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Some Dynamical Properties of the Duffing Equation

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Abstract- **We consider the modified form of the forced spring equation , commonly known as Duffing equation**

$$\ddot{x} + \gamma(p1 + p2x)\dot{x} + \alpha x + \beta x^3 = \beta\ F\ Cos\ \omega\ t$$

where $\alpha, \beta, \gamma, F, p1, p2$ are adjustable parameters. In this paper a detail study of the nonlinear Duffing oscillator with damping and external excitation is presented. The system under study consists of Duffing oscillator which is perturbed by the addition of a linear term $(p1 + p2x)$ where $p1, p2 > 0$. We have considered the case when $\alpha < 0$ and $\beta > 0$ i.e., we are dealing with inverted Duffing oscillator. The dynamical behavior of the proposed model is investigated analytically. We observe that the time continuous Duffing oscillator shows repetition of chaotic behavior. The tools of theoretical approach are the bifurcation diagram, the phase portraits, the Poincare sections, time series and the strange attractors. Next we observed occurrence of homoclinic orbits in our model when damping coefficient is taken as zero with varying F.

Keywords— Nonlinear Duffing Oscillator, Chaos, Poincare sections, Strange Attractors, Homoclinic, Bifurcation.

INTRODUCTION

In the last decades, research activities in systems of nonlinear oscillator resulted in a lot of publications on phenomena that such systems exhibit. Also, the interesting dynamical behavior, which these systems have shown, has triggered an investigation in possible applications of such systems in various scientific field, such as secure communication, cryptography, broadband communication systems, random numbers, radars, robots and in variety of complex physical, chemical and biological systems.

The forced Duffing equation is given by

$$\ddot{x} + \gamma\dot{x} + \alpha x + \beta x^3 = \ F\ Cos\ \omega\ t,\ \ F \geq 0$$

where γ is the damping coefficient, α is the linear stiffness parameter, β is the nonlinear (cubic) stiffness parameter, F is the strength of the driving force and ω is the frequency. It has been studied by a large number of authors since its introduction by Duffing. Duffing considered modeling forced oscillations using several different differential equations aiming at reproducing his observations of the behavior of machines. He was an engineer interested in solving very practical problems: to model forces and frictions including the complicated oscillations. [2]

We have considered a nonlinear and non-autonomous system with linear damping in the following set of equations. We have modified the Duffing equation as

$$\ddot{x} + \gamma(p1 + p2x)\dot{x} + \alpha x + \beta x^3 = \beta\ F\ Cos\ \omega\ t$$

where $\dot{x}$ is the velocity, $\ddot{x}$ is the acceleration, γ is the damping coefficient, α is the linear stiffness parameter, β is the nonlinear (cubic) stiffness parameter. Here we have added a linear term $(p1 + p2x)$ to the equation and assumed $\alpha < 0$ and $\beta > 0$, hence we are studying about inverted Duffing oscillator and its chaotic properties. We find that our model undergoes chaotic behavior as well as it also shows homoclinic properties within a certain range of parametric values. Our study consist of the following: First, we have fixed the parameters $\alpha, \beta, \gamma, p1, p2$ and studied the behavior for different values of F. We obtained bifurcation diagram for $0.1 < F < 15$. Secondly we have obtained phase portrait and Poincare sections in this regard. Thirdly we have shown $x\ vs\ t$ graph for our model within some range of parameters. Fourthly, we observed the behavior of the strange attractors, how they vary with driving force and damping factor. Lastly we observed the homoclinic behavior of the oscillator when damping coefficient is taken as zero.

