Solution of Incompressible Viscous Flow in a Lid-Driven Cavity using Crank-Nicholson Scheme

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***Abstract*-The stream function-vorticity form of the unsteady Navier-Stokes equation is solved using second order accurate in space and first order accurate in time Crank-Nicholson scheme in a finite difference mesh. The scheme shows implicit in nature. The geometry of the problem is a lid-driven cavity. The solution is obtained up to maximum Reynolds number 22,500 in the grid size 129x129 and 257x257. The flow pattern is studied in the form of stream function & vorticity contours, central velocity profiles and location of the eddies.**

Keywords - Unsteady, Incompressible, Crank-Nicholson scheme, Finite difference scheme.

I. INTRODUCTION

The lid-driven cavity is a square cavity in which all the walls except the top lid are fixed. The top lid is allowed to move towards right with non-dimensional velocity unity. Initially, the cavity is filled with fluid which is at rest. When the lid starts moving towards right the fluid flow in the cavity is set up. The fluid is assumed to be viscous and incompressible.

In 1966, Burggraf[1] studied the viscous flow in two-dimensional lid-driven cavity. He solved the stream function and vorticity form of the Navier-Stokes equation up to Reynolds number 400 in the grid size 40x40 and 60x60. He has proved analytically that the vorticity value at the centre of the primary vortex will not exceed -1.8859. Later Erturk *et al*.[2] solved the stream function-vorticity form of unsteady Navier-Stokes equation up to maximum Reynolds number 21000 in the grid sizes 401x401, 501x501 and 601x601. Their solution for vorticity value at the centre of the primary vortex crosses the theoretical limit of -1.8859 within Reynolds number 21000. Erturk & Gockol[3] solved the stream function-vorticity form of the Navier-Stokes equation using fourth order accurate alternating direction implicit method upto Reynolds number 20,000 in the grid sizes 401x401, 501x501 and 601x601. Here Erturk and Gockol's[3] solution for vorticity value at the centre of the primary vortex does not cross the theoretical limit of -1.8859. Erturk[4] solved the stream function-vorticity form of the Navier-Stokes equation using Gauss-Seidel iteration techniques in a grid size 1025x1025. He observed that the vorticity value at the centre of the primary eddy does not cross the theoretical limit of -1.8859 within Reynolds number 20000.

In this paper, the Navier-Stokes equation is solved in a lid-driven cavity using Crank-Nicholson scheme. The Crank-Nicholson scheme has the flexibility of taking the average value at n+1 and n time steps. This scheme is implicit in nature.

II. FORMULATION

The governing equation in stream function-vorticity form is expressed as:

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{\mathrm{Re}}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right) \tag{1}$$

The stream function equation is expressed as:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega \tag{2}$$

Where ψ, ω and Re represent the stream function, vorticity and Reynolds number respectively. u and v represent the components of velocities along x and y-directions respectively.

The boundary conditions for the lid-driven cavity are:

$$\begin{aligned}&\text{At } y=0,\,0\le x\le 1,\,u=v=0;\\&\text{At } y=1,\,0\le x\le 1,\,u=1,\,v=0;\\&\text{At } x=0,1;\,0\le y\le 1,\,u=v=0\end{aligned} \tag{3}$$

Eq.(1) & (2) are discretised using Crank-Nicholson scheme in finite difference uniform mesh.

$$\frac{\omega_{ij}^{n+1}-\omega_{ij}^{n}}{\Delta t}+\frac{u_{ij}^{n}}{2}\left(\frac{\omega_{ij+1}^{n+1}-\omega_{ij-1}^{n+1}}{2\Delta x}+\frac{\omega_{ij+1}^{n}-\omega_{ij-1}^{n}}{2\Delta x}\right)$$
$$+\frac{v_{ij}^{n}}{2}\left(\frac{\omega_{i+1j}^{n+1}-\omega_{i-1j}^{n+1}}{2\Delta y}+\frac{\omega_{i+1j}^{n}-\omega_{i-1j}^{n}}{2\Delta y}\right)=$$

$$\frac{1}{\mathrm{Re}}\left(\frac{\omega_{ij+1}^{n+1}-2.0\omega_{ij}^{n+1}+\omega_{ij-1}^{n+1}}{2\Delta x^2}+\frac{\omega_{ij+1}^{n}-2.0\omega_{ij}^{n}+\omega_{ij-1}^{n}}{2\Delta x^2}\right.$$

$$\left.+\frac{\omega_{i+1j}^{n}-2.0\omega_{ij}^{n+1}+\omega_{i-1j}^{n+1}}{2\Delta y^2}+\frac{\omega_{i-1j}^{n}-2.0\omega_{ij}^{n}+\omega_{i-1j}^{n}}{2\Delta y^2}\right) \qquad (4)$$

The discretised equation (3) is first order accurate in time and second order accurate in space.

