Similarity Solution of MHD Boundary Layer Flow of Prandtl-Eyring Fluids

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Abstract— **This paper investigates the magneto-hydrodynamics (MHD) boundary layer flow of non-Newtonian Prandtl-Eyring fluids past a right angle wedge. The use of Lie group transformation method so-called scaling group symmetry analysis is made to derive possible similarity transformations of the present flow problem. The important conclusion drawn from the present analysis is that for all those non-Newtonian fluids whose shearing stress is composite function of rate of strain, the similarity solutions exist only for the flows past 90 degree wedge. The solution of similarity equation, which is non-linear ordinary differential, is obtained by Keller-Box method for the various values of flow parameters. Effect of imposed magnetic field on the flow of non-Newtonian Prandtl-Eyring fluids is also studied.**

Keywords—Non-Newtonian Fluids; Prandtl-Eyring Fluids; MHD Boundary layer flow; similarity solution; Scaling group symmetry; Non-Newtonian Fluids; Prandtl-Eyring Fluid;

I. INTRODUCTION

The partial differential equations governing the motion of fluid flow problems are usually non-linear in nature and hence cannot be solved easily. Whenever possible these differential equations are reduced to ordinary differential equations by employing transformations to obtain similar solutions. Ames [1], Bluman-Kumai[2],Hansen[3],Seshadri-Na[4] and Stephani[5] have discussed application of groups and symmetries to partial differential equations arising from natural phenomena and technological problems. Symmetry groups are invariant transformations which do not alter the structural form of the equation under investigation. The advantage of the symmetry method is that it can be applied successfully to non-linear partial differential equations governing the motion of fluid. Sophus Lie developed a transformation, currently known as Lie group of transformation, which maps a given differential equation to itself. The differential equations remain invariant under some continuous group of transformations usually known as symmetries of a differential equation.

The classical theory of Newtonian fluid depends upon the hypothesis of linear relationship between stress tensor and rate of strain tensor. The fluids which do not follow such a linear relationship are called non-Newtonian fluids. Non-Newtonian fluids are generally divided in to two categories like isotropic and homogeneous and when they are subjected to a shear the resultant stress depends only on the rate of shear. However, such types of fluids show diverse behavior in response to applied stress. Numbers of rheological models have been proposed to explain such a diverse behavior. Some of this models are ; Power-law fluids, Sisko fluids, Ellis fluids, Prandtl fluids Williamson fluids, Sutterby fluids Reiner-Rivlin fluids, Bingham plastic , Prandtl-Eyring fluids , Powell-Eyring fluids , Reiner-Philippoff etc.

When we consider electrically conducting non-Newtonian fluids flowing under the influence of external magnetic field, the study becomes interesting .This is because in such situation magnetic forces produced in it could influence the motion of the fluids in significant way and hence such interaction problems have great practical applications. The problem of two-dimensional magneto hydrodynamic boundary layer equation for laminar incompressible flow past flat plate has been investigated by Rossow [6] and Greenspan et al [7].Rossow[6] has considered transverse magnetic field where as Greenspan et al [7] have considered longitudinal magnetic fields on the velocity and temperature distributions. Timol et al [8] have investigated three-dimensional magneto hydrodynamic boundary layer flow with pressure gradient and fluid injection. Similarity transformation for both steady and unsteady three-dimensional MHD boundary layer flow of purely viscous non-Newtonian fluid has been derived by Manisha et al[9].They have also derived Similarity Analysis in MHD Heat and Mass Transfer of Non-Newtonian Power Law Fluids Past a Semi-infinite Flat Plate [10].

To investigate the non-Newtonian effects, the class of solutions known as similarity solutions place an important role. This is because that is the only class of the exact solution for the governing equations which are usually non-linear partial differential equations (PDEs) of the boundary layer type. Further this also serves as a reference to check approximate solutions.

It is well known that similarity solutions for the PDEs governing the flow of Newtonian and non-Newtonian fluids exist only for limited classes of main stream velocities at the edge of the boundary layer. For example, for two dimensional laminar boundary layer flow of Newtonian fluids, similarity solutions are limited to the well known Falkner-Skan solution Rajagopal et al [11]. Most of the generalization of the Falkner-Skan solutions and approximate solutions in the literature are limited to the power law fluids; this is because they are mathematically the easiest to be treated among most of the non-Newtonian fluids.

