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Solution of Nonlinear ordinary Differential Equations by New Sumudu Variational Iteration Method

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***Abstract*:- In this work we propose a combined Sumudu transform (ST) and the variational iteration method (VIM) to solve nonlinear ordinary differential equations. The elegant coupling is called the Sumudu transform variational iteration method (STVIM). The strategy is outlined and then illustrated through a number of test examples.it is possible to find the exact solutions or better approximate solutions of these equations. In this method, a correction functional is constructed by a general Lagrange multiplier, which can be identified via variational theory. The solutions obtained by this method show the accuracy and efficiency of the method.**

1. INTRODUCTION

In the last two decades, many analytical approximate methods have been presented to solve nonlinear ordinary differential equations. Most of these problems generally occur commonly in many areas of engineering, physics, chemistry, and applied mathematics. Recently, many researchers have introduced various methods to obtain approximate solutions for nonlinear differential equations (NDEs), such as variational iteration method (VIM), which was developed by JiHuan He for solving linear, nonlinear initial and BVPs (He, 1997; He, 1999; He, 2000.

Motivated and inspired by Wu's thinking, and combing with the Sumudu transform (ST), we give a new modified variational iteration method (VIM),which is based on variational iteration theory and Sumudu transform (ST).The balance in this paper is as follows: the Sumudu transform (ST), variational iteration method (VIM), and the combination of Sumudu transform (ST) and variational iteration method (VIM) are presented in sections 2, 3, and 4. In section 5, numerical application of the method is illustrated by two test examples to demonstrate the efficiency of the method. Section 6 includes a conclusion that briefly summarizes the results**.**

2. SUMUDU TRANSFROM (ST)

Watugula (Watugula,1993) introduced Sumudu transform as a new integral and Sumudu transform is defined over the set of functions,

$$A=\left\{f(t)\backslash \exists M,\tau_1,\tau_2>0,|f(x)|<Me^{\frac{|t|}{\tau_j}}, if\ t\epsilon(-1)^j\times[0,\infty)\right\}$$

by,

$$G(u)=\mathcal{S}\{f(t)\}=\int_0^\infty f(ut)e^{-t}dt,\quad u\in(-\tau_1,\tau_2) \qquad (1)$$

or equivalently,

$$G(u)=\mathcal{S}\{f(t)\}=\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}f(t)dt,\ u\in(-\tau_1,\tau_2) \qquad (2)$$

The inverse of Sumudu transform of function $G(u)$ is denoted by symbol $\mathcal{S}^{-1}[G(u)]=f(t)$ and is defined with Bromwich contour integral by

$$\mathcal{S}^{-1}[G(u)]=f(t)=\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT}e^{st}\,G(u)du$$

Proposition 1 It deals with the effect of the differentiation of the function $f(t)$, k times on the Sumudu transform $G(u)$ if $\mathcal{S}\{f(t)\}=G(u)$ then

(i) $\mathcal{S}(f'(t))=\frac{1}{u}G(u)-\frac{1}{u}f(0)$

(ii) $\mathcal{S}(f''(t))=\frac{1}{u^2}G(u)-\frac{1}{u^2}f(0)-\frac{1}{u}f'(0)$

(iii) $\mathcal{S}(f^n(t))=u^{-n}\left(G(u)-\sum_{k=0}^{n-1}u^kf^k(0)\right)$

Where $f^{(0)}(0)=f(0)$ and $f^{(k)}(0);\ k=1,2,3,\dots,n-1$
are the kth-order derivatives of the function $f(t)$ evaluated at $t=0$

Proposition 2 Let $f(t)$ and $g(t)$ be a functions with the Laplace transforms $F(s)$ and $G(s)$ respectively and Sumudu transform $M(u)$ and $N(u)$, respectively. Then the Sumudu transform of the convolution of f and g.

$$(f*g)(t)=\int_0^\infty f(t)g(t-\tau)d\tau$$

is given by: $\mathcal{S}\big((f*g)(t)\big) = u\,M(u)\,N(u)$

3. THE VARIATIONAL ITERATION METHOD (VIM)

To clarify the base idea of the Variational Iteration Method (VIM). Consider the nonlinear differential equation

$$Lw + Nw = f(x), \qquad (3)$$

Where L and N are linear and nonlinear operators respectively, and $f(x)$ is the source inhomogeneous term. We can construct a correction functional according to the variational iteration method (VIM) for Eq. (3) as follows:

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda(\varepsilon)(Lw_n(\varepsilon) + N\,\widetilde{w_n}(\varepsilon) - f(\varepsilon))\,d\varepsilon, \quad n \geq 0 \qquad (4)$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, w_n is the nth approximate solution and $\widetilde{w_n}$ is a restricted variation which means $\delta\widetilde{w_n} = 0$.

