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Solutions of Boundary Value Problems for Systems of Integro-Differential Equations

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***Abstract*:- In this work, we deals with approximate, uniqueness and stability of a solutions of boundary value problems for new systems of integro-differential equations .We provide a scheme of numerical- analytic method based upon successive approximation for investigate the periodic solution of ordinary differential equations, which are given by Samoilenko. We give sufficient conditions for the solvability of the problem and prove the uniform convergence of the approximations to the limit functions . Also these investigations lead us to the improving and extending the results of Butris.**

Keywords: Numerical-analytic methods, existence, uniqueness and stability solution, integro-differential equations, boundary value problem.

1. INTRODUCTION

Theory of differential equations has been of great interest for many years. It plays an important role in different subjects, such as physics, biology, chemistry, etc. and the study of periodic solutions for non-linear system of differential equations with boundary conditions is very important branch in the differential equation theory [3,11,14,16,25]. Many results about the existence and approximation of periodic solutions for system of nonlinear differential equations have been obtained by numerical-analytic methods that were proposed by samoilenko [21,24]. The periodic solutions for some nonlinear systems of integro-differential equations with boundary conditions have been used to study numerous problems for example [1,2,5,9,10,23,24].

The so-called numerical-analytic method for investigating a periodic solution, is widely used for studying solvability of nonlinear boundary value problem and constructing approximate solutions [4,19,20], and it is convenient for finding harmonic oscillations arising in various systems described by ordinary differential equations, differential equations with retarder argument and with impulsive action, integro-differential equations, partial differential equations and differential equations with boundary conditions [5,6,7,12,13,15,17,18,].

In [8] Butris and Taher, studied the periodic solution of integro-differential equations depended on special function with singular kernels having the following form

$$\left.\begin{aligned}\frac{dx}{dt} &= f\left(t,\gamma(t,\alpha),x(t),\mu,\int_{-\infty}^{t} R(t,\tau)\big(x(\tau)-y(\tau)\big)d\tau\right)\\ \frac{dy}{dt} &= g\left(t,\gamma(t,\alpha),y(t),\omega,\int_{-\infty}^{t} G(t,\tau)\big(x(\tau)-y(\tau)\big)d\tau\right)\end{aligned}\right\}$$

where

$\mu = \int_a^b \gamma(\tau,\alpha)x(\tau)d\tau$ and $\omega = \int_c^d \gamma(\tau,\alpha)y(\tau)d\tau$, $x \in D_1 \subset R^n, y \in D_2 \subset R^n$,
where D_1 and D_2 are compact domains where $\gamma(\tau,\alpha)$ is the Gamma function.

Consider the following problem :-

$$\left.\begin{aligned}\frac{dx}{dt} &= f\left(t,\beta(t,\alpha),x(t),\mu,\int_{-\infty}^{t} R(t,\tau)\big(x(\tau)-y(\tau)\big)d\tau\right)\\ \frac{dy}{dt} &= g\left(t,\beta(t,\alpha),y(t),\omega,\int_{-\infty}^{t} G(t,\tau)\big(x(\tau)-y(\tau)\big)d\tau\right)\end{aligned}\right\} \quad \dots (1.1)$$

with boundary conditions

$$\left.\begin{aligned}A_1x(\tau)+A_2x(\tau+T) &= e_1\\ B_1y(\tau)+B_2y(\tau+T) &= e_2\end{aligned}\right\} \quad \dots (1.2)$$

where

$\mu = \int_a^b \beta(t,\alpha)x(\tau)d\tau$ and $\omega = \int_c^d \beta(t,\alpha)y(\tau)d\tau$, $x \in D_1 \subset R^n, y \in D_2 \subset R^n$,
where D_1 and D_2 are compact domains .

Let the vector functions $f(t,\beta(t,\alpha),x,\mu,u)$ and $g(t,\beta(t,\alpha),y,\omega,v)$ are defined and continuous on the domain :-

$$\left.\begin{aligned}(t,\beta(t,\alpha),x,\mu,u) &\in R^1 \times G_1 = (-\infty,\infty)\times D \times D_1 \times D_\mu \times D_u\\ (t,\beta(t,\alpha),y,\omega,v) &\in R^1 \times G_2 = (-\infty,\infty)\times D \times D_2 \times D_\omega \times D_v\end{aligned}\right\} \quad \dots (1.3)$$

where D_μ, D_ω, D_u and D_v are bounded domains subset of Euclidean space R^n . Also

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$u = \int_{-\infty}^{t} R(t,\tau)\big(x(\tau) - y(\tau)\big)d\tau, v = \int_{-\infty}^{t} G(t,\tau)\,\big(x(\tau) - y(\tau)\big)d\tau, D = [\tau, \tau + T] \times (0,1]$. And also $A_1 = (A_{1ij}),\ A_2 = (A_{2ij}),\ B_1 = (B_{1ij})$ and $B_2 = (B_{2ij})$ are $(n \times n)$ non-negative constant matrices and $e_1 = (e_{11}, e_{12}, e_{13}, \dots, e_{1n})$, $e_2 = (e_{21}, e_{22}, e_{23}, \dots ,$
$e_{2n})$ are positive constant vectors.

Suppose that the functions $f(t, \beta(t,\alpha), x, \mu, u)$ and $g(t, \beta(t,\alpha), y, \omega, v)$ satisfy the following inequalities :

$$\left.\begin{array}{l} \| f(t,\beta(t,\alpha),x,\mu,u) \| \le \|\beta(t,\alpha)\| \| f(t,x,\mu,u)\| \le M_\beta M \\ \| g(t,\beta(t,\alpha),y,\omega,v) \| \le \|\beta(t,\alpha)\|\ \| g(t,y,\omega,v) \| \le N_\beta N \end{array}\right\} \quad \dots (1.4)$$

$$\| f(t,\beta(t,\alpha),x_1,\mu_1,u_1) - f(t,\beta(t,\alpha),x_2,\mu_2,u_2) \| \le M_\gamma(K_1 \| x_1 - x_2 \| + K_2 \| \mu_1 - \mu_2 \| + K_3 \| u_1 - u_2 \|) \quad \dots (1.5)$$

$$\| g(t,\beta(t,\alpha),y_1,\omega_1,v_1) - g(t,\beta(t,\alpha),y_2,\omega_2,v_2) \| \le N_\gamma(L_1 \| y_1 - y_2 \| + L_2 \| \omega_1 - \omega_2 \| + L_3 \| v_1 - v_2 \|) \quad \dots (1.6)$$

for all $t \in R^1, x, x_1, x_2 \in D_1, y, y_1, y_2 \in D_2, \mu, \omega, u$ and v are belong to D_μ, D_ω, D_u and D_v respectively, where $M, N, K_1, K_2, K_3, L_1, L_2$ and L_3 are positive constants. Also $\beta(t,\alpha)$ is said to be special function(Beta function). The isolated singular kernels $R(t,\tau)$ and $G(t,\tau)$ satisfying the following conditions :-

$$\left.\begin{array}{l} \| R(t,\tau) \| \le h e^{-\alpha(t-\tau)} \\ \| G(t,\tau) \| \le \sigma e^{-\beta(t-\tau)} \end{array}\right\} \quad \dots (1.7)$$

where $-\infty < 0 \le \tau \le t \le \tau + T, \alpha$ and δ are positive constants.