Hansen and Na [12] were probably first to derive similarity analysis of laminar incompressible boundary layer equations of the non-Newtonian fluids whose stress and rate of strain are related by arbitrary continuous function, while extending the work of Lee and Ames [13].They have used linear group of transformations to derive their similarity equations which includes many of the non-Newtonian viscoinelastic fluids. Timol and Kalthia [14] are probably first to derive similarity solution for the class of three dimensional boundary layer flow of visco-in-elastic non-Newtonian fluids.

For the derivation of the constitutive equations governing the motion of non-Newtonian fluids, the mathematical structure of stress-strain relationship, which is non linear, is important in when found in functional form. This relationship may be implicit or explicit. In the present paper we have consider such relationship in the form of general arbitrary continuous function of the type:

$$\Omega\left(\tau_{yx,}\frac{\partial u}{\partial y}\right) = 0 \tag{1}$$

Here τ is the shearing stress and $\partial u/\partial y$ is the rate of the strain of the fluids.

For the similarity analysis many techniques are available, among them the similarity methods which invoke the invariance under the group of transformations are known as group theoretic methods. These methods are more recent and are mathematically elegant and hence they are widely used in different fields. The group theoretic methods involve mainly two different types of groups of transformations, namely, assumed group of transformations, spiral group transformations are the assumed group of transformations and are mainly due to Birkhoff [15] and Morgan [16].

So Motivated by these , we represent in the present paper Using Scaling group transformation technique ,class of similarity solution for steady ,two dimensional laminar MHD boundary layer flows of incompressible non-Newtonian is derived .From the present analysis it is interesting to observe that for non-Newtonian viscoinelastic fluids of any model , which is characterized by the property that its stress and the rate of strain can be related by arbitrary continuous function given by equation (1), the similarity solutions exist only for the flows past 90^0 wedge , as shown in Fig.1.

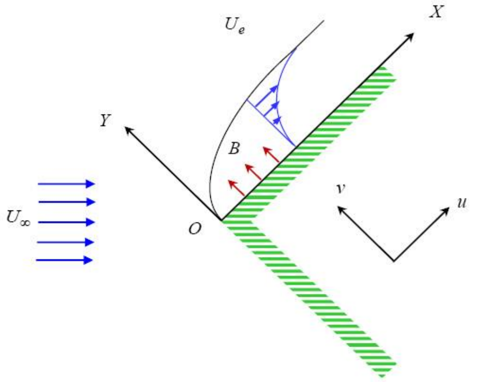

Figure 1: Schematic Diagram of flow past 90° wedge

II. GOVERNING EQUATIONS

The equation of motion for incompressible electrically conducting non-Newtonian fluid can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho}\frac{\partial \tau}{\partial y} + U\frac{\partial U}{\partial x} - \frac{\sigma B_0^2}{\rho}u \tag{3}$$

With stress-strain relationship is given by,

$$\Omega\left(\tau_{yx'}\frac{\partial u}{\partial y}\right) = 0 \tag{4}$$

Together with boundary conditions,

$$y = 0, \qquad u(x,0) = v(x,0) = 0$$

$$y = \infty, \qquad u(x,\infty) = U(x) \tag{5}$$

$$x^* = \frac{x}{L};\ y^* = \frac{y}{L}(Re)^{\frac{1}{2}};\ u^* = \frac{u}{U_\infty};\ v^* = \frac{v}{U_\infty}(Re)^{\frac{1}{2}}$$

$$\tau^*_{yx} = \frac{\tau_{yx}}{\rho U_\infty^2}(Re)^{\frac{1}{2}};\ U^* = \frac{U}{U_\infty} R_e = \frac{U_\infty L}{v};\ S^* = \frac{L}{U_\infty} S \tag{6}$$

Where $R_e = \frac{U_\infty L}{v}$ Reynolds number and $S^*(x) = \frac{\sigma B_0^2(x)}{\rho}$ magnetic parameter

Substitute these quantities in equation (1) to (5) and dropping the asterisk, for simplicity