4. SUMUDU TRANSFORM VARIATIONAL ITERATION METHOD (STVIM)

In a wide range of problems that appear in the literature, the general form of Lagrange multiplier is found to be of the form:

$$\lambda = \bar{\lambda}(x - \varepsilon).$$

In this section, we will make the assumption that λ is expressed in this latter way. In such a case, the integration is basically the convolution; hence Sumudu transform (**ST**) is appropriate to use. Applying Sumudu transform (**ST**) on both sides of (4) the correction functional will be constructed in the following manner:

$$\mathcal{S}\big(w_{n+1}(x)\big) = \mathcal{S}\big(w_n(x)\big) + \mathcal{S}\big(\int_0^x \bar{\lambda}(\varepsilon)(Lw_n(\varepsilon) + N\,\widetilde{w_n}(\varepsilon) - f(\varepsilon))\,d\varepsilon\big), \ n \geq 0 \qquad (5)$$

Therefore

$$\mathcal{S}\big(w_{n+1}(x)\big) = \mathcal{S}\big(w_n(x)\big) + \mathcal{S}\left(\bar{\lambda}(x) * (Lw_n(x) + N\,\widetilde{w_n}(x) - f(x))\right) \qquad (6)$$

$$\mathcal{S}\big(w_{n+1}(x)\big) = \mathcal{S}\big(w_n(x)\big) + u\mathcal{S}\left(\bar{\lambda}(x)\right) * \mathcal{S}(Lw_n(x) + N\,\widetilde{w_n}(x) - f(x)) \qquad (7)$$

To find the optimal value of $\bar{\lambda}(x-\varepsilon)$ we first take the variation with respect to $w_n(x)$ .Thus

$$\frac{\delta}{\delta w_n}\,\mathcal{S}\big(w_{n+1}(x)\big) = \frac{\delta}{\delta w_n}\mathcal{S}\big(w_n(x)\big) + u\frac{\delta}{\delta w_n}\mathcal{S}\left(\bar{\lambda}(x)\right) * \mathcal{S}(Lw_n(x) + N\,\widetilde{w_n}(x) - f(x)) \qquad (8)$$

And hence upon applying the variation this simplifies to

$$\mathcal{S}\big(\delta w_{n+1}(x)\big) = \mathcal{S}\big(\delta w_n(x)\big) + u\mathcal{S}\left(\bar{\lambda}(x)\right) * \mathcal{S}\big(\delta w_n(x)\big) \qquad (9)$$

We assume that L is a linear differential operator with constant coefficients given by

$$L(w) = a_n w^{(n)} + a_{n-1}w^{(n-1)} + a_{n-2}w^{(n-2)} + \cdots + a_2 w'' + a_1 w' + a_0 w, \qquad (10)$$

Where are a_i 's constants. It is important to note that if the coefficients contain only non-constant terms x^k, then the Sumudu variational approach is still valid.

The Sumudu transform of the first term of the operator L is given by

$$\mathcal{S}\big(a_n w^{(n)}\big) = \frac{a_n}{u^n}\mathcal{S}(w) - \frac{a_n}{u^n}\sum_{k=0}^{n-1} u^k\, w^{(k)}(0), \qquad (11)$$

so the variation with respect to w is

$$\delta\mathcal{S}\big(a_n w^{(n)}\big) = \frac{a_n}{u^n}\mathcal{S}(\delta w). \qquad (12)$$

The other term in the operator L, namely $a_{n-1}w^{(n-1)} + a_{n-2}w^{(n-2)} + a_2 w'' + a_1 w' + a_0 w$, yields similar results. Hence using Eq. (12) , Eq. (9) reduces to

$$\mathcal{S}\big(\delta w_{n+1}(x)\big) = \mathcal{S}\big(\delta w_n(x)\big) + u\mathcal{S}\left(\bar{\lambda}(x)\right)\left(\sum_{k=0}^{n-1}\frac{a_n}{u^n}\right)\mathcal{S}\big(\delta w_n(x)\big), \qquad (13)$$

$$\mathcal{S}\big(\delta w_{n+1}(x)\big) = \left[1 + \mathcal{S}\left(\bar{\lambda}(x)\right)\left(\sum_{k=0}^{n-1}\frac{a_n}{u^n}\right)\right]\mathcal{S}\big(\delta w_n(x)\big). \qquad (14)$$