1. STUDY OF OUR MODEL

Our modified Duffing equation is

$$\ddot{x} + \gamma(p1 + p2x)\dot{x} + \alpha x + \beta x^3 = \beta\ F\ Cos\ \omega\ t$$

with signs having their usual meaning. We have obtained the bifurcation diagram for our model. Bifurcation theory attempts to provide a systematic classification of the sudden changes in the qualitative behavior of the dynamical system. It is the mathematical study of changes in the qualitative or topological structure of a given family. In order to understand the various type of qualitative behavior that are exhibited by a physical system, it is necessary to describe the various bifurcations that occur in the system of differential equations modeling the physical system and to determine the parameter values, called bifurcation values , at which bifurcation occurs. Going back to the Duffing equation, we tried for different values of the parameter γ and ω and observed where the

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period doubling route to chaos occurs. We have fixed $\gamma = 0.1,\ \alpha = -2,\ \beta = 2, \omega = 1.2,\ x(0) = 1,\ \dot{x}(0) = 1$.
F is taken within a range of 0.1 to 15 to obtain the bifurcation diagram.

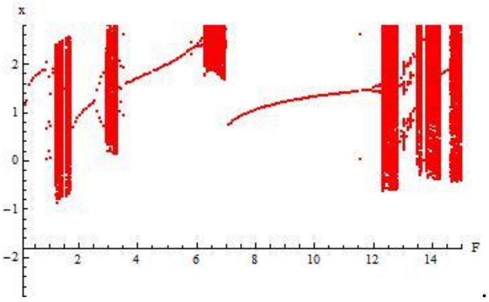


Fig-1: period-doubling route to chaos with γ=0.1, α=-2, β=2, ω=1.2, $F\leq$ 15 is plotted.

Clearly we see period doubling route to chaos. With parameters chosen in the region of limit cycles the system is either in the well of positive ***x*** or in the well of negative ***x***, depending on the precise value of *F*, but does not hop between the wells. We see that there is a repetition of period one, period two behavior which ultimately leads to chaotic behavior.

2. PERIODIC AND CHAOTIC BEHAVIOR

Next we concentrate our study in phase portrait and Poincare sections along with *x vs t* graph.
A useful way of analyzing chaotic motion is to look at the phase plane and Poincare section. Phase plane gives us a continuous curve indicating a periodic orbits whereas Poincare section is just the discrete set of phase space points of the particle of every points of the driving force i.e., at $\frac{2\tau}{\omega}, \frac{4\tau}{\omega}, \frac{8\tau}{\omega}$ etc. Clearly for a periodic orbit the Poincare section is a single dot, when the period is doubled there is double dot and the process goes on. [3]

Phase portrait along with their respective Poincare return maps are presented in the figures shown below. When 0.1< F < 0.4123, there is a period-one harmonic solution of period $\frac{2\tau}{\omega}$. which is depicted as a closed curve in the phase plane and as a single dot in the Poincare section. When *F=0.6402* period-two cycle of period $\frac{4\tau}{\omega}$ appears which is represented by double dot in the Poincare section. A period three cycle of period $\frac{8\tau}{\omega}$ is observed when *F=0.9553*, that is centered at O and surrounds both -1 and 1. When *F=1.229* our model shows chaotic behavior. A single trajectory plotted in the phase plane intersect itself many times and the portrait soon becomes messy. Again at *F=1.901* there is once more a stable period one solution. At *F=2.693* there is a period two cycle, at *F=2.899* there is a period four cycle and again chaotic behavior is observed at *3.341*.Thus we observe that there is a repetition of periodic orbits and chaotic behavior as the driving force increases.