Similarly, the stream function equation (2) is discretised using Crank-Nicholson scheme as:

$$\omega_{ij}^{n+1}=-\frac{1}{2}\left(\frac{\psi_{ij+1}^{n+1}-2.0\psi_{ij}^{n+1}+\psi_{ij-1}^{n+1}}{\Delta x^2}\right.$$

$$+\frac{\psi_{ij+1}^{n}-2.0\psi_{ij}^{n}+\psi_{ij-1}^{n}}{\Delta x^2}+\frac{\psi_{i+1j}^{n+1}-2.0\psi_{ij}^{n+1}+\psi_{i-1j}^{n+1}}{\Delta y^2}$$

$$\left.+\frac{\psi_{i+1j}^{n}-2.0\psi_{ij}^{n}+\psi_{i-1j}^{n}}{\Delta y^2}\right) \qquad (5)$$

The equation (4) and (5) are second order accurate in space. Equation (4) and (5) produces the vorticity and stream function values respectively in new time steps (n+1). Applying Eq.(3) in Thom's formula[1], the first order accurate boundary values for vorticity can be obtained. The discretised boundary conditions are represented as:

$$\omega_{ij}=\frac{2}{h^2}\left(\psi_{ij+1}-\psi_{ij}\right)+O(h)\ \text{ for j} = 0 \text{ and all } i;$$

$$\omega_{ij}=\frac{2}{h^2}\left(\psi_{ij}-\psi_{ij-1}\right)+O(h)\ \text{ for j} = N \text{ and all } i;$$

$$\omega_{ij}=\frac{2}{h^2}\left(\psi_{i+1j}-\psi_{ij}\right)+O(h)\ \text{ for } i = 0 \text{ and all } j\ \&$$

$$\omega_{ij}=\frac{2}{h^2}\left(\psi_{ij+1}-\psi_{ij}\right)+O(h)\ \text{ for } i = N \text{ and all } j$$

(6)

Where 'N' is the total number of grid points along i and j directions respectively. Here the boundary conditions are first order accurate in space. Since the discretised equations are implicit in nature, the change of time step does not have any effect to the convergence. The solution of the discretised equation (4) and (5) are obtained using Gauss-Seidel iteration techniques with relaxation parameter. The convergence of the solution is assumed to be the residue of discretised equation (1) and (2) approaches towards 10^{-10}. The grid meshes used for the solution are 129x129, 257x257.

III. RESULTS & DISCUSSION

The solution for the incompressible viscous flow in the lid-driven cavity is obtained up to Reynolds number 22,500 in both grid sizes 129x129 and 257x257. It is observed that at lower Reynolds number 100, the stream function contour forms closed structures and its centre is towards right near the lid. This is named as primary eddy of the cavity. The location and strength of the primary eddy for various Reynolds numbers are shown in Table.1. There are small eddies at the corners along the lower wall of the cavity. As the Reynolds number increases, the stream function contours in the primary eddy become circular in nature and its centre moves towards the geometric centre of the cavity. As the Reynolds number increases more and more, the stream functions in the primary eddy become more circular in nature and the centre of the primary eddy moves towards the geometric centre of the cavity as shown in Fig.1. The number of secondary eddies increase in the lower corner of the cavity. The number of secondary eddies become more than the right corner as the Reynolds number increases. The number of secondary eddies also increase in the left vertical wall of the cavity after Reynolds number 2500 as shown in Fig.1. The vorticity contours also show large variation near to the walls of the cavity. The centre of the primary eddy becomes constant vorticity core with increase of Reynolds number as shown in Fig.2. The radius of the constant vorticity core increases with increase of Reynolds number.

