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Solvable Lie Algebra and Lie's Theorem

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Abstract **:A simply connected Lie group is completely determined by its Lie algebra. By this, of course, we mean that it is determined to within isomorphism. We now discuss the possibilities for Lie groups which are not simply connected. We say a Lie group is connected if every two points of it can be joined by an arc lying in the group. If a Lie group is connected, we call it simply connected when every simple closed curve in the group can be continuously shrunk to a point without any part of it passing outside the group in the process. By a component of a Lie group we mean a maximal connected subset, i.e., all the elements which can be connected to some given element by arcs in the group. In this research we consider the situation where the Lie group is not connected. It can be shown that the component containing the identity is always a closed normal subgroup of the Lie group and that the components are precisely the cosets of this normal subgroup. We can regard this collection of cosets forming the quotient group, as an abstract group. (Indeed, if we take it to be a discrete group the natural mapping is analytic.) The study of the algebraic structure of a Lie group which is not connected can almost be broken into two parts:the structure of the connected subgroup forming the component of the identity and the structure of the discrete quotient group.**

INTRODUCTION

The first step in the classification of Lie algebras was the consideration of classical Lie algebras. They were totally outlined over C and R, but the problem is not so easy over an algebraically closed field of characteristic different from zero. This "new" Lie theory emerged around 1935 from the studies by Witt, who defined a simple Lie algebra (now called the Witt algebra W1) whose behavior was totally different from the Lie algebras studied till then, over C or R. Less than ten years later, Jacobson and Zassenhaus put some order in these new algebras, but it was not until the 21st century when a clear classification came. In fact, is a survey on these specific classifications of simple finite-dimensional Lie algebras over algebraically closed fields of characteristic $p>0$. Roughly speaking, and being p a prime greater than 3, the simple Lie algebras are either classical or finite dimensional Cartan Lie type (and their deformations) or Melikyian algebras. If the characteristic is big enough, some other interesting properties hold; for instance, for finite-dimensional Lie algebras over an algebraically closed field of characteristic = > 7, the existence of non-singular Casimir operators (i.e., dealing with a restricted Lie algebra) is equivalent to the decomposition of the algebra as a direct sum of classical simple Lie algebras.

Literature review

One of the most fundamental concepts in mathematics is that of a group. Germs of group was present, even in ancient times, in the study of motions in space, in the study of congruences of geometric figures. In the beginning of nineteenth century development of group theory started**.**

Dedekind [1897] studied about groups all of whose subgroups are normal.

Miller and Moreno [1903] studied groups all of whose proper subgroups are abelian. They studied that all such groups are solvable. Their orders cannot be divided by more than two distinct primes.

Schmidt [1924]worked on groups every proper subgroups of which is special.

Malcev [1945]determined the classification of complex solvable Lie algebras.

Golfond [1948] determined groups all of whose proper subgroups are special.

Chandra Harish [1955]representation of semi simple lie group.

METHODOLOGY

Definitions Solvable group**:**A group G is called solvable if it has a subnormal series whose factor group are all abelian, that is, if there are subgroup $\{1\} = G_0 \leq G_1 \leq \ldots\ldots \leq G_k = G$ such that G_{j-1} is normal G_j, and $G_j \,/\, G_{j-1}$ is an abelian group, for $j = 1,2,3 \ldots\ldots\ldots k$.

DefinitionsFunction space:A function space $f(I)$is the collection of all real-valued continuous functions defined on some interval I. $f^n(I)$ is the collection of all function which belongs $f(I)$ with continuous n^{th}derivatives. A function space is a topological vector space.

DefinitionsOperator**:** An operator $A: f^n(I) \to f(I)$ assigns to every function $f \in f^n(I)$. It is therefore a mapping between two function spaces.

Definitions Commutator**:** Let G be a group. An element of the form $aba^{-1}b^{-1}$ which is denoted by $[a, b]$ is called a commutator. The subgroup $G'$of G generated by all commutator of G is called commutator subgroup or the derived subgroup of G.

Or

Let $\tilde{A}, \tilde{B}$ ... ... be operators. Then the commutator of $\tilde{A}$ and $\tilde{B}$ is defined as $[\tilde{A}, \tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A}$.

***DefinitionsJacobi identity*:** The Jacobi identity is the relationship $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ between three elements A, B and C, where $[A, B]$ is the commutator. The elements of a lie algebra satisfy this identity.