We defined the non-empty sets as follows :-

$$\left.\begin{array}{l} D_\chi = G_1 - \left(\dfrac{T}{2} M_\beta M + B_1\right) \\ \\ D_{1\chi} = G_2 - \left(\dfrac{T}{2} N_\beta N + B_2\right) \end{array}\right\} \quad \dots (1.8)$$

where
$B_1 = \|e_1 A_2^{-1} - (A_1 A_2^{-1} + E)x_0\|$ and $B_2 = \|e_2 B_2^{-1} - (B_1 B_2^{-1} + E)y_0\|$.

Furthermore, we suppose that the largest eigen-value of the matrix

$$\Lambda = \begin{pmatrix} \frac{T}{2} M_\beta C_1 & \frac{T}{2} M_\beta C_2 \\ \\ \frac{T}{2} N_\beta C_3 & \frac{T}{2} N_\beta C_4 \end{pmatrix}$$ less than one, i.e.

$$\lambda_{max}(\Lambda) = \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4(\varphi_2 - \varphi_3)}}{2} < 1 \quad \dots (1.9)$$

where $\varphi_1 = \frac{T}{2} M_\beta C_1 + \frac{T}{2} N_\beta C_4,\ \varphi_2 = \left(\frac{T}{2} M_\beta C_2\right)\left(\frac{T}{2} N_\beta C_3\right)$ and
$\varphi_3 = \left(\dfrac{T}{2} M_\beta C_1\right)\left(\dfrac{T}{2} N_\gamma C_4\right)$

Define the sequence of functions on the domain (1.3) by the following

$$x_{m+1}(t,x_0,y_0) = x_0 + \int_\tau^t \left[f(\tau,\beta(\tau,\alpha),x_m(\tau,x_0,y_0),\mu_m,u_m) - \frac{1}{T}\int_\tau^{\tau+T} f(\tau,\beta(\tau,\alpha) ,x_m(\tau,x_0,y_0),\mu_m,u_m) \right] d\tau + \frac{t-\tau}{T}\,[e_1 A_2^{-1} - (A_1 A_2^{-1} + E)x_0\,] \quad \dots (1.10)$$

and

$$y_{m+1}(t,x_0,y_0) = y_0 + \int_\tau^t \left[g(\tau,\beta(\tau,\alpha),y_m(\tau,x_0,y_0),\omega_m,v_m) - \frac{1}{T}\int_\tau^{\tau+T} g(\tau,\beta(\tau,\alpha) ,y_m(\tau,x_0,y_0),\omega_m,v_m) \right] d\tau + \frac{t-\tau}{T}\,[e_2 B_2^{-1} - (B_1 B_2^{-1} + E)y_0\,] \quad \dots (1.11)$$

with
$x_0(t,x_0,y_0) = x_0,\ y_0(t,x_0,y_0) = y_0, \qquad m = 0,1,2,\dots$.
where

$$\mu_m = \int_a^b \beta(\tau,\alpha)x_m(\tau,x_0,y_0)d\tau, \omega_m = \int_c^d \beta(\tau,\alpha)y_m(\tau,x_0,y_0)d\tau$$

$$u_m = \int_{-\infty}^t R(t,\tau)(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0))d\tau$$

$$v_m = \int_{-\infty}^{t} G(t,\tau)(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0))d\tau$$

Lemma 1.1.[19] Let $f(t,\gamma(t,\alpha),x,\mu,u)$ and $g\ (t,\gamma(t,\alpha),y,\omega,v)$ be vectors which are defined on the interval $[\tau,\tau+T]$, then

$$\begin{pmatrix} \| Q_1(t,x_0,y_0) \| \\ \\ \| Q_2(t,x_0,y_0) \| \end{pmatrix} \le \begin{pmatrix} \alpha(t)M_\beta M + B_1 \\ \\ \alpha(t)N_\beta N + B_2 \end{pmatrix} \qquad \dots (1.12)$$

satisfies for $\tau \le t \le \tau + T$ and $\alpha(t) \le \frac{T}{2}$,

where $\alpha(t) = 2(t-\tau)\left(1 - \frac{t-\tau}{T}\right)$ for all $t \in [\tau,\tau+T]$,

$$Q_1(t,x_0,y_0) = \int_{\tau}^{t}\left[f(\tau,\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,u) - \frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,u)d\tau\right]d\tau + \frac{t-\tau}{T}[e_1A_2^{-1} - (A_1A_2^{-1}+E)x_0]$$

$$Q_2(t,x_0,y_0) = \int_{\tau}^{t}\left[g(\tau,\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,v) - \frac{1}{T}\int_{\tau}^{\tau+T} g(\tau,\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,v)d\tau\right]d\tau + \frac{t-\tau}{T}[e_2B_2^{-1} - (B_1B_2^{-1}+E)y_0]$$

Proof .

$$\| Q_1(t,x_0,y_0) \| \le (1 - \frac{t-\tau}{T})\int_{\tau}^{t}\|\beta(\tau,\alpha)\|\left\|f(\tau,x(\tau,x_0,y_0),\mu,u)\right\|d\tau + \frac{t-\tau}{T}\int_{t}^{\tau+T}\|\beta(\tau,\alpha)\|\left\|f(\tau,x(\tau,x_0,y_0),\mu,u)\right\|d\tau + \left\|e_1A_2^{-1} - (A_1A_2^{-1}+E)x_0\right\|$$

$$\le \alpha(t)M_\beta M + B_1$$

so that

$$\| Q_1(t,x_0,y_0) \| \le \alpha(t)M_\beta M + B_1 \qquad \dots (1.13)$$

and

$$\| Q_2(t,x_0,y_0) \| \le \alpha(t)N_\beta N + B_2 \qquad \dots (1.14)$$

From (1.13) and (1.14) we conclude that the inequality (1.12) holds .