We get,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{7}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\partial}{\partial y^*}\left(\tau^*_{y^*x^*}\right) + U^*\frac{\partial U^*}{\partial x} - S^*(x)\,u^* \tag{8}$$

With stress-strain relationship is given by,

$$\Omega\left(\tau^*_{yx}, \frac{\partial u^*}{\partial y^*}\right) = 0 \qquad (9)$$

Introducing stream function ψ such that,

$$u^* = \frac{\partial \psi^*}{\partial y^*}, \qquad v^* = -\frac{\partial \psi^*}{\partial x^*} \qquad (10)$$

Equation of continuity (7) gets satisfied identically,

Equation (8) and (9) becomes,

$$\frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}} = \frac{\partial}{\partial y^*}(\tau^*_{yx}) + U^*\frac{\partial U^*}{\partial x} - S^*(x)\frac{\partial \psi^*}{\partial y^*} \qquad (11)$$

$$\Omega\left(\tau^*_{yx}, \frac{\partial^2 \psi^*}{\partial y^{*2}}\right) = 0 \qquad (12)$$

With boundary conditions,

$$y = 0, \qquad \frac{\partial \psi}{\partial y}(x,0) = \frac{\partial \psi}{\partial x}(x,0) = 0$$

$$y = \infty, \qquad \frac{\partial \psi}{\partial y}(x,y) = U(x) \qquad (13)$$

III. SIMILARITY SOLUTION OF THE PROBLEM

By using scaling linear group transformation

$$\bar{x}^* = A^{\alpha_1}x^*, \qquad \bar{y}^* = A^{\alpha_2}y^*$$
$$\bar{\psi}^* = A^{\alpha_3}\psi^*, \qquad \bar{\tau}^*_{yx} = A^{\alpha_4}\tau^*_{yx}$$
$$\bar{U}^* = A^{\alpha_5}U^*, \qquad \bar{S}^* = A^{\alpha_6}S^* \qquad (14)$$

Where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ and A are Constants for the dependent and independent variables. From equation (14) one obtains

$$\left(\frac{\bar{x}^*}{x^*}\right)^{\frac{1}{\alpha_1}} = \left(\frac{\bar{y}^*}{y^*}\right)^{\frac{1}{\alpha_2}} = \left(\frac{\bar{\psi}^*}{\psi^*}\right)^{\frac{1}{\alpha_3}} = \left(\frac{\bar{\tau}^*_{yx}}{\tau^*_{yx}}\right)^{\frac{1}{\alpha_4}} = \left(\frac{\bar{U}^*}{U^*}\right)^{\frac{1}{\alpha_5}} = \left(\frac{\bar{S}^*}{S^*}\right)^{\frac{1}{6}} = A \qquad (15)$$

Introducing the linear transformation, givequation (15), into the Eqs. (11-12) results in

$$A^{2\alpha_3-2\alpha_2-\alpha_1}\frac{\partial \bar{\psi}^*}{\partial \bar{y}^*}\frac{\partial^2 \bar{\psi}^*}{\partial \bar{x}^* \partial \bar{y}^*} - A^{2\alpha_3-2\alpha_2-\alpha_1}\frac{\partial \bar{\psi}^*}{\partial \bar{x}^*}\frac{\partial^2 \bar{\psi}^*}{\partial \bar{y}^{*2}}$$

$$= A^{\alpha_4-\alpha_2}\frac{\partial}{\partial \bar{y}^*}(\bar{\tau}^*_{yx}) + A^{2\alpha_5-\alpha_1}\bar{U}^*\frac{\partial U^*}{\partial \bar{x}^*} - A^{\alpha_6+\alpha_3-\alpha_2}\frac{\partial \bar{\psi}^*}{\partial \bar{x}^*} \qquad (16)$$

And

$$\Omega\left(A^{\alpha_4}\tau^*_{yx}, A^{\alpha_3-2\alpha_2}\frac{\partial^2 \psi^*}{\partial y^{*2}}\right) = 0 \qquad (17)$$

