}\} \). Let \( f(X, \tau_1) \to (Y, \tau_2) \) be the identity map then \( f \) is \( \alpha^c g \)-continuous, because for the inverse image of every closed in \( Y \) is \( \alpha^c g \)-closed in \( X \), but not \( \alpha^c g \)-irresolute. Because for the inverse image of every \( \alpha^c g \)-closed in \( Y \) is not \( \alpha^c g \)-closed in \( Y \), (ie) for the \( \alpha^c g \)-closed set \( \{b\} \) in \( Y \) the inverse image \( f^{-1}(\{b\}) = \{b\} \) is not \( \alpha^c g \)-closed in \( X \).

Theorem 4.4: Let \( X, Y, \) and \( Z \) be any topological spaces.

For any \( \alpha^c g \)-irresolute map \( f : X \to Y \) and any \( \alpha^c g \)-continuous map \( g : Y \to Z \) the composition \( g \circ f : X \to Z \) is \( \alpha^c g \)-continuous.

Proof: Let \( F \) be any closed set in \( Z \). Since \( g \) is \( \alpha^c g \)-continuous, \( g^{-1}(F) \) is \( \alpha^c g \)-closed in \( Y \). Since \( f \) is \( \alpha^c g \)-irresolute \( f^{-1}(g^{-1}(F)) \) is \( \alpha^c g \)-closed \( f^{-1}(g^{-1}(F)) \) is \( \alpha^c g \)-continuous.

Therefore \( g \circ f \) is \( \alpha^c g \)-continuous.

Theorem 4.5: If \( f : X \to Y \) from topological space \( X \) into a topological space \( Y \) is bijective, \( \alpha^c g \)-open set and \( \alpha^c g \)-continuous then \( f \) is \( \alpha^c g \)-irresolute.
Proof: Let A be a αc⁠g -closed set in Y. Let \( f^{-1}(A) \subseteq O \). Where O is C^*-set in X. Therefore A \( \subseteq f(O) \) holds. Since \( f(O) \) is αc⁠g-open set and A is αc⁠g-closed in Y, acl(A) \( \subseteq f(O) \), \( f^{-1}\text{ (acl(A))} \subseteq f(O) \).

Since f is αc⁠g-continuous and acl(A) is closed in Y. \( \alpha cl(f^{-1}(\alpha cl(A)) \subseteq O \).

Therefore \( f^{-1}(A) \) is αc⁠g-closed in X. Hence f is αc⁠g-irresolute.

The following examples show that no assumption of the above theorem can be removed.

Example 4.6: Consider the topological space \( X = Y = \{a, b, c\} \) with topology \( \tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{\varnothing, Y, \{a\}\} \). Then the defined identity map \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) is αc⁠g-continuous, bijective and not αc⁠g-open. So f is not αc⁠g-irresolute. Since for the αc⁠g-closed set \{a\} in Y the inverse image \( f^{-1}(\{a\}) = \{a\} \) is not αc⁠g-closed in X.

Example 4.7: Consider the topological space \( X = Y = \{a, b, c\} \) with topology \( \tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{\varnothing, Y, \{a\}\} \). Then the map \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) be defined by \( f(a) = a, f(b) = b, f(c) = a \). Then f is αc⁠g-continuous, αc⁠g-open and not bijective. So f is not αc⁠g-irresolute. Since for the αc⁠g-closed set \{b\} in Y the inverse image \( f^{-1}(\{b\}) = \{b\} \) is not αc⁠g-closed in X.

Example 4.8: Consider the topological space \( X = Y = \{a, b, c\} \) with topology \( \tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{\varnothing, Y, \{a\}, \{a, b\}\} \) Then the defined identity map \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) is bijective, αc⁠g-open and not αc⁠g-continuous. So f is not αc⁠g-irresolute. Since for the αc⁠g-closed set \{b\} in Y the inverse image \( f^{-1}(\{b\}) = \{b\} \) is not αc⁠g-closed in X.

Remark 4.9: The following two examples show that the concepts of irresolute maps and αc⁠g-irresolute maps are independent of each other.

Example 4.10: Consider the topological space \( X = Y = \{a, b, c\} \) with topology \( \tau_1 = \{\varnothing, X, \{a\}, \{b\}\} \), \( \tau_2 = \{\varnothing, Y, \{a\}, \{a, b\}\} \). Then the defined identity map \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) is irresolute but not αc⁠g-irresolute. Since \{b\} is αc⁠g-closed set in Y has its inverse image \( f^{-1}(\{b\}) = \{b\} \) is not αc⁠g-closed in X.