2. APPROXIMATION SOLUTION FOR THE SYSTEM (1.1) WITH BOUNDARY CONDITIONS (1.2).

In this section, we shall the investigation of approximation solution of the system (1.1) with boundary conditions (1.2) by the following :-

Theorem 2.1: If the system (1.1) with boundary conditions (1.2) defined in the domain (1.3), continuous in t,x,y and satisfy the inequalities (1.4) to (1.7) and the conditions (1.8) and (1.9). Then the sequence of functions (1.10) and (1.11) convergent uniformly as $m \to \infty$ on the domain :-

$$(t,x_0,y_0) \in [\tau,\tau+T] \times D_\chi \times D_{1\chi} \qquad \dots (2.1)$$

to the limit function $\begin{pmatrix} x(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix}$ defined on the domain (2.1) which is periodic in t of period T and satisfying the following integral equations :-

$$x(t,x_0,y_0) = x_0 + \int_{\tau}^{t}\left[f(\tau,\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,u) - \frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,u)\right]d\tau + \frac{t-\tau}{T}[e_1A_2^{-1} - (A_1A_2^{-1}+E)x_0] \qquad \dots (2.2)$$

and

$$y(t,x_0,y_0) = y_0 + \int_{\tau}^{t}\left[g(\tau,\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,v) - \frac{1}{T}\int_{\tau}^{\tau+T} g(\tau,\beta(\tau,\alpha),\right.$$

$$y(\tau, x_0, y_0), \omega, v)\Bigg] d\tau + \frac{t-\tau}{T}\,[e_2 B_2^{-1} - (B_1 B_2^{-1} + E) y_0\,] \qquad \dots (2.3)$$

which are unique solutions on the domain (2.1), provided that

$$\begin{pmatrix} \| x(t, x_0, y_0) - x_0 \| \\ \\ \| y(t, x_0, y_0) - y_0 \| \end{pmatrix} \le \begin{pmatrix} \frac{T}{2} M_\beta M + B_1 \\ \\ \frac{T}{2} N_\beta N + B_2 \end{pmatrix} \qquad \dots (2.4)$$

and

$$\begin{pmatrix} \| x(t, x_0, y_0) - x_m(t, x_0, y_0) \| \\ \\ \| y(t, x_0, y_0) - y_m(t, x_0, y_0) \| \end{pmatrix} \le \Lambda^m (E - \Lambda)^{-1} \eta \qquad \dots (2.5)$$

for all $m \ge 0, x_0 \in D_\chi, y_0 \in D_{1\chi}$ and $t \in R^1$, where E is identity matrix.

Proof. Setting $m = 0$ in the sequence of functions (1.10),(1.11) and using Lemma 1.1, we have

$$\|x_1(t, x_0, y_0) - x_0\| = \Bigg\| \int_\tau^t \Bigg[f(\tau, \beta(\tau, \alpha), x_0, \int_a^b \gamma(\tau, \alpha)\, x_0 d\tau, \int_{-\infty}^t R(t, \tau)$$
$$(x_0 - y_0) d\tau) - \frac{1}{T} \int_\tau^{\tau+T} f(\tau, \beta(\tau, \alpha), x_0, \int_a^b \gamma(\tau, \alpha)\, x_0 d\tau, \int_{-\infty}^t R(t, \tau)$$
$$(x_0 - y_0) d\tau) + \frac{t-\tau}{T} [e_1 A_2^{-1} - (A_1 A_2^{-1} + E) x_0] \Bigg\|$$

Hence

$$\|x_1(t, x_0, y_0) - x_0\| \le \alpha(t) M_\beta M + B_1 \le \frac{T}{2} M_\beta M + B_1 \qquad \dots (2.6)$$

for all $x_1(t, x_0, y_0) \in G_1$, for all $t \in [\tau, \tau + T], x_0 \in D_\chi$.
and similarly, we get

$$\|y_1(t, x_0, y_0) - y_0\| \le \alpha(t) N_\beta N + B_2 \le \frac{T}{2} N_\beta N + B_2 \qquad \dots (2.7)$$

for all $y_1(t, x_0, y_0) \in G_2$, for all $t \in [\tau, \tau + T],\ y_0 \in D_{1\chi}$.
Thus, by mathematical induction, we can prove that

$$\left.\begin{aligned} \| x_m(t, x_0, y_0) - x_0 \| &\le \frac{T}{2} M_\beta M + B_1 \\ \\ \| y_m(t, x_0, y_0) - y_0 \| &\le \frac{T}{2} N_\beta N + B_2 \end{aligned}\right\} \qquad \dots (2.8)$$

i.e. $x_m(t, x_0, y_0) \in G_1, y_m(t, x_0, y_0) \in G_2,\ x_0 \in D_\chi, y_0 \in D_{1\chi}$, for all $t \in [\tau, \tau + T]$, $m = 0,1,2, \dots$.
Rewrite the inequality (2.8) by the vector from, then we get (2.4).

We claim that the sequence of functions (1.10) and (1.11) are uniformly convergent on the domain (2.1).

We begin by finding an estimate for $\|x_{m+1}(t, x_0, y_0) - x_m(t, x_0, y_0)\|$ and $\|y_{m+1}(t, x_0, y_0) - y_m(t, x_0, y_0)\|$, since

$$x_{m+1}(t, x_0, y_0) - x_m(t, x_0, y_0) = \int_\tau^t \Bigg[[f(\tau, \beta(\tau, \alpha), x_m(\tau, x_0, y_0), \mu_m, u_m)$$
$$- f(\tau, \beta(\tau, \alpha), x_{m-1}(\tau, x_0, y_0), \mu_{m-1}, u_{m-1})] - \frac{1}{T} \int_\tau^{\tau+T} [f(\tau, \beta(\tau, \alpha),$$
$$x_m(\tau, x_0, y_0), \mu_m, u_m) - f(\tau, \beta(\tau, \alpha), x_{m-1}(\tau, x_0, y_0), \mu_{m-1}, u_{m-1})] d\tau \Bigg] d\tau$$

and

$$y_{m+1}(t, x_0, y_0) - y_m(t, x_0, y_0) = \int_\tau^t \Bigg[[g(\tau, \beta(\tau, \alpha), y_m(\tau, x_0, y_0), \omega_m, v_m)$$
$$- g(\tau, \beta(\tau, \alpha), y_{m-1}(\tau, x_0, y_0), \omega_{m-1}, v_{m-1})] - \frac{1}{T} \int_\tau^{\tau+T} [g(\tau, \gamma(\tau, \alpha),$$

$$y_m(\tau,x_0,y_0),\omega_m,v_m) \; - \; g(\tau,\beta(\tau,\alpha),y_{m-1}(\tau,x_0,y_0),\omega_{m-1},v_{m-1})]d\tau\Big] d\tau$$

where

$$\mu_{m-1} = \int_a^b \beta(\tau,\alpha)x_{m-1}(\tau,x_0,y_0)d\tau, \omega_{m-1} = \int_c^d \beta(\tau,\alpha)y_{m-1}(\tau,x_0,y_0)d\tau$$

$$u_{m-1} = \int_{-\infty}^t R(t,\tau)(x_{m-1}(\tau,x_0,y_0) - y_{m-1}(\tau,x_0,y_0))d\tau$$

$$v_{m-1} = \int_{-\infty}^t G(t,\tau)(x_{m-1}(\tau,x_0,y_0) - y_{m-1}(\tau,x_0,y_0))d\tau$$

