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# Shear Waves in Functionally Graded Electro-Magneto-Elastic Media

David K. Gasparyan
Institute of Mechanics
National Academy of Sciences
Yerevan, Armenia

Karen B. Ghazaryan
Institute of Mechanics
National Academy of Sciences
Yerevan, Armenia

***Abstract-*** **In the framework of quasi-static approach the shear wave propagation is considered in functionally graded transversally isotropic hexagonal 6mm symmetry magnetoelectroelastic media (MEE). Assuming that in functionally graded MEE material elastic and electromagnetic properties vary in the same proportion in direction perpendicular to the MEE polling direction, special classes of inhomogeneity functions were found, admitting exact solutions for the coupled wave field and allowing to estimate the effects of inhomogeneity on the wave behavior. Exact solutions defining the coupled shear wave field in MEE can be used in many problems, e.g. shear surface waves propagation along the surface of a semi-infinite space, interfacial waves in a multilayered and periodic structure, Love type waves in a layer overlying a half-space, guided waves in an inhomogeneous waveguide, etc. Based on exact solutions, the localized wave propagation is studied for MEE layer with quadratic and inverse quadratic inhomogeneity profiles of material parameters varying continuously along the layer thickness direction. Dispersion equations are deduced analytically and for the BaTiO3–CoFe2O4 MEE crystal the numerical results estimating effects of inhomogeneity are presented.**

***Keywords- piezomagnetic, piezoelectric, shear wave, waveguide, inhomogeneity***

## I. INTRODUCTION

Recently, the propagation of coupled electromagnetic and elastic waves in magneto-electro-elastic (MEE) structures attracted much attention due to the wide range of application of these materials in smart structures. MEE materials are a class of new artificial composites that consist of simultaneous piezoelectric and piezomagnetic phases. The magnetoelectric effect of piezoelectric–piezomagnetic composites was first reported in [1]. Such materials rarely occur in nature and demonstrate weak magneto-electro-elastic effects. However in artificial composite materials the magneto-electro-elastic effect is notable, which makes such materials highly valuable in technological usage. Magneto-electro-elastic composites are built up by combining piezoelectric and piezomagnetic phases [2] to obtain a smart composite that presents not only the electro-mechanical and magneto-mechanical coupling, characteristic of constituent phases, but also a strong magnetoelectric coupling. In [2] the theoretical estimates are shown to be in agreement with available experimental results, and also show the interesting magnetoelectric behaviour of the composites. Review of the physics of such materials is given in [3], which begins with a brief summary of the historical perspective of the magnetoelectric composites since their appearance in 1972. The review concludes with an outlook of the exciting future possibilities and scientific challenges in the field of multiferroic magnetoelectric composites.

A new method for the synthesis of artificial crystal with a strong electro-magneto-elastic interaction by combining electro-elastic $BaTiO_3$ and magneto-elastic $CoFe_2O_4$ in a single crystal is presented in [4].

Quasi-static approximation was used to research surface and bulk wave propagation in such materials [5-12]. The propagation of Bleustein-Gulyaev surface wave is investigated in [5] for transversely isotropic MEE materials. In [6] the existence of a new surface SH wave is stated for a cubic anisotropic MEE material. An analytical approach was used to investigate Love wave propagation in a layered MEE structure [7], where a solution of dispersion relations was obtained for magnetoelectrically open and short boundary conditions. In [8] Rayleigh waves are investigated in MEE half plane, the material of which is assumed to possess hexagonal 6 mm symmetry. In [9] it is shown that shear surface waves with twelve different velocities in cases of different magneto-electrical boundary conditions can be guided by the interface of two identical MEE half-spaces. The existence of shear surface wave travelling along the interface of two half spaces of different MEE materials is studied in [10]. The study of SH waves in a hetero-structure made from three different MEE materials with 6mm symmetry is given in [11]. In [12] the dispersion relation of the MEE three-dimensional, anisotropic and multi-layered thick plate vibration frequency has been derived. Applied problems of MEE beam and plate behaviour are studied in [13-17]. In [13-14] based on Timoshenko's beam theory an analytical model for MEE bimorph beam [13] is employed, to study its response to mechanical and electro-magnetic time varying loads. In [14] it was shown that the effect of shear deformation has a great influence on piezo-electro-elastic beam's natural frequencies and mode shapes. The pyroelectric and pyromagnetic effects on magneto-electro-elastic plate with different boundary conditions under uniform temperature rise is studied in [15]. Based on Kirchhoff thin-plate theory, a closed form expressions for bending problem of MEE rectangular thin plates are derived in [16] and the exact solutions for the deformation behaviours of the fiber-reinforced the $BaTiO_3$–$CoFe_2O_4$ composites subjected to certain types of surface loads are analytically obtained. In [17] exact solutions are derived for anisotropic, simply-supported, multi-layered rectangular MEE plates under static loadings.