Example 4.11: Consider the topological space \( X = Y = \{a, b, c\} \) with topology \( \tau_1 = \{\varnothing, X, \{a\}, \{a, b\}\} \), \( \tau_2 = \{\varnothing, Y, \{a\}, \{a, b\}\} \). Then the defined identity map \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) is αc⁠g-irresolute but not irresolute. Since the closed set \{a,c\} in Y has its inverse image \( f^{-1}(\{a,c\}) = \{a,c\} \) is not closed in X.

Remark 4.12: From the following diagram we can conclude that αc⁠g-irresolute map is independent with irresolute map.

\[ \text{αc⁠g-irresolute map} \quad \text{irresolute map} \]

5. αc⁠g-homeomorphism maps in Topological Spaces

Several mathematicians have generalized homeomorphism in topological spaces. Biswas[14], Crossley and Hildebrand[9], Gentry and Hoyle[13] and Umehara and Maki[12] have introduced and investigated semi-homeomorphism, which also a generalization of homeomorphism. Sundaram[2] introduced g-homeomorphism and gc-homeomorphism is topological spaces.

In this section we introduce the concept of αc⁠g-homeomorphism and study some of their properties.

Definition 5.1: A bijection \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) is called αc⁠g-homeomorphism if f is both αc⁠g-open and αc⁠g-continuous.

Theorem 5.2: Every homeomorphism is a αc⁠g-homeomorphism but not conversely.

Proof: Since every continuous function is αc⁠g-continuous and every open map is αc⁠g-open the proof follows.
The converse of the above theorem need not be true as seen from the following example.

**Example 5.3:** Let \( X = Y = \{a, b, c\} \) with \( \tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{\varnothing, X, \{a, b\}\} \), then \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) is \( \alpha\gamma^* \) -homeomorphism but not homeomorphism.

**Theorem 5.4:** For any bijection \( f: X \rightarrow Y \) the following statements are equivalent.

i) \( f^{-1}: Y \rightarrow X \) is \( \alpha\gamma^* \) -continuous.

ii) \( f \) is a \( \alpha\gamma^* \) -open map.

iii) \( f \) is a \( \alpha\gamma^* \) -closed map.

**Proof:** (i) \( \Rightarrow \) (ii) Let \( G \) be any open set in \( X \). Since \( f^{-1} \) is \( \alpha\gamma^* \) -continuous, the inverse image of \( G \) under \( f^{-1} \) namely \( f(G) \) is \( \alpha\gamma^* \) -open in \( Y \). So \( f \) is \( \alpha\gamma^* \) -open map.

(ii) \( \Rightarrow \) (iii) Let \( F \) be any closed set in \( X \). Then \( F^c \) is open in \( X \). Since \( f \) is \( \alpha\gamma^* \) -open map \( f(F^c) \) is \( \alpha\gamma^* \) -open map in \( Y \). But \( f(F^c) = Y - f(F) \) and so \( f(F) \) is \( \alpha\gamma^* \) -open map in \( Y \). Therefore \( f \) is a \( \alpha\gamma^* \) -closed map.

(iii) \( \Rightarrow \) (i) Let \( F \) be any closed set in \( X \). Then \( (f^{-1})^{-1}F = f(F) \) is \( \alpha\gamma^* \) -closed map in \( Y \). Therefore \( f^{-1}: Y \rightarrow X \) is \( \alpha\gamma^* \) -continuous.

**Theorem 5.5:** Let \( f(X, \tau) \rightarrow (Y, \sigma) \) be a bijective and \( \alpha\gamma^* \) -continuous map the following statement are equivalent.

i) \( f \) is a \( \alpha\gamma^* \) -open map.

ii) \( f \) is a \( \alpha\gamma^* \) -homeomorphism.

iii) \( f \) is a \( \alpha\gamma^* \) -closed map.

**Proof:** The proof easily follows from definitions and assumptions.

The following examples shows that the composition of two \( \alpha\gamma^* \) -homeomorphism need not be \( \alpha\gamma^* \) -homeomorphism.

**Example 5.6:** Let \( X = Y = Z = \{a, b, c\} \) with topologies \( \tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{\varnothing, Y, \{a\}, \{a, b\}\} \), \( \tau_3 = \{\varnothing, Z, \{a, b\}\} \) then \( f(X, \tau_1) \rightarrow (Y, \tau_2) \rightarrow (Z, \tau_3) \) is \( \alpha\gamma^* \) -homeomorphism but not \( \alpha\gamma^* \) -homeomorphism.

Let \( f \) and \( g \) be identity maps such that \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) then \( f \) and \( g \) are \( \alpha\gamma^* \) -homeomorphism, but their composition \( g \cdot f: X \rightarrow Z \) is not \( \alpha\gamma^* \) -homeomorphism.