The differential equation are completely invariant to the proposed linear transformation, the following coupled algebraic equations are obtained

$$2\alpha_3 - 2\alpha_2 - \alpha_1 = \alpha_4 - \alpha_2 = 2\alpha_5 - \alpha_1 = \alpha_6 + \alpha_3 - \alpha_2 \qquad (18)$$
$$\alpha_3 - 2\alpha_2 = 0 \qquad (19)$$
$$\alpha_4 = 0 \qquad (20)$$

By solving above equations with , we get

$$\frac{\alpha_2}{\alpha_1} = \frac{1}{3}, \quad \frac{\alpha_3}{\alpha_1} = \frac{2}{3}, \quad \frac{\alpha_4}{\alpha_1} = 0, \quad \frac{\alpha_5}{\alpha_1} = \frac{1}{3}, \quad \frac{\alpha_6}{\alpha_1} = -\frac{2}{3} \qquad (21)$$

Introducing equation (21) into equation (15) result in

$$\eta = \frac{y^*}{x^{*\frac{1}{3}}}, \quad \psi^* = f(\eta)x^{*\frac{2}{3}}, \quad U^* = G(\eta)x^{*-\frac{1}{3}}$$

$$\tau^*_{yx} = H(\eta) \quad \text{and} \quad S(x) = S_0 x^{*-\frac{2}{3}} \qquad (22)$$

With the boundary conditions, equation (13) becomes

$$\eta = 0, \qquad f(0) = f'(0) = 0$$
$$\eta \to \infty, \qquad f'(\infty) = 1 \qquad (23)$$

Introducing equations (22) in equation (11)-(13), we get following similarity equation

$$f'^2(\eta) - 2f(\eta)f''(\eta) - 3H'(\eta) + \eta G(\eta)G'(\eta) - G^2(\eta) + S_0\left(f'(\eta)\right) = 0 \qquad (24)$$

But U^* is independent of y, $G(\eta)$ must be constant .Therefore $G(\eta)$ assume Unity

i.e. $G(\eta)=1,\ G'(\eta)=0$ And also assume that $S_0 = 1$

$$f'^2(\eta) - 2f(\eta)f''(\eta) - 3H'(\eta) - 1 = 0 \qquad (25)$$

With the boundary conditions,

$$\eta = 0, \qquad f(0) = f'(0) = 0$$
$$\eta \to \infty, \qquad f'(\infty) = 1 \qquad (26)$$

And the Stress-Strain functional relationship is given by,

$$\Omega(H, f'') = 0 \qquad (27)$$

IV. NUMERICAL SOLUTION OF PROBLEM

The transformed highly non-linear ordinary differential equation (25) subject to the boundary conditions (26) is solved numerically by using Keller-Box method (Cebeci and Bradshaw [17]; Keller [18]). This method is second order accurate and allows uniform and non-uniform grid size. The numerical algorithm to solve the two-point non-linear boundary value problem is as explained below.

The boundary value problem of (24) in *f* is reduced to a first order system of three simultaneous ordinary differential equations:

$$\frac{df_i}{d\eta} = F_i(\eta, f_1, f_2, \ldots f_n); \quad (i = 1,2,3 = n)$$

(28)

Where, $f_1 = f, \quad f_2 = f', \quad f_3 = f''$

The boundary conditions (24) become,

$$f_1(0) = 0, \quad f_2(0) = 0, \quad f_2(\eta_\infty) = U_0$$

Then, after choosing η_∞, the numerical infinity, a grid for the closed interval $[0, \eta_\infty]$ is chosen and the above system of first order equations are transformed into a system of finite difference equations (FDEs) by replacing the differential terms by forward difference approximation and the non-differential terms by the average of two adjacent grid points. The numerical method gives approximate values of f, f', f'' at all the grid points. By adding the boundary conditions (28) to the system of FDEs, we obtain a non-linear system of algebraic equations in which the number of equations and unknowns are the same. Subsequently, the linearization of these FDEs was done by Newton's method. The resulting systems of linear equations were solved by a block tri-diagonal solver. The step size $\Delta\eta$ in η and the position of the edge of the boundary layer in η_∞ are to be adjusted for different values of the parameters to maintain accuracy. For brevity, further details on the solution process are not presented here. It is worth mentioning that a uniform grid of $\Delta\eta = 0.01$ was found to be satisfactory for a convergence criterion of 10^{-6} in all most all the cases.