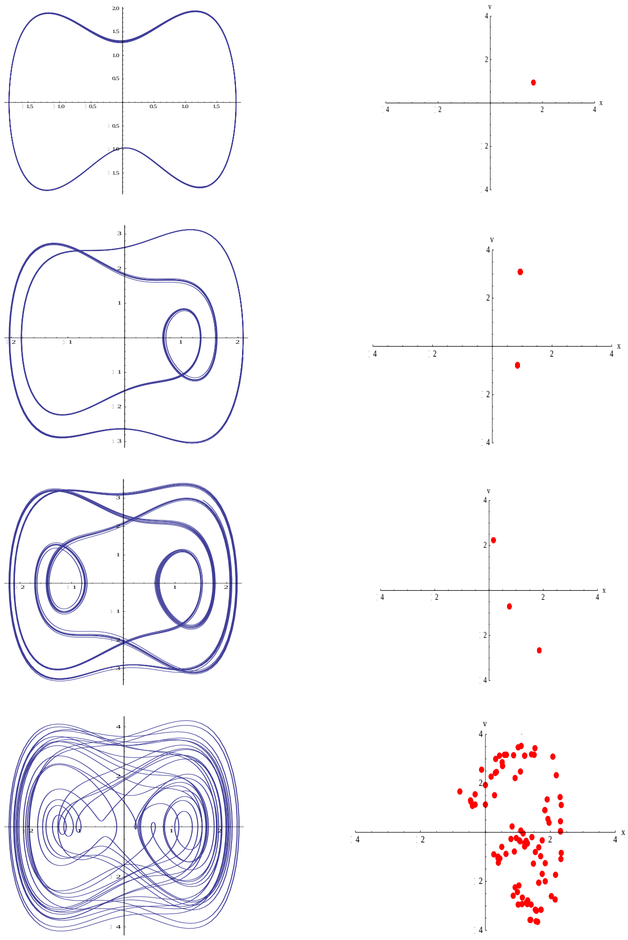
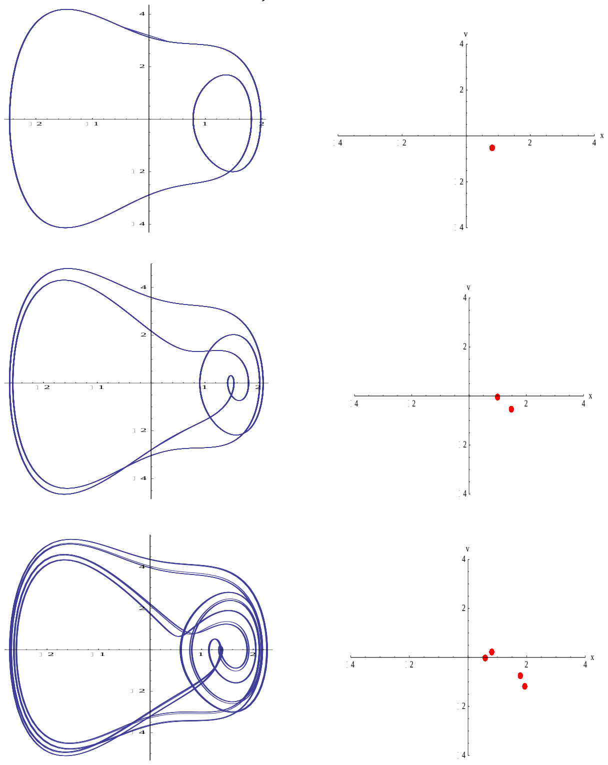


Fig-2: left shows phase portrait and right shows Poincare sections at γ=0.1, α=-2, β=2, ω=1.2, *p1=1 ,p2=0.4* is plotted at F=0.4123, F=0.6402,F=0.9553 and F=1.229 **.**



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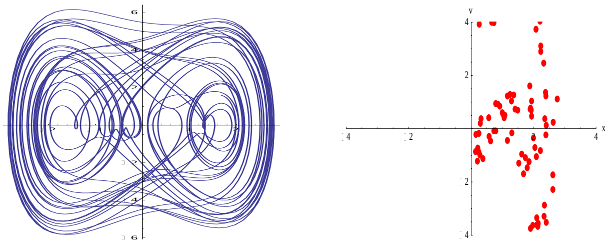
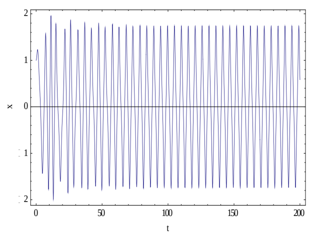


Fig-3: left shows phase portrait and right shows poincare sections at γ=0.1, α=-2, β=2, ω=1.2, p1=1, p2=0.4 at F=1.901,F=2.693,F=2.899 and F=3.341

3. TIME SERIES

A plot of $x(t)$ over the time interval $t = 0.....200$ for the inverted Duffing equation is shown below. Here γ=0.1, α=-2, β=2, ω=1.2, *p1=1, p2=0.4* and the initial conditions are taken as $x(0) = 1, \dot{x}(0) = 0$. The figure shows period-one, period-two, period-three, period-four for different values of F. At *F=1.229*, *3.341*, etc. are quite irregular with no obvious pattern emerging even if a longer time range is chosen. It is chaotic solutions.