RESULTS

In this section k and K are field satisfying $k \subseteq K \subseteq C$, (where C is a complex field) and all Lie algebras have k the underlying field and are finite dimensional.

THEOREM 1: An n-dimensional Lie algebra g is solvable if and only if there exists a sequence of subalgebras

$$g = a_0 \supseteq a_1 \supseteq \cdots a_n = 0$$

Such that, for each i, a_{i+1} is ideal in a_i and dim $(a_i/a_{i+1}) = 1$.

PROOF: Let g be solvable. Form the commutator series g^j and interpolate subspace a_i in the sequence so that dim (a_i/a_{i+1})=1 for all i. We have

$g = a_0 \supseteq a_1 \supseteq \cdots a_n = 0$.

For any i, we can find j such that $g^j \supseteq a_i \supseteq a_{i+1} \supseteq g^{j+1}$. Then

$[a_i, a_i] \subseteq [g^j, g^j] = g^{j+1} \subseteq a_{i+1}$.

Hence a_i is a subalgebra for each i, and a_{i+1} is an ideal a_i.

Conversely let the sequence exist. Choose x_i so that $a_i = kx_i + a_{i+1}$.

We show by induction that $g^i \subseteq a_i$, so that $g^n = 0$. In fact, $g^0 = a_0$ .If $g^i \subseteq a_i$, then

$g^{i+1} = [g^i, g^i] \subseteq [kx_i + a_{i+1}, kx_i + a_{i+1}] \subseteq [kx_i, a_{i+1}] + [a_{i+1}, a_{i+1}] \subseteq a_{i+1}$,

and the induction is complete. Hence g is solvable

The kind of sequence in the theorem is called an elementary sequence. The existence of such a sequence has the following implication. Write $a_i = kx_i \oplus a_{i+1}$. Then kx_i is a 1-demensional subspace of a_i , hence a subalgebra. Also a_{i+1} is ideal a_i . In a view of proposition 1.22 ....

a_i is exhibited as a semidirect product of a 1-dimensional Lie algebra and a_{i+1} . The theorem says that solvable Lie algebra are exactly those that can be obtain from semidirectproduct , starting from 0 and adding one dimension at a time .

Let V be vector space over K, and g be a Lie algebra. A representation of g on V is homomorphism of Lie algebra $\alpha: g \to (End_k V)^k$, which we often write simply as $\alpha: g \to End_k V$. Because of the definition of bracket in $End_k V$, the conditions on α are that it be k linear and satisfy

$\alpha([X, Y]) = \alpha(X)\alpha(Y) - \alpha(Y)\alpha(X)$ for all $X, Y \in g$. (1)

***THEOREM 2*:** Let g be solvable, let $V \neq 0$ be a fintedimensional vector space over K, and let $\alpha : g \to End_K V$ be a representation. If K is algebraically closed, then there is a simultaneous eigenvector $v \neq 0$ for all the members $\alpha(g)$. More generally (for K).there is a simultaneous eigenvector if all the eigenvalues of all $\alpha(X), X\epsilon\, g$, lie in K.

REMARK

1. When g is a solvable Lie algebra and α is representation, $\alpha(g)$ is solvable. This follows immediately from
2. The theorem is the base step in an induction that will show that V has a basis in all the matrices of $\alpha(g)$ are triangular. This conclusion appears as theorem 3 below. If g is solvable lie algebra of matrices and α is the identity and one of the conditions on K is satisfied, then g can be conjugated so as to be triangular.

PROOF: We induct on $\dim g = 1$, then $\alpha(g)$ consist of the multiples of a single transformation, and the results follows.

Assume the theorem for all solvable Lie algebras of dimension less than dim g satisfying the eigenvalue condition. Since g is solvable, $[g, g] \subset g$. choose a subspace h of codimension 1 in g with $[g, g] \subset h$. then $[h, g] \subseteq [g, g] \subseteq h$, and h is an ideal. So h is solvable. (also the eigenvalue condition holds for h if it holds for g.)

By inductive hypothesis we can choose $e \in V$ with $\alpha(H)e = \beta(H)e$ for all $H \in h$, where $\beta(H)e$ is a scalar valued function defined for $H \in h$.

$$e_{-1} = 0, \qquad e_0 = e \qquad e_p = \alpha(X)e_{p-1}$$

and let $E = span\{e_0 \ldots . e_p \ldots .\}$. Then $(X)E \subseteq E$. Let v be an eigenvalue for $\alpha(X)$ in E. We show that v is an eigenvalue for each (H) , $H \in h$.