V. RESULTS AND DISCUSSION

Using the algorithm discussed above (which is unconditionally stable), the numerical solutions are obtained for several series of values of the magnetic parameter *Mn*, the flow parameters, α and γ the numerical values are plotted in Figs. 2-7. Without loss of generality U_0 is taken as unity. These figures depict the influence of flow parameters under effect of magnetic field on the velocity component along the wedge of surface, so-called the wedge velocity (Figs. 2, 3) and on the variation of local shear-stress, hence skin friction at the wedge (Figs. 4, 5). Fig. 6, 7 shows the magnetic effect on wedge velocity profile and shear-stress at wedge. Figs. 2,3 are the graphical representation of the velocity profiles f' along the wedge for different values of α and γ. It shows the effect of magnetic field on velocity profile. These figures distinguish the velocity with and without magnetic fields. It is worth to observe that the velocity along the wedge increases under the effect of magnetic field for the given values of the flow parameters.

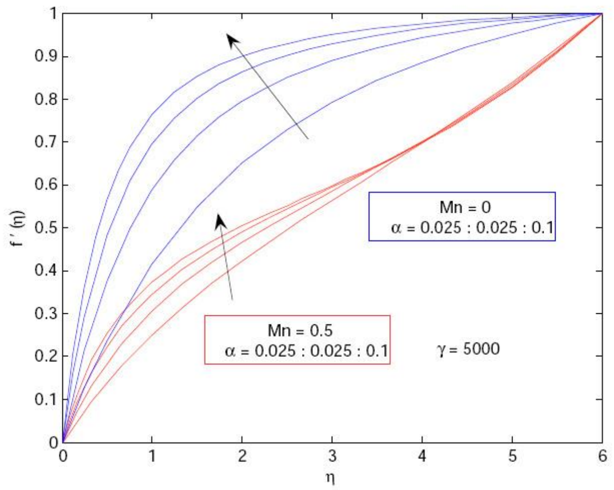

Figure 2: Comparison of wedge velocity for various α

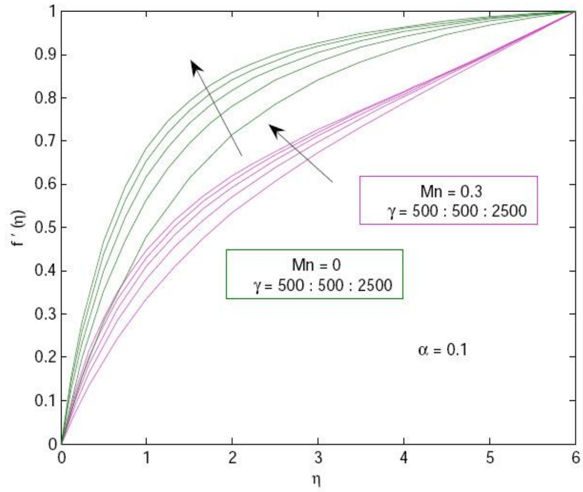

Figure 3: Comparison of wedge velocity for various γ

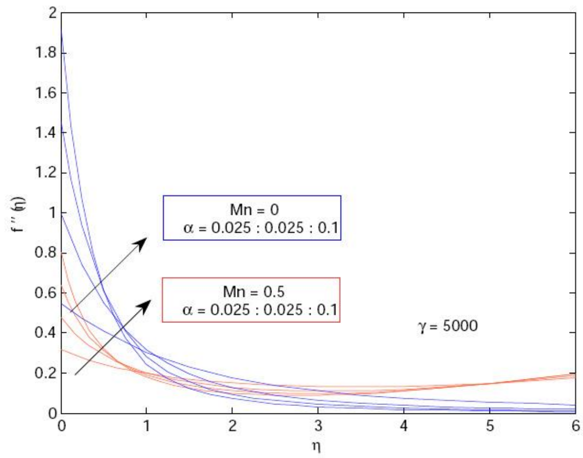

Figure 4: Comparison of local shear-stress for various α

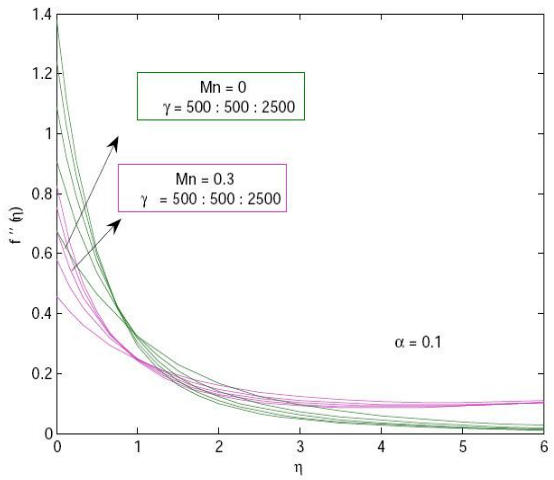

Figure 5: Comparison of local shear-stress for various γ

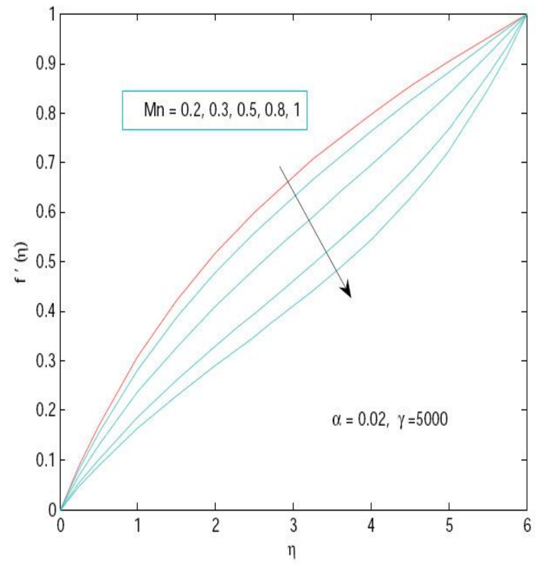

Figure 6: Influence of magnetic field on wedge velocity

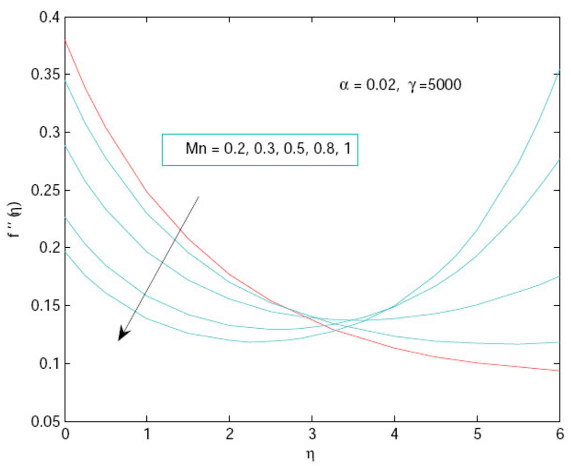

Figure 7: Effect of magnetic field on local shear-stress