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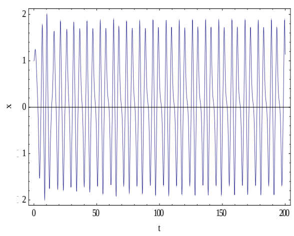

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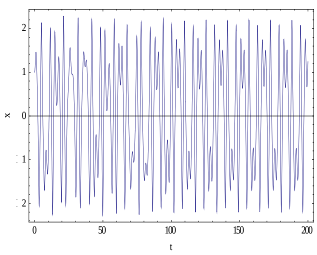

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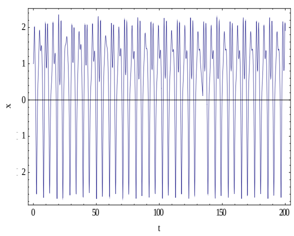

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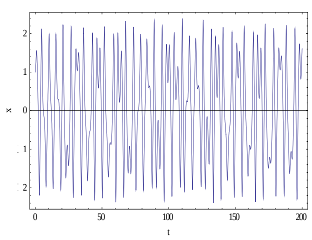

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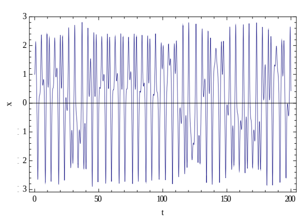


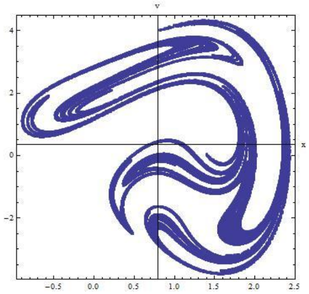
(f)

Fig-4: graph for with initial conditions γ=0.1, α=-2, β=2, ω=1.2, p1=1, p2=0.4 at (a) F =0.413,(b) F=0.6402, (c) F=0.9553 (d) F=2.899 shows periodic behavior and (e) F=1.229 (f) F=3.341 shows chaotic behavior.

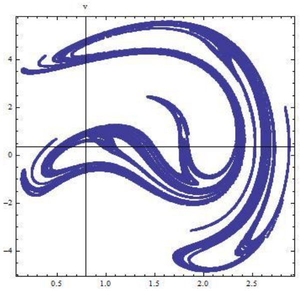
4. STRANGE ATTRACTORS

The strange attractors are a set of limiting points to which the trajectory tends (after the initial transient) every period of the driving force. The term strange is most often used as a name for attractors that exhibit chaotic behavior i.e., sensitivity to initial conditions. Here we have observed the chaotic behavior at five different values of *F* keeping all other terms constant i.e.

γ=0.1, α=-2, β=2, ω=1.2, *p1=1, p2=0.4* with initial conditions $x(0) = 1, \dot{x}(0) = 0$.



(a)



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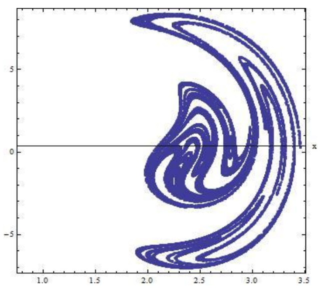

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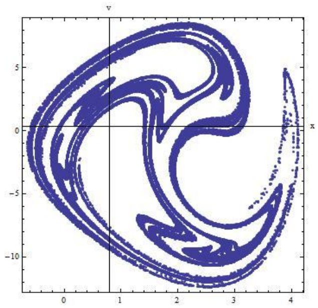

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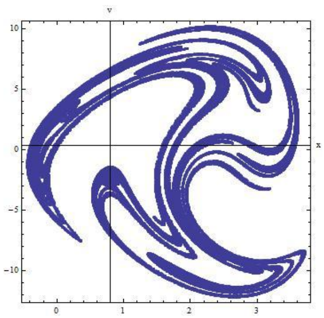