First we show that

$$\alpha(H)e_p \equiv \beta(H)e_p \bmod span\{e_0, \ldots \ldots, e_{p-1}\} \quad (1)$$

For all $H \in h$. We do so by induction on p. Formula (1) is valid for $p = 0$ by definition of e_0. Assume (1) for p. then

$$\alpha(H)e_{p+1} = \alpha(X)\alpha(H)e_p$$

$$= \alpha([H, X]) + \alpha(X)\alpha(H)e_p$$

$$\equiv \beta([H, X])e_p + \alpha(X)\alpha(H)e_p \quad \bmod span\{e_0, \ldots \ldots, e_{p-1}\}$$

(By induction)

$$\equiv \beta([H, X])e_p + \beta(H)\alpha(X)e_p \quad \bmod span\{e_0, \ldots \ldots, e_{p-1}, \alpha(X)e_0, \ldots \ldots, \alpha(X)e_{p-1}\}$$

(By induction) $\equiv \beta(H)\alpha(X)e_p \bmod span\{e_0, \ldots \ldots, e_p\}$

$$= \beta(H)e_{p+1} \bmod span\{e_0, \ldots \ldots, e_p\}$$

This proves (1) for p+1 and completes the induction.

Next we show that

$$\beta([H, X\}) = 0 \quad for\ all\ H\epsilon\, h. \quad (2)$$

In fact, (1) says that $\alpha(H)E \subseteq E$ and that, relative to the basis $e_0, e_1, \ldots .,$ the linear transformation $\alpha(H)$ has matrix

$$\alpha(h) = \begin{pmatrix} \beta(H) & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta(H) \end{pmatrix}$$

Thus Tr $\alpha(H) = \beta(H)$ dim E, and we obtain

$\beta([H, X])$ dim E = Tr $\alpha([H, X\})$ =Tr$[\alpha(H), \alpha(X)] = 0$.

Since our fields have characteristic 0, (2) follow,

Now we can sharpen (1) to

$\alpha(H)e_p = \beta(H)e_p$ for all $H \in h$. (3)

To prove (3), we induct on p. for $p = 0$, the formula is the definition of e_0. Assume (3) for p. then

$$\alpha(H)e_{p+1} = \alpha(H)\alpha(X)e_p$$
$$= \alpha([H,X])e_p + \alpha(X)\alpha(H)e_p$$
$$= \beta([H,X])e_p + \alpha(X)\beta(H)e_p$$

(by induction)

$= 0 + \beta(H)e_{p+1}$ by (2)

$= \beta(H)e_{p+1}$.

This completes the induction and proves (3). Because of (3), $\alpha(H)x = \beta(H)x$ for all $x \in E$ and in particular for $x = v$. Hence the eigenvector v of $\alpha(X)$ is also an eigenvector of $\alpha(h)$. The theorem follows.

Before carrying out the induction indicated in Remark 2, we observe something about eigenvalues in connection with representations. Let α be a representation of g on a finite dimensional V, and let $U \subseteq V$ be an invariant subspace: $(g)U \subseteq U$. Then the formula $\alpha(X)(v + U) = \alpha(X)v + U$ defines a quotient representation of g on V/U. The characteristic polynomial of $\alpha(X)$ on V is the product of the characteristic polynomial on U and that on V/U, and hence the eigenvalues for V/U are a subset of those for V.

CONCLUSION

THEOREM 1: An n-dimensional Lie algebra g is solvable if and only if there exists a sequence of subalgebras

$$g = a_0 \supseteq a_1 \supseteq \cdots a_n = 0$$

Such that, for each i, a_{i+1} is ideal in a_i and dim $(a_i/a_{i+1}) = 1$.

THEOREM 2 **:** Let g be solvable, let $V \neq 0$ be a finte dimensional vector space over K, and let $\alpha : g \to End_K V$ be a representation. If K is algebraically closed, then there is a simultaneous eigenvector $v \neq 0$ for all the members $\alpha(g)$. More generally (for K).there is a simultaneous eigenvector if all the eigenvalues of all $\alpha(X), X \epsilon\, g$, lie in K.

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