Using the inequalities (1.5) and (1.6), we have

$$\|x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0)\| \le \alpha(t)M_\beta C_1\|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| + \alpha(t)M_\beta C_2\|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\| \quad \text{... (2.9)}$$

where $C_1 = K_1 + K_2M_\beta(b-a) + \frac{h}{\alpha}K_3$ and $C_2 = \frac{h}{\alpha}K_3$

And

$$\|y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0)\| \le \alpha(t)N_\beta C_3\|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| + \alpha(t)N_\beta C_4\|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\| \quad \text{... (2.10)}$$

where $C_3 = \frac{\sigma}{\beta}L_3$ and $C_4 = L_1 + L_2N_\beta(d-c) + \frac{\sigma}{\beta}L_3$

Rewrite the inequalities (2.9) and (2.10) in vector form

$$\begin{pmatrix} \| x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0) \| \\ \\ \| y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0) \| \end{pmatrix} \le \Lambda \begin{pmatrix} \| x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0) \| \\ \\ \| y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0) \| \end{pmatrix} \quad \text{... (2.11)}$$

If we set $m = 1$ in (1.23) and (1.24), we get

$$\begin{pmatrix} \| x_2(t,x_0,y_0) - x_1(t,x_0,y_0) \| \\ \\ \| y_2(t,x_0,y_0) - y_1(t,x_0,y_0) \| \end{pmatrix} \le \Lambda \begin{pmatrix} \| x_1(t,x_0,y_0) - x_0 \| \\ \\ \| y_1(t,x_0,y_0) - y_0 \| \end{pmatrix} \quad \text{... (2.12)}$$

Next, set $m = 2$ and use (2.12), we have

$$\begin{pmatrix} \| x_3(t,x_0,y_0) - x_2(t,x_0,y_0) \| \\ \\ \| y_3(t,x_0,y_0) - y_2(t,x_0,y_0) \| \end{pmatrix} \le \Lambda^2 \begin{pmatrix} \| x_1(t,x_0,y_0) - x_0 \| \\ \\ \| y_1(t,x_0,y_0) - y_0 \| \end{pmatrix}$$

Setting $m = 2$, yields in the same way

$$\begin{pmatrix} \| x_4(t,x_0,y_0) - x_3(t,x_0,y_0) \| \\ \\ \| y_4(t,x_0,y_0) - y_3(t,x_0,y_0) \| \end{pmatrix} \le \Lambda^3 \begin{pmatrix} \| x_1(t,x_0,y_0) - x_0 \| \\ \\ \| y_1(t,x_0,y_0) - y_0 \| \end{pmatrix}$$

and by mathematical induction, we obtain that

$$\begin{pmatrix} \| x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0) \| \\ \\ \| y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0) \| \end{pmatrix} \le \Lambda^m \begin{pmatrix} \| x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0) \| \\ \\ \| y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0) \| \end{pmatrix}$$

Now, from $m = 1,2,\ldots$ and $p \ge 1$, we have

$$\begin{pmatrix} \| x_{m+p}(t,x_0,y_0) - x_m(t,x_0,y_0) \| \\ \\ \| y_{m+p}(t,x_0,y_0) - y_m(t,x_0,y_0) \| \end{pmatrix} \le \begin{pmatrix} \| x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0) \| \\ \\ \| y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0) \| \end{pmatrix}$$
$$+ \begin{pmatrix} \| x_{m+2}(t,x_0,y_0) - x_{m+1}(t,x_0,y_0) \| \\ \\ \| y_{m+2}(t,x_0,y_0) - y_{m+1}(t,x_0,y_0) \| \end{pmatrix} + \cdots + \begin{pmatrix} \| x_{m+p}(t,x_0,y_0) - x_{m+p-1}(t,x_0,y_0) \| \\ \\ \| y_{m+p}(t,x_0,y_0) - y_{m+p-1}(t,x_0,y_0) \| \end{pmatrix}$$
$$\le \Lambda^m \begin{pmatrix} \| x_1(t,x_0,y_0) - x_0 \| \\ \\ \| y_1(t,x_0,y_0) - y_0 \| \end{pmatrix} + \Lambda^{m+1} \begin{pmatrix} \| x_1(t,x_0,y_0) - x_0 \| \\ \\ \| y_1(t,x_0,y_0) - y_0 \| \end{pmatrix} + \cdots +$$
$$\Lambda^{m+p-1} \begin{pmatrix} \| x_1(t,x_0,y_0) - x_0 \| \\ \\ \| y_1(t,x_0,y_0) - y_0 \| \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \| x_{m+p}(t,x_0,y_0) - x_m(t,x_0,y_0) \| \\ \\ \| p(t,x_0,y_0) - y_m(t,x_0,y_0) \| \end{pmatrix} \le \Lambda^m \,(E + \Lambda + \cdots + \Lambda^{p-1})\eta \qquad \text{... (2.13)}$$

where $\eta = \begin{pmatrix} \frac{T}{2} M_\beta M + B_1 \\ \\ \frac{T}{2} N_\beta N + B_2 \end{pmatrix}$

For all $t \in [\tau, \tau+T], x_0 \in D_\chi$ and $y_0 \in D_{1\chi}$.

Since $\frac{\varphi_1 + \sqrt{\varphi_1^2 + 4(\varphi_2 - \varphi_3)}}{2} < 1$ and $\lim_{m\to\infty} \Lambda^m = 0$, so that the right side of (2.13) tends to zero. Therefore, the sequence of function $\begin{pmatrix} x\,(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix}$ is convergent uniformly on the domain (2.1) to the limit function $\begin{pmatrix} x\,(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix}$ which is defined on the same domain.

Let

$$\lim_{m\to\infty} \begin{pmatrix} x_m(t,x_0,y_0) \\ \\ y_m(t,x_0,y_0) \end{pmatrix} = \begin{pmatrix} x\,(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix} \qquad \text{... (2.14)}$$

Since the sequence of functions (1.10) and (1.11) are conations in t of period T, then the limiting function $\begin{pmatrix} x\,(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix}$ is also conations. Also, by the Lemma 1.1 and the inequality (2.13) the inequalities (2.4) and (2.5) are holds for all $m \ge 0$.

By using the relation (2.14) and proceeding in (1.10) and (1.11) to limit, when $m \to 0$, convinces us that the limiting function $\begin{pmatrix} x\,(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix}$ is the a solution of the integral equations (1.10) and (1.11).