(e)

Fig-5: Chaotic attractors at γ=0.1, α=-2, β=2, ω=1.2, p1=1, p2=0.4 with initial condition $x(0) = 1, \dot{x}(0) = 0$ and F=1.307, F=3.164, F=6.5, F=12.34 and F=14.86

5. DETECTION OF HOMOCLINIC BIFURCATION IN OUR MODEL

Some of the theory involved in the bifurcations to chaos for flows and maps is a result of the behavior of the stable and unstable manifolds of saddle points. The stable and unstable manifolds can form homoclinic and heteroclinic orbits as a parameter is varied. Homoclinic bifurcation occurs if a

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stable (unstable) branch of a saddle point crosses the unstable (stable) branch of the same saddle point. In our model we noticed appearance of homoclinic orbits as the parameter values are increased or decreased. We found that if we take damping factor as zero i.e., $\gamma = 0$ and decrease the value of the nonlinear (cubic) stiffness parameter to 0.001 i.e., $\beta = 0.001$ keeping $\alpha = -2, \omega = 1.2$ and varying the forced parameter to a larger value, it always shows homoclinic behavior. There may be further changes in its behavior for even higher of *F* but we have observed that the range within which we have chose the value of F, it always shows homoclinic nature.

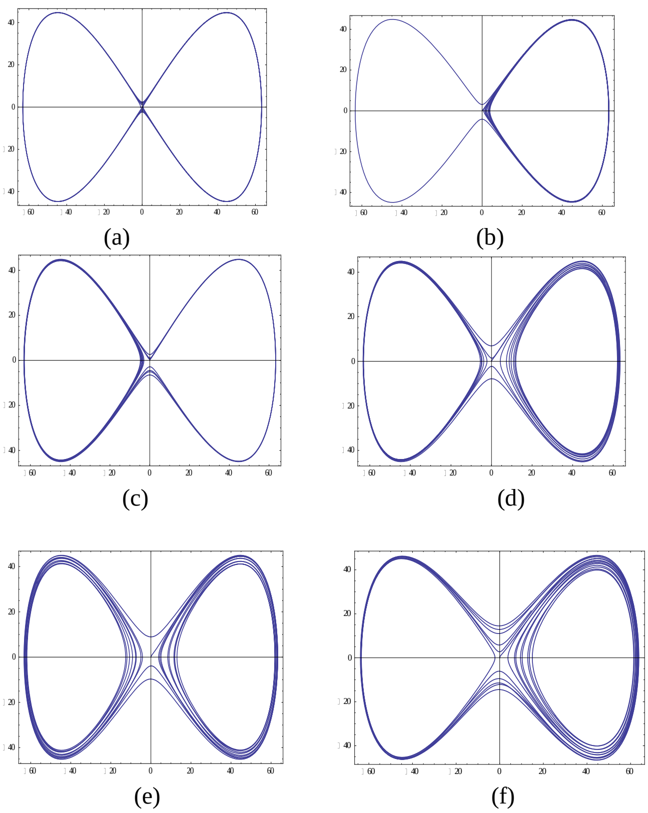

Fig-6: Homoclinic bifurcation occurs at γ=0, α= -2.8, β=0.001, ω=1.2 with initial condition $x(0) = 1, \dot{x}(0) = 0$ and F=10, F=100, F=250, F=500, F=750 and F=1000

CONCLUSION

This paper reported a systematic investigation in the phase space of the double well Duffing oscillator. We used bifurcation diagram to show the region characterized by the parameters for which one finds periodic solutions, aperiodic solutions. When driving force is increased there is a series of parallel "islands" of parameters characterized by aperiodic attractors with wide basis of attractors. We have found that even the model is perturbed by linear term it shows periodic and chaotic behavior. Next we observed that when damping coefficient is taken as zero and the nonlinear stiffness parameter is taken sufficiently small, the model shows homoclinic nature for whatever be the value of force.

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