Figs. 4,5 are represent the comparison of shear stress $f''(0)$ at wedge with and without magnetic field. It suggests that under the influence of magnetic field the shear stress at wedge decreases and hence the local skin friction coefficient C_f decreases for given values of the flow parameters. These figures are also depicting the effect of flow parameters on local skin friction.

Fig. 6,7 depicts the magnetic effect on fluid flow. It is interesting to notice that as the magnetic field strength increase there is sharp fall down in the shear stress at wedge and hence the local skin friction decreases sharply. This shows the essential effect of magnetic field on non-Newtonian Prandtl-Eyring fluids.

VI. CONCLUDING REMARKS:

- The similarity solutions for MHD laminar incompressible boundary layer equations of all non-Newtonian Prandtl-Eyring fluids are derived.
- It is interesting to note that the deductive group theoretic method based on general group of transformation is applied to derive proper similarity transformations for the non-linear partial differential equation with the stress-strain functional relationship condition, governing the flow under consideration.
- It is to be observing that similarity solutions for all non-Newtonian fluids exist only for the flow past 90° wedge.
- The present similarity equation is solved by Keller-Box method.
- The numerical solutions have produced and compare for fluid flow with and without magnetic field.
- It is worth to note that all solutions have derived for non-dimensional quantities and hence these results are applicable for all types of under considered non-Newtonian fluids.

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