TABLE I: Location and strength of the centre of the primary eddy

Re	Grid	x	y	ψ	ω	Ref.
100	129x129	0.617	0.734	-0.100	-3.046	
	257x257	0.617	0.738	-0.102	-3.121	
400	129x129	0.563	0.609	-0.107	-2.156	
	257x257	0.555	0.606	-0.111	-2.231	
1000	129x129	0.531	0.563	-0.107	-1.855	
	257x257	0.531	0.566	-0.113	-1.966	
	601x601	0.530	0.565	-0.119	-2.067	Ref[2]
2500	129x129	0.523	0.539	-0.101	-1.632	
	257x257	0.523	0.543	-0.112	-1.814	
	601x601	0.520	0.543	-0.121	-1.974	Ref[2]
5000	129x129	0.524	0.531	-0.101	-1.632	
	257x257	0.511	0.535	-0.108	-1.706	
	601x601	0.515	0.536	-0.122	-1.936	Ref[2]
7500	129x129	0.523	0.531	-0.084	-1.312	
	257x257	0.516	0.531	-0.104	-1.633	
	601x601	0.513	0.532	-0.122	-1.919	Ref[2]
10000	129x129	0.523	0.531	-0.078	-1.211	
	257x257	0.516	0.527	-0.101	-1.573	
	601x601	0.512	0.530	-0.122	-1.908	Ref[2]
12500	129x129	0.523	0.523	-0.073	-1.129	
	257x257	0.516	0.527	-0.098	-1.522	
	601x601	0.512	0.528	-0.121	-1.899	Ref[2]
15000	129x129	0.523	0.523	-0.069	-1.060	
	257x257	0.516	0.527	-0.095	-1.477	
	601x601	0.510	0.528	-0.121	-1.893	Ref[2]
17500	129x129	0.523	0.523	-0.065	-1.001	
	257x257	0.516	0.527	-0.093	-1.436	
	601x601	0.510	0.527	-0.121	-1.887	Ref[2]
20000	129x129	0.523	0.523	-0.062	-0.095	
	257x257	0.516	0.523	-0.090	-1.399	
	601x601	0.510	0.527	-0.121	-1.881	Ref[2]
22500	129x129	0.523	0.523	-0.059	-0.906	

The u and v-velocity profiles along the centre of the cavity at different Reynolds numbers are shown in Fig.3 and 4 respectively. Both the velocity profiles match very well up to Reynolds number 5000. At Reynolds number 10,000 and 20,000 the computed velocity profiles do not match perfectly with Erturk *et al*[2]. This happens due to large grid size difference with Erturk *et al*[2]. The computed velocity profiles are at grid size 257x257 whereas Erturk *et al*'s.[2] grid size is 401x401. Here the important point is to be observed that the vorticity contour plot at Reynolds number 20,000 show zig-zag nature near the top right corner of the cavity in the grid size 257x257 which shows the insufficient grid resolution at that Reynolds number because the *u* & *v*-velocity profiles do not match with Erturk *et al.*[2]. So the Crank-Nicholson scheme seems to produce better result than Erturk *et al*'s[2] result but accuracy is the main point of concern.

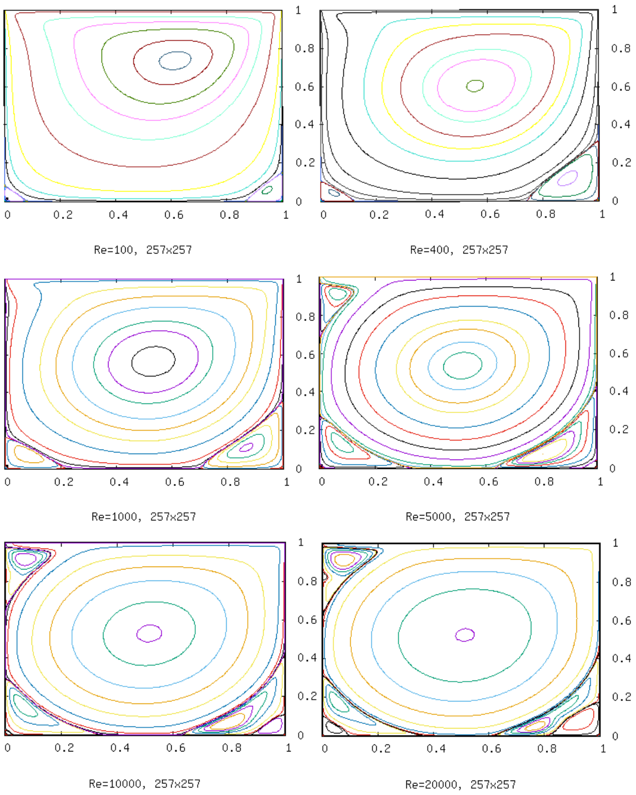
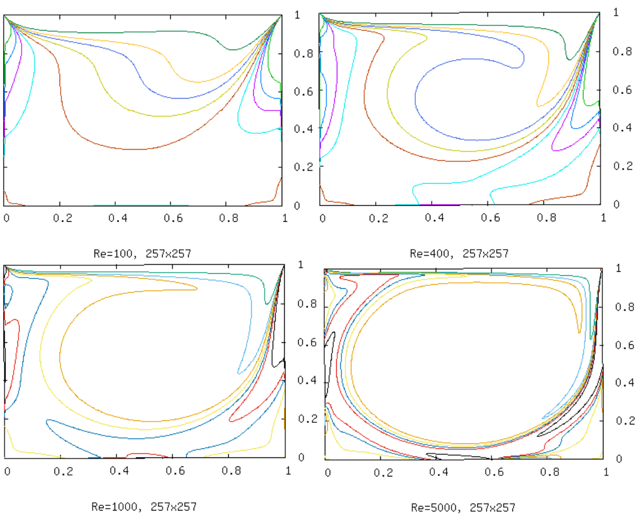