The extremum condition of w_{n+1} requires that $\delta w_{n+1} = 0$. This means that the right -hand side Eq. (14) should be set to zero. Hence, we have the stationary condition

$$\mathcal{S}\left(\bar{\lambda}(x)\right) = -\frac{1}{\sum_{k=0}^{n-1}\frac{a_n}{u^{n-1}}} \qquad (15)$$

Taking the Sumudu inverse of the last equation gives the optimal value of $\bar{\lambda}$. For this value of $\bar{\lambda}$, we have the following formulation:

$$\mathcal{S}\big(w_{n+1}(x)\big) = \mathcal{S}\big(w_n(x)\big) + \mathcal{S}\big(\int_0^x \bar{\lambda}(x-\varepsilon)(Lw_n(\varepsilon) + N\,\widetilde{w_n}(\varepsilon) - f(\varepsilon))\,d\varepsilon\big), \ n \geq 0 \qquad (16)$$

5. APPLICATIONS

In this section, we apply the Sumudu variational iteration method for solving nonlinear ordinary differential equations.

Example1. Consider the following nonlinear differential equation

$$w^{(5)}(x) = e^{-x}w^2(x), \qquad (17)$$

$w(0) = 1,\ \ w'(0) = 1,\ \ w''(0) = 1,\ \ w(1) = e\,,\ w'(1) = e$

The exact solution for the given differential equation is $w = e^x$

For this case the Sumudu variational iteration correction functional will be constructed in the following manner:

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$w_{n+1}(x) = w_n(x) + \int_0^x \bar{\lambda}(x-\varepsilon)\left(w_n{}^{(5)}(\varepsilon) - e^{-\varepsilon}\widetilde{w_n}{}^{(2)}(\varepsilon)\right) d\varepsilon,\ n \ge 0$ (18)

Next, by applying Sumudu transform, we have:

$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) + \mathcal{S}\left(\int_0^x \bar{\lambda}(x-\varepsilon)\left(w_n{}^{(5)}(\varepsilon) - e^{-\varepsilon}\widetilde{w_n}{}^{(2)}(\varepsilon)\right) d\varepsilon\right),\ n \ge 0$ (19)

or equivalent, by applying the convolution property, we get:

$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) + u\mathcal{S}(\bar{\lambda}) * \mathcal{S}\left(w_n{}^{(5)}(x) - e^{-x}\widetilde{w_n}{}^{(2)}(x)\right),$ (20)

$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) + u\mathcal{S}(\bar{\lambda})\left[\frac{1}{u^5}\mathcal{S}(w_n(x)) - \frac{w_n(0)}{u^5} - \frac{w_n{}'(0)}{u^4} - \frac{w_n{}''(0)}{u^3} - \frac{w_n{}'''(0)}{u^2} - \frac{w_n{}^{(4)}(0)}{u} - \mathcal{S}\left(e^{-x}\widetilde{w_n}{}^{(2)}(x)\right)\right].$

(21)

Applying the variation on the Eq. (21), we get

$\frac{\delta}{\delta w_n}\ \mathcal{S}(w_{n+1}(x)) = \frac{\delta}{\delta w_n}\mathcal{S}(w_n(x)) + \frac{\delta}{\delta w_n} u\ \mathcal{S}\left(\bar{\lambda}(x)\right) * \left[\frac{1}{u^5}\mathcal{S}(w_n(x)) - \frac{w_n(0)}{u^5} - \frac{w_n{}'(0)}{u^4} - \frac{w_n{}''(0)}{u^3} - \frac{w_n{}'''(0)}{u^2} - \frac{w_n{}^{(4)}(0)}{u} - \right.$
$\left.\mathcal{S}\left(e^{-x}\widetilde{w_n}{}^{(2)}(x)\right)\right].$ (22)

By simplifying Eq. (22), we get

$\mathcal{S}(\delta w_{n+1}(x)) = \mathcal{S}(\delta w_n(x)) + \mathcal{S}\left(\bar{\lambda}(x)\right)\frac{1}{u^4}\ \mathcal{S}(\delta w_n(x))$ (23)

$\mathcal{S}(\delta w_{n+1}(x)) = \mathcal{S}(\delta w_n(x))\left[1 + \frac{1}{u^4}\mathcal{S}\left(\bar{\lambda}(x)\right)\right]$ (24)