**Theorem 5.7:** Every \( \alpha \) -homeomorphism is a \( \alpha\gamma^* \) -homeomorphism.

**Proof:** Let \( f: X \rightarrow Y \) be a \( \alpha \) -homeomorphism then \( f \) is \( \alpha \) -continuous and \( \alpha \) -closed. Since every \( \alpha \) -continuous is \( \alpha\gamma^* \) -continuous and every \( \alpha \) -closed is \( \alpha\gamma^* \) -closed, \( f \) is \( \alpha\gamma^* \) -continuous and \( \alpha\gamma^* \) -closed. Therefore \( f \) is \( \alpha\gamma^* \) -homeomorphism.

The converse of the above theorem need not to be true as seen from the following example.

**Example 5.8:** Consider the topological space \( X = Y = \{a, b, c\} \) with topology \( \tau_1 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\} \), \( \tau_2 = \{\varnothing, Y, \{a\}, \{a, c\}\} \). Then the defined identity map \( f(X, \tau_1) \rightarrow (Y, \tau_2) \) is \( \alpha\gamma^* \) -homeomorphism but not \( \alpha \) -homeomorphism. Since for the open set \( \{a\} \) in \( X \) the inverse image \( f^{-1}(\{a\}) = \{a\} \) is not \( \alpha \) -open in \( Y \).

From the above observations we get the following diagram:

\[
\text{homeomorphism} \quad \longrightarrow \quad \alpha \text{-homeomorphism} \quad \downarrow \\
\alpha\gamma^* \text{-homeomorphism} \quad \uparrow \\
\text{\( \alpha\gamma^* \) -irresolute map}
\]

**Definition 5.9:** A bijection \( f(X, \tau) \rightarrow (Y, \sigma) \) is said to be \( (\alpha\gamma^*)^* \) homeomorphism if \( f \) and its inverse \( f^{-1} \) are \( \alpha\gamma^* \) -irresolute map.

**Notation 5.10:** Let the family of all \( (\alpha\gamma^*)^* \) -homeomorphism from \( (X, \tau) \) onto itself be denoted by \( (\alpha\gamma^*)^* \) and the family of all \( \alpha\gamma^* \) -homeomorphism from \( (X, \tau) \) onto itself be denoted by \( (\alpha\gamma) \). The family of all
homeomorphism from \((X, \tau)\) onto itself be denoted by \(h(X, \tau)\).

**Theorem 5.11:** Let \(X\) be a topological space. Then

1. The set \((\alpha c^* g)^* h(X)\) is group under composition of maps.  
   
2. \(h(x)\) is a subgroup of \((\alpha c^* g)^* h(X)\)  
3. \((\alpha c^* g)^* h(X)\) is closed under the composition of maps.

**Proof for (i):** Let \(f, g \in (\alpha c^* g)^* h(X)\), then 
\(g \cdot f \in (\alpha c^* g)^* h(X)\) and so \((\alpha c^* g)^* h(X)\) is closed under the composition of maps. The composition of maps is associative. The identity map \(I : X \to X\) is a \((\alpha c^* g)^*\)-homeomorphism and so \(I \in (\alpha c^* g)^* h(X)\). Also \(f \cdot I = I \cdot f = f\) for every \(f \in (\alpha c^* g)^* h(X)\). If \(f \in (\alpha c^* g)^* h(X)\), then \(f^{-1} \in (\alpha c^* g)^* h(X)\) and 
\(f \cdot f^{-1} = f^{-1} \cdot f = I\). Hence \((\alpha c^* g)^* h(X)\) is a group under the composition of maps.

**Proof for (ii):** Let \(f(X, \tau) \to (Y, \sigma)\) be a homeomorphism. Then by theorem 4.5. Both of \(f\) and \(f^{-1}\) are \((\alpha c^* g)^*\)- irresolute and so \(f\) is a \((\alpha c^* g)^*\)-homeomorphism. Therefore every homeomorphism is a \((\alpha c^* g)^*\)-homeomorphism. Therefore \(h(x)\) is a subgroup of \((\alpha c^* g)^* h(X)\). Also \(h(x)\) is a group under composition of maps.

**Proof for (iii):** Since every \((\alpha c^* g)^*\)-irresolute map is \(\alpha c^* g\)-continuous, \((\alpha c^* g)^* h(X)\) is a subset of \((\alpha c^* g)^* h(X)\).

**REFERENCES**


[5] Kavitha.A., \(\alpha cg, \alpha c^*g, \alpha c(s)g\)-closed sets in Topological spaces, In the 99th Indian Science Congress, Bhubaneswar(2012).