Figure: 1 Stream function contour plots at various Reynolds numbers



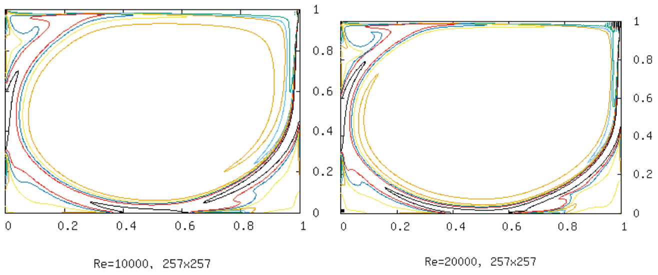


Figure: 2 Vorticity contour plots at various Reynolds numbers

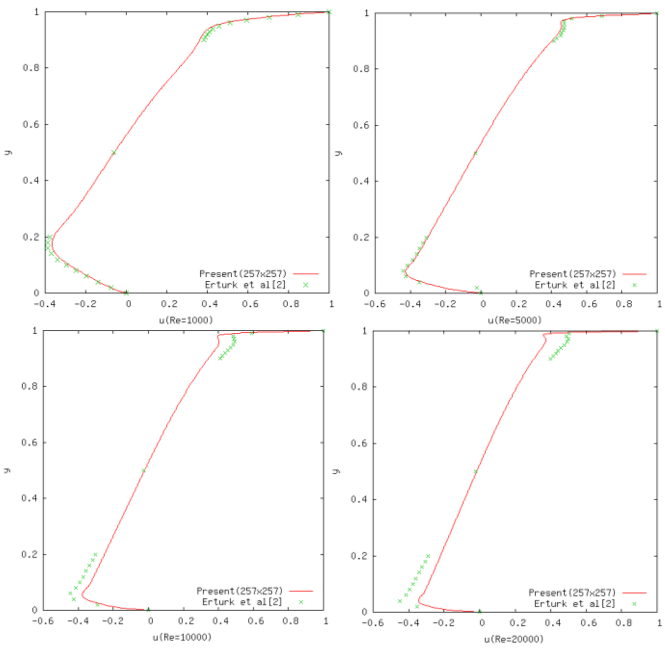


Figure: 3 *u*-velocity profiles along the centerline of the cavity at different Reynolds numbers

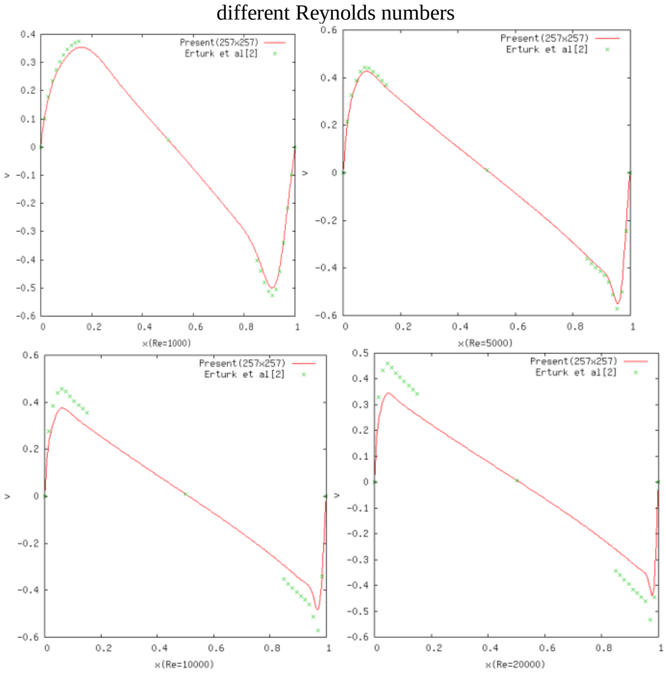


Figure: 4 *v*-velocity profiles along the centerline of the cavity at different Re