The extremum condition of w_{n+1} requires that $\delta w_{n+1} = 0$, then

$\mathcal{S}(\delta w_n(x))\left[1 + \frac{1}{u^4}\mathcal{S}\left(\bar{\lambda}(x)\right)\right] = 0$ (25)

$\mathcal{S}\left(\bar{\lambda}(x)\right) = -u^4$ (26)

Applying the inverse Sumudu transform, we get:

$$\bar{\lambda}(x) = -\frac{1}{24}\ x^4 \quad (27)$$

Substituting Eq. (27) into Eq. (19), we get

$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) - \frac{1}{24}\mathcal{S}\left(\int_0^x (x-\varepsilon)^4\left(w_n{}^{(5)}(\varepsilon) - e^{-\varepsilon}\widetilde{w_n}{}^{(2)}(\varepsilon)\right) d\varepsilon\right),\ n \ge 0$ (28)

or

$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) - \frac{1}{24}u\mathcal{S}(x^4) * \mathcal{S}\left(w_n{}^{(5)}(x) - e^{-x}\widetilde{w_n}{}^{(2)}(x)\right),$ (29)

Suppose that $w_0(x) = w(0) + xw'(0) + \frac{1}{2!}\ x^2w''(0) + \frac{1}{3!}\ x^3w'''(0) + \frac{1}{4!}\ x^4w^{(4)}(0)$
$= 1 + x + \frac{1}{2!}\ x^2 + A\ x^3 + B\ x^4$

Therefore

$\mathcal{S}(w_1(x)) = \mathcal{S}\left(1 + x + \frac{1}{2!}\ x^2 + A\ x^3 + B\ x^4\right) - \frac{1}{24}u\mathcal{S}(x^4) * \mathcal{S}\left(-e^{-x}\left(1 + x + \frac{1}{2!}\ x^2 + A\ x^3 + B\ x^4\right)^2\right),$

(30)

Applying the inverse of Sumudu transform, the resulting expression for w_1 is too kong so we choose not to include it. Upon using the two conditions $w(1) = e$ and $w'(1) = e$, the values of A and B are respectively $A = 0.1666728160$ and $B = 0.04166338259$.

The following Taylor series expansion of w_1 was obtained:

$$w_1 = 1.00000 + 1.000006x + 0.5x^2 + 0.1666728160x^3 + 0.04166338259x^4 + \frac{1}{120}x^5 + \cdots, \quad (31)$$

which matches highly accurately with Taylor expansion of the exact solution of the given nonlinear differential equation.

Example2. Consider the following nonlinear differential equation

$w'(x) - 2w(x) + w^2(x) = 1 + \delta(x-2),$ (32)
$w(0) = 0.$

The exact solution for the given differential equation is $w = 1 + \sqrt{2}\ \tanh\left(\sqrt{2}\,x - \tanh^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

For this case the Sumudu variational iteration correction functional will be constructed in the following manner:

$w_{n+1}(x) = w_n(x) + \int_0^x \bar{\lambda}(x-\varepsilon)\left(w_n{}'(\varepsilon) - 2w_n(\varepsilon) - 1 - \delta(\varepsilon-2) + \widetilde{w_n}{}^{(2)}(\varepsilon)\right) d\varepsilon,\ n \ge 0$ (33)

Next, by applying Sumudu transform, we have:

$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) + \mathcal{S}\left(\int_0^x \bar{\lambda}(x-\varepsilon)\left(w_n{}'(\varepsilon) - 2w_n(\varepsilon) - 1 - \delta(\varepsilon-2) + \widetilde{w_n}{}^{(2)}(\varepsilon)\right) d\varepsilon\right),\ n \ge 0$

(34)

or equivalent, by applying the convolution property, we get:

$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) + u\mathcal{S}(\bar{\lambda}) * \mathcal{S}\left(w_n{}'(x) - 2w_n(x) - 1 - \delta(x-2) + \widetilde{w_n}{}^{(2)}(x)\right),$ (35)

$$\mathcal{S}(w_{n+1}(x)) = \mathcal{S}(w_n(x)) + u\mathcal{S}(\bar{\lambda})\left[\left(\frac{1}{u} - 2\right)\mathcal{S}(w_n(x)) - \frac{w_n(0)}{u} - 1 - \frac{1}{u}e^{-\frac{2}{u}} - \mathcal{S}\left(\widetilde{w_n}{}^{(2)}(x)\right)\right]. \quad (36)$$