Finally, we must still prove that $\begin{pmatrix} x\,(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix}$ is a unique solution of (1.1) with boundary conditions (1.2). Assume that $\begin{pmatrix} \hat{x} \\ \\ \hat{y} \end{pmatrix}$ is another solution for the system (1.1) with boundary conditions (1.2), i.e.

$$\hat{x}(t,x_0,y_0) = x_0 + \int_\tau^t \Bigg[f(\tau, \beta(\tau,\alpha), \hat{x}(\tau,x_0,y_0), \mu, u) - \frac{1}{T}\int_\tau^{\tau+T} f(\tau, \beta(\tau,\alpha), \hat{x}(\tau,x_0,y_0), \mu, u) \Bigg] d\tau + \frac{t-\tau}{T}\,[e_1 A_2^{-1} - (A_1 A_2^{-1} + E)x_0\,] \qquad \text{... (2.15)}$$

and

$$\hat{y}(t,x_0,y_0) = y_0 + \int_\tau^t \Bigg[g(\tau, \beta(\tau,\alpha), \hat{y}(\tau,x_0,y_0), \omega, v) - \frac{1}{T}\int_\tau^{\tau+T} g(\tau, \beta(\tau,\alpha), \hat{y}(\tau,x_0,y_0), \omega, v) \Bigg] d\tau + \frac{t-\tau}{T}\,[e_2 B_2^{-1} - (B_1 B_2^{-1} + E)y_0\,] \qquad \text{... (2.16)}$$

where

$$\mu = \int_a^b \beta(\tau,\alpha)\hat{x}(\tau,x_0,y_0)d\tau, \omega = \int_c^d \beta(\tau,\alpha))\hat{y}(\tau,x_0,y_0)d\tau$$

$$u = \int_{-\infty}^t R(t,\tau)(\hat{x}(\tau,x_0,y_0) - \hat{y}(\tau,x_0,y_0))d\tau$$

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$$v=\int_{-\infty}^{t} G(t,\tau)(\hat{x}(\tau,x_0,y_0)-\hat{y}(\tau,x_0,y_0))d\tau$$

Now, we shall prove that $\begin{pmatrix} x\,(t,x_0,y_0) \\ \\ y(t,x_0,y_0) \end{pmatrix}=\begin{pmatrix} \hat{x}\,(t,x_0,y_0) \\ \\ \hat{y}(t,x_0,y_0) \end{pmatrix}$ for all $t\in[\tau,\tau+T]$ and $x_0\in D_\chi$, $y_0\in D_{1\chi}$. And to do this we need to prove the following inequality :-

$$\begin{pmatrix} \| \hat{x}(t,x_0,y_0)-x_m(t,x_0,y_0) \| \\ \\ \| \hat{y}(t,x_0,y_0)-y_m(t,x_0,y_0) \| \end{pmatrix} \le \Lambda^m(E-\Lambda)^{-1}\eta \qquad \dots (2.17)$$

where

$$M=\max_{t\in[\tau,\tau+T]}|f(t,\beta(\tau,\alpha),\hat{x}(t,x_0,y_0),\mu,u)|$$

and

$$N=\max_{t\in[\tau,\tau+T]}|g(t,\beta(\tau,\alpha),\hat{y}(t,x_0,y_0),\omega,v)|$$

From (1.10), (1.11) and suppose $m=0$ in (2.13), we have

$$\| \hat{x}(t,x_0,y_0)-x_0 \| = \Bigg\| x_0+\int_{\tau}^{t}\Bigg[f\Big(\tau,\beta(\tau,\alpha),\hat{x}(\tau,x_0,y_0),\int_a^b \beta(\tau,\alpha)\,\hat{x}(\tau,x_0,y_0)d\tau,$$

$$\int_{-\infty}^{t} R(t,\tau)(\hat{x}(\tau,x_0,y_0)-\hat{y}(\tau,x_0,y_0))d\tau\Big)-\frac{1}{T}\int_{\tau}^{\tau+T} f\Big(\tau,\beta(\tau,\alpha),\hat{x}(\tau,x_0,y_0),$$

$$\int_a^b \beta(\tau,\alpha)\,\hat{x}(\tau,x_0,y_0)d\tau,\int_{-\infty}^{t} R(t,\tau)(\hat{x}(\tau,x_0,y_0)-\hat{y}(\tau,x_0,y_0))d\tau\Big)d\tau\Bigg]d\tau$$

$$+\frac{t-\tau}{T}[e_1A_2^{-1}-(A_1A_2^{-1}+E)x_0]-x_0\Bigg\|$$

Hence

$$\| \hat{x}(t,x_0,y_0)-x_0 \| \le \alpha(t)M_\beta M+B_1\le \frac{T}{2}M_\beta M+B_1 \qquad \dots (2.18)$$

By the same way, we get

$$\| \hat{y}(t,x_0,y_0)-y_0 \| \le \alpha(t)N_\beta N+B_2\le \frac{T}{2}N_\beta N+B_2 \qquad \dots (2.19)$$

Thus (2.17) is true for $m=0$, suppose that (2.18) and (2.19) are true for $m=p$, we have

$$\| \hat{x}(t,x_0,y_0)-x_p(t,x_0,y_0) \| \le \alpha(t)M_\beta C_1 \| \hat{x}(t,x_0,y_0)-x_p(t,x_0,y_0) \| + \alpha(t)M_\beta C_2 \| \hat{y}(t,x_0,y_0)-y_p(t,x_0,y_0) \| \qquad \dots (2.20)$$

And similarly

$$\| \hat{y}(t,x_0,y_0)-y_p(t,x_0,y_0) \| \le \alpha(t)N_\gamma C_3 \| \hat{x}(t,x_0,y_0)-x_p(t,x_0,y_0) \| + \alpha(t)N_\gamma C_3 \| \hat{y}(t,x_0,y_0)-y_p(t,x_0,y_0) \| \qquad \dots (2.21)$$

Rewrite the inequalities (1.34) and (1.35) in a vector form, we get

$$\begin{pmatrix} \| \hat{x}(t,x_0,y_0)-x_p(t,x_0,y_0) \| \\ \\ \| \hat{y}(t,x_0,y_0)-y_p(t,x_0,y_0) \| \end{pmatrix} \le \Lambda^p(E-\Lambda)^{-1}\eta \qquad \dots (2.22)$$

Then, when $m=p+1$, we have

$$\| \hat{x}(t,x_0,y_0)-x_{p+1}(t,x_0,y_0) \| \le \alpha(t)M_\beta C_1 \| \hat{x}(t,x_0,y_0)-x_p(t,x_0,y_0) \| + \alpha(t)M_\beta C_2 \| \hat{y}(t,x_0,y_0)-y_p(t,x_0,y_0) \| \qquad \dots (2.23)$$

And similarly

$$\| \hat{y}(t,x_0,y_0)-y_{p+1}(t,x_0,y_0) \| \le \alpha(t)N_\beta C_3 \| \hat{x}(t,x_0,y_0)-x_p(t,x_0,y_0) \| + \alpha(t)N_\beta C_3 \| \hat{y}(t,x_0,y_0)-y_p(t,x_0,y_0) \| \qquad \dots (2.24)$$