From Fig.5 it is seen that the vorticity value in the grid size 257x257 crosses the theoretical limit value of -1.8859 near to Re=2000 which is much earlier than Erturk *et al*[2]. So, it seen that though the high Reynolds number solution is obtained at lower grid size but the accuracy is very poor. From Fig.6 comparing the magnitude of the velocity values along the centre of the cavity, it seen that as the Reynolds number increases the length of the central portion of linear profile increases. The linear nature of the velocity profile shows the constant vorticity region which increases with increase of Reynolds number. The velocity profiles become closer to the wall with increase of Reynolds number which indicates the presence of boundary layers on the walls. This type of flow feature can also be verified from the vorticity contour plot in Fig.2.

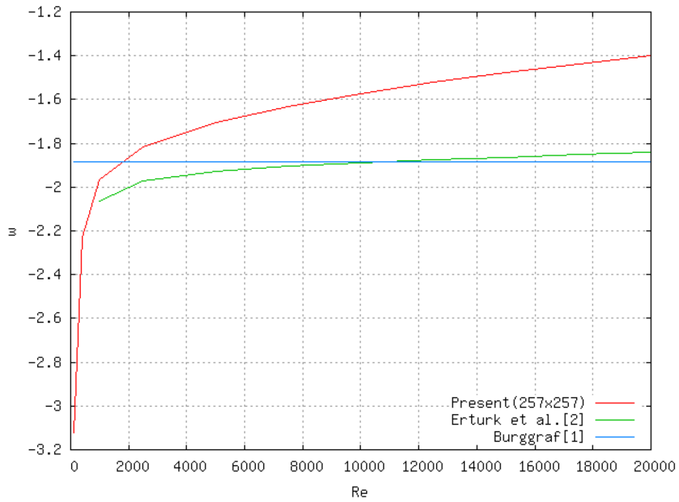


Figure: 5 Vorticity value at the centre of the primary eddy with Re

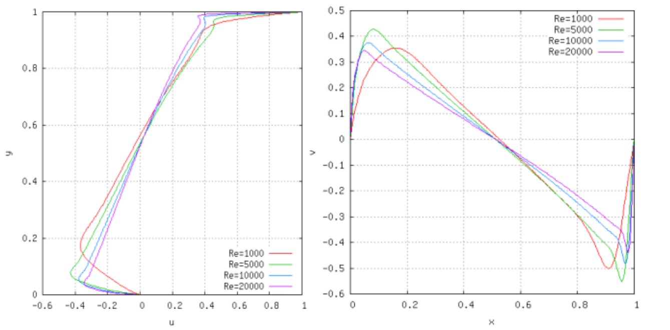


Figure: 6 *u* and *v*-velocity profiles along the centreline of the cavity with Re

IV CONCLUSION

The stream function vorticity form of the Navier-Stokes equation is solved using Crank-Nicholson scheme. The solution is obtained up to maximum Reynolds number 22,500 using the grid size 129x129 and 257x257. The Crank-Nicholson scheme produces better result at lower grid size than that of Erturk *et al* but the accuracy of the former scheme is very poor.

REFERENCES

[1] R Burggraf, "Analytical and numerical studies of the structure of steady separated flows," *Journal of Fluid Mechanics* **24** *pp.*113-151 (1966).

[2] E. Erturk, T. C. Corke and C Gockol, "Numerical solutions of 2-D steady incompressible driven cavity flow at higher Reynolds numbers," *Int. Journal of Numerical Methods in Fluids* **48** *pp*. 747-774 (2005).

[3] E Erturk and C. Gokcol, "Fourth-order compact formulation of Navier-Stokes equations and driven cavity flow at high Reynolds numbers," *Int. J. for Num. Methods in Fluids.* **50** *pp.* 421-436 (2006).

[4] E Erturk, "Discussions on driven cavity flow." *Int. J. for Num. Methods in Fluids.* **60** *pp*. 275-294 (2009).