Applying the variation on the Eq. (36), we get

$$\frac{\delta}{\delta w_n}\mathcal{S}(w_{n+1}(x))=\frac{\delta}{\delta w_n}\mathcal{S}(w_n(x))+\frac{\delta}{\delta w_n}u\,\mathcal{S}\left(\bar{\lambda}(x)\right)*\left[\left(\frac{1}{u}-2\right)\mathcal{S}(w_n(x))-\frac{w_n(0)}{u}-1-\frac{1}{u}e^{-\frac{2}{u}}-\mathcal{S}\left(\widetilde{w_n}^{(2)}(x)\right)\right].$$
(37)

By simplifying Eq. (37), we get

$$\mathcal{S}(\delta w_{n+1}(x))=\mathcal{S}(\delta w_n(x))+u\mathcal{S}\left(\bar{\lambda}(x)\right)\left(\frac{1}{u}-2\right)\mathcal{S}(\delta w_n(x)) \quad (38)$$

$$\mathcal{S}(\delta w_{n+1}(x))=\mathcal{S}(\delta w_n(x))\left[1+(1-2u)\mathcal{S}\left(\bar{\lambda}(x)\right)\right] \quad (39)$$

The extremum condition of w_{n+1} requires that $\delta w_{n+1}=0$, then

$$\mathcal{S}(\delta w_n(x))\left[1+(1-2u)\mathcal{S}\left(\bar{\lambda}(x)\right)\right]=0 \quad (40)$$

$$\mathcal{S}\left(\bar{\lambda}(x)\right)=-\frac{1}{1-2u} \quad (41)$$

Applying the inverse Sumudu transform, we get:

$$\bar{\lambda}(x)=-e^{2x} \quad (42)$$

Substituting Eq. (42) into Eq. (34), we get

$$\mathcal{S}(w_{n+1}(x))=\mathcal{S}(w_n(x))-\mathcal{S}\left(\int_0^x e^{2(x-\varepsilon)}\left(w_n'(\varepsilon)-2w_n(\varepsilon)-1-\delta(\varepsilon-2)+\widetilde{w_n}^{(2)}(\varepsilon)\right)d\varepsilon\right),\ n\geq 0$$
(43)

$$\mathcal{S}(w_{n+1}(x))=\mathcal{S}(w_n(x))-u\mathcal{S}(e^{2x})*\mathcal{S}\left(w_n'(x)-2w_n(x)-1-\delta(x-2)+\widetilde{w_n}^{(2)}(x)\right), \quad (44)$$

Suppose that $w_0(x)=w(0)=0$, since $w(0)=0$, then :

$$\mathcal{S}(w_1(x))=\mathcal{S}(w_0(x))-u\mathcal{S}(e^{2x})*\mathcal{S}\left(w_0'(x)-2w_0(x)-1-\delta(x-2)+\widetilde{w_0}^{(2)}(x)\right), \quad (45)$$

$$\mathcal{S}(w_1(x))=0+\frac{u}{1-2u}*\left(1+\frac{1}{u}e^{-\frac{2}{u}}\right), \quad (46)$$

or

$$\mathcal{S}(w_1(x))=\frac{u}{1-2u}+\frac{1}{1-2u}e^{-\frac{2}{u}} \quad (47)$$

Applying the inverse of Sumudu transform, we have

$$w_1(x)=\frac{1}{2}e^{2x}-\frac{1}{2}+e^{2x-4}(1-H(2-x)) \quad (48)$$

Where H is the Heaviside unit step function. In a similar fashion we obtain the higher iterates, for instance the second iterate is

$$w_2(x)=-\frac{3}{8}-\frac{1}{8}e^{4x}+\frac{1}{2}e^{2x-8}+e^{3x-8}\sinh x-e^{3x-2}\sinh(x-2)H(x-2)+\frac{1}{2}(1+x)e^{2x}+\frac{1}{2}\left(1-H(2-x)\right)(2x-1)e^{2x-4}+\left(\frac{1}{2}H(2-x)-1\right)e^{4x-8} \quad . \quad (49)$$

The first few terms of Taylor expansion of w_2, for $x\neq 2$, is

$$w_2(x)=x+x^2+\frac{1}{3}x^3-\frac{1}{3}x^4-\cdots$$

which matches with the exact solution of the given nonlinear differential equation.

6. CONCLUSION

In this paper, Sumudu transform variational iteration method has been efficiently applied for solving nonlinear ordinary differential equations to give rapid convergent successive approximations without any linearization, discretization or restrictive assumptions that may change the physical behavior of the problem and absorb the positive features of the coupled techniques.

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