From (2.23) and (2.24) in vector form, we get

$$\begin{pmatrix}\| \hat{x}(t,x_0,y_0)-x_{p+1}(t,x_0,y_0)\| \\ \\ \| \hat{y}(t,x_0,y_0)-y_{p+1}(t,x_0,y_0)\|\end{pmatrix} \le \Lambda \begin{pmatrix}\| \hat{x}(t,x_0,y_0)-x_{p}(t,x_0,y_0)\| \\ \\ \| \hat{y}(t,x_0,y_0)-y_{p}(t,x_0,y_0)\|\end{pmatrix}$$

$$\le \Lambda\,\Lambda^p (E-\Lambda)^{-1}\eta$$

Hence

$$\begin{pmatrix}\| \hat{x}(t,x_0,y_0)-x_{p+1}(t,x_0,y_0)\| \\ \\ \| \hat{y}(t,x_0,y_0)-y_{p+1}(t,x_0,y_0)\|\end{pmatrix} \le \Lambda^{p+1}(E-\Lambda)^{-1}\eta \qquad \dots(2.25)$$

Thus (2.17) is true in general.

So

$$\lim_{m\to\infty}\begin{pmatrix}\| \hat{x}(t,x_0,y_0)-x_{m}(t,x_0,y_0)\| \\ \\ \| \hat{y}(t,x_0,y_0)-y_{m}(t,x_0,y_0)\|\end{pmatrix} \le (E-\Lambda)^{-1}\eta \lim_{m\to\infty}\Lambda^m.$$

Because $\lim_{m\to\infty}\Lambda^m = 0$, and therefore

$$\lim_{m\to\infty}\begin{pmatrix}\| \hat{x}(t,x_0,y_0)-x_{m}(t,x_0,y_0)\| \\ \\ \| \hat{y}(t,x_0,y_0)-y_{m}(t,x_0,y_0)\|\end{pmatrix} = 0$$

And hence

$$\begin{pmatrix}\hat{x}(t,x_0,y_0) \\ \\ \hat{y}(t,x_0,y_0)\end{pmatrix} = \lim_{m\to\infty}\begin{pmatrix}x_m(t,x_0,y_0) \\ \\ y_m(t,x_0,y_0)\end{pmatrix} = \begin{pmatrix}x(t,x_0,y_0) \\ \\ y(t,x_0,y_0)\end{pmatrix}$$

and this proves that the solution $\begin{pmatrix}x(t,x_0,y_0) \\ \\ y(t,x_0,y_0)\end{pmatrix}$ is a unique on the domain (2.15).

3. EXISTENCE SOLUTION OF (1.1) WITH BOUNDARY CONDITION (1.2).

The problem of the existence solution for the system (1.1) with boundary condition (1.2) is uniquely connected with the existence of zeros of the vector function :-

$$\begin{pmatrix}\Delta_1^*(0,x_0,y_0) \\ \\ \Delta_2^*(0,x_0,y_0)\end{pmatrix} = \begin{pmatrix}\frac{1}{T}\int_\tau^{\tau+T} f(\tau,\gamma\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,u)d\tau + \frac{1}{T}[A_1A_2^{-1}+E)x_0 - e_1A_2^{-1}] \\ \\ \frac{1}{T}\int_\tau^{\tau+T} g(\tau,\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,v)d\tau + \frac{1}{T}[B_1B_2^{-1}+E)y_0 - e_2B_2^{-1}]\end{pmatrix}$$

$$\dots(3.1)$$

the vector function $\begin{pmatrix}\Delta_1^*(0,x_0,y_0) \\ \\ \Delta_2^*(0,x_0,y_0)\end{pmatrix}$ is approximately determined by the following :-

$$\begin{pmatrix}\Delta_{1m}^*(0,x_0,y_0) \\ \\ \Delta_{2m}^*(0,x_0,y_0)\end{pmatrix} = \begin{pmatrix}\frac{1}{T}\int_\tau^{\tau+T} f(\tau,\beta(\tau,\alpha),x_m(\tau,x_0,y_0),\mu_m,u_m)d\tau + \frac{1}{T}[A_1A_2^{-1}+E)x_0 - e_1A_2^{-1}] \\ \\ \frac{1}{T}\int_\tau^{\tau+T} g(\tau,\beta(\tau,\alpha),y_m(\tau,x_0,y_0),\omega_m,v_m)d\tau + \frac{1}{T}[B_1B_2^{-1}+E)y_0 - e_2B_2^{-1}]\end{pmatrix}$$

$$\dots(3.2)$$

Theorem 3.1: If the hypotheses and all conditions of the theorem (3.1) are given then the following inequality are satisfy :-

$$\begin{pmatrix}\| \Delta_1^*(0,x_0,y_0)-\Delta_{1m}^*(0,x_0,y_0)\| \\ \\ \| \Delta_2^*(0,x_0,y_0)-\Delta_{2m}^*(0,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\langle \begin{pmatrix}d_1\\d_2\end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta\,\rangle \\ \\ \langle \begin{pmatrix}d_3\\d_4\end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta\,\rangle\end{pmatrix} \qquad \dots(3.3)$$

where

$d_1 = M_\beta C_1,\ d_2 = M_\beta C_2,\ \ d_3 = N_\beta C_3,\ \ d_4 = N_\beta C_4.$

Proof . From the equations (3.1) and (3.2), we have

$$\| \Delta_1^*(0,x_0,y_0) - \Delta_{1m}^*(0,x_0,y_0) \| \le \langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \qquad \dots (3.4)$$

and

$$\| \Delta_2^*(0,x_0,y_0) - \Delta_{2m}^*(0,x_0,y_0) \| \le \langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \qquad \dots (3.5)$$

From (3.4) and (3.5), we get (3.3).

Now, we prove the following theorem taking into account that the inequality (3.3) will be satisfy for all $m \ge 0, x_0 \in D_\chi$, $y_0 \in D_{1\chi}$.

Theorem 3.2: Let the vector function $f(t,\gamma(t,\alpha),x,\mu,u)$ and $g(t,\gamma(t,\alpha),y,\omega,v)$ of the system (1.1) with boundary condition (1.2) are defined on the interval $[a,b]$ and $[c,d]$ in R^1. Assume that for any integer $m \ge 0$, the sequence of function (3.2) satisfies the function inequalities :-

$$\left.\begin{aligned} \min_{x_0\in J_1, y_0\in J_2} \Delta_{1m}^*(0,x_0,y_0) &\le -\langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \\ \max_{x_0\in J_1, y_0\in J_2} \Delta_{1m}^*(0,x_0,y_0) &\ge \langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \end{aligned}\right\} \qquad \dots (3.6)$$

$$\left.\begin{aligned} \min_{x_0\in J_1, y_0\in J_2} \Delta_{2m}^*(0,x_0,y_0) &\le -\langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \\ \max_{x_0\in J_1, y_0\in J_2} \Delta_{2m}^*(0,x_0,y_0) &\ge \langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \end{aligned}\right\} \qquad \dots (3.7)$$

where $x_0 \in J_1 = \left[a + \frac{T}{2} M_\beta M + B_1 , b - \frac{T}{2} M_\beta M - B_1\right]$

and

$y_0 \in J_2 = [c + \frac{T}{2} N_\beta N + B_1, d - \frac{T}{2} N_\beta N - B_2]$.

Then (1.1) with boundary condition (1.2) has a periodic solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$$

for $x_0 \in [a + J_1, b - J_1]$ and $y_0 \in [c + J_2, d - J_2]$.

Proof . Let x_1, x_2 and y_1, y_2 be any points belonging on the intervals I_1 and I_2 respectively, such that

$$\left.\begin{aligned} \Delta_{1m}^*(0,x_1,y_1) &= \min_{x_0\in I_1, y_0\in I_2} \Delta_{1m}^*(0,x_0,y_0) \\ \Delta_{1m}^*(0,x_2,y_2) &= \max_{x_0\in I_1, y_0\in I_2} \Delta_{1m}^*(0,x_0,y_0) \end{aligned}\right\} \qquad \dots (3.8)$$

$$\left.\begin{aligned} \Delta_{2m}^*(0,x_1,y_1) &= \min_{x_0\in I_1, y_0\in I_2} \Delta_{2m}^*(0,x_0,y_0) \\ \Delta_{2m}^*(0,x_2,y_2) &= \max_{x_0\in I_1, y_0\in I_2} \Delta_{2m}^*(0,x_0,y_0) \end{aligned}\right\} \qquad \dots (3.9)$$

By using the inequalities (3.4), (3.5), (3.6), (3.7) and (3.8), we obtains

$$\left.\begin{aligned} \Delta_1^*(0,x_1,y_1) &= \Delta_{1m}^*(0,x_1,y_1) + (\Delta_1^*(0,x_1,y_1) - \Delta_{1m}^*(0,x_1,y_1)) < 0 \\ \Delta_1^*(0,x_2,y_2) &= \Delta_{1m}^*(0,x_2,y_2) + (\Delta_1^*(0,x_2,y_2) - \Delta_{1m}^*(0,x_2,y_2)) > 0 \end{aligned}\right\} \qquad \dots (3.10)$$

$$\left.\begin{aligned} \Delta_2^*(0,x_1,y_1) &= \Delta_{2m}^*(0,x_1,y_1) + (\Delta_2^*(0,x_1,y_1) - \Delta_{2m}^*(0,x_1,y_1)) < 0 \\ \Delta_2^*(0,x_1,y_1) &= \Delta_{2m}^*(0,x_2,y_2) + (\Delta_2^*(0,x_2,y_2) - \Delta_{2m}^*(0,x_2,y_2)) > 0 \end{aligned}\right\} \qquad \dots (3.11)$$

and from the continuity of the functions $\Delta_1^*(0,x_0,y_0)$ and $\Delta_2^*(0\,,x_0,y_0)$ and the inequalities(3.10) and (3.11), then there exist and isolated points $x^0 \in [x_1,x_2]$ and $y^0 \in [y_1,y_2]$ such that $\Delta_1^*(0,x_0,y_0) = 0$ and $\Delta_2^*(0,x_0,y_0) = 0$. This means that (1.1) has a solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$.

4. STABILITY OF SOLUTION OF (1.1) WITH BOUNDARY CONDITION (1.2).

In this section, we study the stability theorem of a periodic solution for (1.1).

Theorem 4.1: Suppose that the function (3.1) be given. Then the following inequalities :

$$\begin{pmatrix} \| \Delta_1^*(0,x_0,y_0) \| \\ \| \Delta_2^*(0,x_0,y_0) \| \end{pmatrix} \le \begin{pmatrix} M_\beta M + \dfrac{B_1}{T} \\ N_\beta N + \dfrac{B_2}{T} \end{pmatrix} \qquad \dots (4.1)$$

and

$$\begin{pmatrix} \| \Delta_1^*(0,x_0^1,y_0^1) - \Delta_1^*(0,x_0^2,y_0^2) \| \\ \\ \| \Delta_2^*(0,x_0^1,x_0^1) - \Delta_2^*(0,x_0^2,y_0^2) \| \end{pmatrix} \le \begin{pmatrix} R_1 & R_2 \\ \\ R_3 & R_4 \end{pmatrix} \begin{pmatrix} \| x_0^1 - x_0^2 \| \\ \\ \| y_0^1 - y_0^2 \| \end{pmatrix} \quad \dots (4.2)$$

are holds for all $x_0, x_0^1, x_0^2 \in D_\chi$ and $y_0, y_0^1, y_0^2 \in D_{1\chi}$, where

$B_1 = \|(A_1 A_2^{-1} + E)x_0 - e_1 A_2^{-1}\|, B_2 = \|(B_1 B_2^{-1} + E)y_0 - e_2 B_2^{-1}\|,$

$B_3 = \|A_1 A_2^{-1} + E\|,\ B_4 = \|B_1 B_2^{-1} + E\|,\ F_1 = [\left(1 - \frac{T}{2} M_\beta C_1\right)\left(1 - \frac{T}{2} N_\beta C_4\right)]^{-1},$

$F_2 = (1 - \frac{T^2}{4} N_\beta M_\beta C_2 C_3 F_1)^{-1},\ w_1 = F_1 F_2 \left(1 - \frac{T}{2} N_\beta C_4\right)(E + B_3),$

$w_2 = \frac{T}{2} M_\beta C_2 F F_{2_1} (E + B_4),\ w_3 = \frac{T}{2} N_\beta C_3 F_1 F_2 (E + B_3),$

$w_4 = F_1 \left(1 - \frac{T}{2} M_\beta C_1\right)(E + B_4)\left[1 + \frac{T^2}{4} N_\beta M_\beta C_2 C_3 F_1 F_2\right],$

$R_1 = M_\beta(C_1 w_1 + C_2 w_3) + \frac{B_3}{T}, R_2 = M_\gamma(C_1 w_2 + C_2 w_4),\ R_3 = N_\beta(C_3 w_1 + C_4 w_3),$

$R_4 = N_\beta(C_3 w_2 + C_4 w_4) + \frac{B_4}{T}$

Proof . From the equation (3.1), we get

$$\| \Delta_1^*(0,x_0,y_0) \| \le M_\beta M + \frac{B_1}{T} \quad \dots (4.3)$$

And

$$\| \Delta_2^*(0,x_0,y_0) \| \le N_\beta N + \frac{B_2}{T} \quad \dots (4.4)$$

From (4.3) and (4.4), we get (4.1).

Now, by using the function (3.1), we get

$$\| \Delta_1^*(0,x_0^1,y_0^1) - \Delta_1^*(0,x_0^2,y_0^2) \| \le M_\beta C_1 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + M_\beta C_2 \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| + \frac{B_3}{T}\|x_0^1 - x_0^2\| \quad \dots (4.5)$$

And

$$\| \Delta_2^*(0,x_0^1,y_0^1) - \Delta_2^*(0,x_0^2,y_0^2) \| \le N_\beta C_3 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + N_\beta C_4 \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| + \frac{B_4}{T}\|y_0^1 - y_0^2\| \quad \dots (4.5)$$

where $x(t,x_0^1,y_0^1), x(t,x_0^2,y_0^2), y(t,x_0^1,y_0^1)$ and $y(t,x_0^2,y_0^2)$ are solutions of the integral equations :-

$$x(t,x_0^k,y_0^k) = x_0^k + \int_\tau^t \left[f(\tau,\beta(\tau,\alpha),x(\tau,x_0^k,y_0^k),\mu,u) - \frac{1}{T}\int_\tau^{\tau+T} f(\tau,\beta(\tau,\alpha) ,x(\tau,x_0^k,y_0^k),\mu,u) \right] d\tau + \frac{t-\tau}{T}[e_1 A_2^{-1} - (A_1 A_2^{-1} + E)x_0^k] \quad \dots (4.7)$$

$$y(t,x_0^k,y_0^k) = y_0^k + \int_\tau^t \left[g(\tau,\beta(\tau,\alpha),y(\tau,x_0^k,y_0^k),\omega,v) - \frac{1}{T}\int_\tau^{\tau+T} g(\tau,\beta(\tau,\alpha) ,y(\tau,x_0^k,y_0^k),\omega,v) \right] d\tau + \frac{t-\tau}{T}[e_2 B_2^{-1} - (B_1 B_2^{-1} + E)y_0^k] \quad \dots (4.8)$$

with

$x(t,x_0^k,y_0^k) = x_0^k, y(t,x_0^k,y_0^k) = y_0^k,$ where $k = 1,2.$

Now, use the equation (4.7) , as follows

$$\| x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2) \| \le \frac{T}{2} M_\beta C_1 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2 y_0)\| + \frac{T}{2} M_\beta C_2 \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2 y_0)\| + (E + B_3)\|x_0^1 - x_0^2\| \quad \dots (4.9)$$

then we can write

$$\| x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2) \| \le (1 - \frac{T}{2} M_\beta C_1)^{-1}(E + B_3)\|x_0^1 - x_0^2\| + \frac{T}{2} M_\beta C_2 (1 - \frac{T}{2} M_\gamma C_1)^{-1} \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2 y_0)\| \quad \dots (4.10)$$

Also, use the equation (4.8), we get

$$\| y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\| \le \frac{T}{2}N_\beta C_3 \|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2 y_0)\|$$
$$+\frac{T}{2}N_\beta C_4 \|y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2 y_0)\|+(E+B_4)\|y_0^1-y_0^2\| \quad \dots (4.11)$$

then we write this equation as follows

$$\| y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\| \le (1-\frac{T}{2}N_\gamma C_4)^{-1}(E+B_4)\|y_0^1-y_0^2\|$$
$$+\frac{T}{2}M_\beta C_3(1-\frac{T}{2}N_\gamma C_4)^{-1}\|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2 y_0)\| \quad \dots (4.12)$$

By substituting inequality (4.12) in (4.10), we get

$$\| x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2)\| \le F_1F_2\left(1-\frac{T}{2}N_\gamma C_4\right)(E+B_3)\|x_0^1-x_0^2\|$$
$$+\frac{T}{2}M_\beta C_2F_1F_2(E+B_4)\|y_0^1-y_0^2\| \quad \dots (4.13)$$

Also, substituting the inequality (4.12) in (4.14), we find that

$$\| y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\| \le \frac{T}{2}N_\beta C_3F_1F_2(E+B_3)\|x_0^1-x_0^2\|+F_1\left(1-\frac{T}{2}M_\gamma C_1\right)$$
$$(E+B_4)\left[1+\frac{T^2}{4}N_\beta M_\beta C_2C_3F_1F_2\right]\|y_0^1-y_0^2\| \quad \dots (4.14)$$

Finally, we substitute the inequalities (4.13) and (4.14) in (4.5), we get (4.2)and substitute the inequalities (4.13) and (4.14) in (4.6), we get (4.2).

Theorem 4.2: Let the system (1.1) with boundary conditions (1.2) be defined in the domain (1.3). Suppose that G_1 and G_2 be closed and bounded domain subset of domain D_χ and $D_{1\chi}$. Then, G_1 and G_2 have points at which the Δ −constant is zero, then for any point $x_0 \in D_\chi$ and $y_0 \in D_{1\chi}$, the following inequality holds :-

$$\begin{pmatrix} \| \Delta_{1m}^*(0,x_0,y_0)\| \\ \| \Delta_{2m}^*(0,x_0,y_0)\| \end{pmatrix} \le \begin{pmatrix} \langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle \\ \langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle \end{pmatrix} + \begin{pmatrix} M_\beta M+\frac{B_1}{T} \\ N_\beta N+\frac{B_2}{T} \end{pmatrix} \quad \dots (4.15)$$

for all $m \ge 0$ and $x_0 \in D_\chi, y_0 \in D_{1\chi}$.

Proof . By using the inequality (4.1), we get

$$\| \Delta_1^*(0,x_0,y_0)\| \le M_\beta M+\frac{B_1}{T}$$

and

$$\| \Delta_2^*(0,x_0,y_0)\| \le N_\beta N+\frac{B_2}{T}$$

Also, from (3.1), we have

$$\| \Delta_{1m}^*(0,x_0,y_0)\| = \| \Delta_{1m}^*(0,x_0,y_0)-\Delta_1^*(0,x_0,y_0)+\Delta_1^*(0,x_0,y_0)\|$$
$$\le \| \Delta_{1m}^*(0,x_0,y_0)-\Delta_1^*(0,x_0,y_0)\| + \| \Delta_1^*(0,x_0,y_0)\|$$
$$\le \langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle + \left(M_\beta M+\frac{B_1}{T}\right) \quad \dots (4.16)$$

and

$$\| \Delta_{2m}^*(0,x_0,y_0)\| = \| \Delta_{2m}^*(0,x_0,y_0)-\Delta_2^*(0,x_0,y_0)+\Delta_2^*(0,x_0,y_0)\|$$
$$\le \| \Delta_{2m}^*(0,x_0,y_0)-\Delta_2^*(0,x_0,y_0)\| + \| \Delta_2^*(0,x_0,y_0)\|$$
$$\le \langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle + \left(N_\beta N+\frac{B_2}{T}\right) \quad \dots (4.17)$$

Rewrite the inequalities (4.16) and (4.17) in a vector form, we get (4.15).

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