In [18] functionally graded materials, whose properties vary continuously in space were used to improve the efficiency of Bleustein–Gulyaev waves for a hexagonal 6 mm piezoelectric crystal. Assuming that the elastic stiffness, the piezoelectric constant, the dielectric constant, the mass density of MEE material vary in the same proportion with a single space variable special classes of the inhomogeneity functions were founded allowing the exact solution of surface wave propagation problem.

Functionally graded materials (FGM) are inhomogeneous elastic bodies whose properties vary continuously with space. The FGM structure has attracted wide and increased attentions of scientists and engineers. FGM plays an essential role in the most advanced integrated systems for vibration control and health monitoring. The progress in the characterization, modelling, analysis and principal developments of FGM was reviewed in [19, 20].

In pure elastic FGM materials surface wave propagation was discussed in [21-26].

In [27, 28] the propagation of shear electroelastic monochromatic localized waves in functionally graded piezoelectric layer is studied, where the influence of inhomogeneity function on dispersion of shear wave is analysed and numerical comparison between wave speeds of homogeneous and inhomogeneous layers is carried out. Surface Love waves are considered in [29] for a layered structure with inhomogeneous piezoelectric layer. The behaviour of Lamb waves in the functionally graded piezoelectric–piezomagnetic plate with material parameters varying continuously along the thickness direction is investigated in [30], where the power series technique is employed to find dispersion curves numerically. In [31] the Bleustein–Gulyaev waves are studied by analytical technique in a functionally graded transversely isotropic MEE half-space, in which all parameters change exponentially along the depth direction. All the above mentioned solutions were based on quasi-static approximation of a problem [32], where the derivatives of time in Maxwell's electro-dynamic equations were ignored. Such approximation allows high precision solutions regarding the influence of the electro-magnetic field on the properties of elastic fields, however is limited in finding out the coupled wave processes in magneto-electro-elastic materials. More specifically, quasi-static definition cannot be used to describe the reflection and refraction of electro-magnetic waves [33], coupling effects of electro-magnetic and elastic fields which causes polariton interaction in MEE periodic structure [34]. Based on the full complete set of electrodynamics dynamics equations and the elasticity theory equations in [35] the two-dimensional equations are derived describing coupled wave process in MEE medium of hexagonal symmetry, where it was stated that contrary to the quasi-static approximation under dynamic approach the plane and anti-plane deformations are coupled.

## II. Statement of the Problem

For a transversely isotropic magneto-electro-elastic hexagonal symmetry 6 mm medium with z-axis normal to the plane of isotropy, polling direction of which coincides with z-axis direction, the anti-plane equations in Cartesian coordinate system $(x, y, z)$ can be written as

$$\begin{aligned} &\partial_x D_x + \partial_y D_y = 0 \\ &\partial_x B_x + \partial_y B_y = 0 \\ &\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = \rho \partial_{t,t} W \end{aligned} \quad (1)$$

$$\begin{aligned} \sigma_{yz} &= G\partial_y W + e\partial_y \varphi + \beta\partial_y \phi \\ D_y &= e\partial_y W - \varepsilon\partial_y \varphi - \alpha\partial_y \phi \\ B_y &= \beta\partial_y W - \alpha\partial_y \varphi - \mu\partial_y \phi \end{aligned} \quad (2)$$

$$\begin{aligned} \sigma_{xz} &= G\partial_x W + e\partial_x \varphi + \beta\partial_x \phi \\ D_x &= e\partial_x W - \varepsilon\partial_x \varphi - \alpha\partial_x \phi \\ B_x &= \beta\partial_x W - \alpha\partial_x \varphi - \mu\partial_x \phi \end{aligned} \quad (3)$$

Here the $W(x,y,t)$ is the elastic displacement directed along $z$-direction, , $\sigma_{xz}(x,y,t), \sigma_{yz}(x,y,t)$ are mechanical stresses, the $D_x(x,y,t), D_y(x,y,t)$, $B_x(x,y,t), B_y(x,y,t)$, $\varphi = \varphi(x,y,t)$, $\phi = \phi(x,y,t)$ are electric displacements, the magnetic inductions, the electric potential and the magnetic potential, correspondingly, $\rho, G = c_{44}, e = e_{15}, \beta = d_{15}, \alpha$ are the bulk density, elastic, piezoelectric, piezomagnetic and magneto-elastic modulus respectively, $\varepsilon$ and $\mu$ are the dielectric permittivity and magnetic permeability coefficients, while $\partial_x = \partial/\partial x; \partial_y = \partial/\partial y; \partial_{tt} = \partial^2/\partial_{tt}^2$ .

To exemplify the problem and provide insights of shear waves propagation in functionally graded piezo-electro-magneto-elastic media the following model is considered.

The material parameters in the MEE medium gradually change along y-direction having the same function variation properties

$$\begin{aligned} G(y) = G_0 f(y); \beta = \beta_0 f(y); e = e_0 f(y) \\ \alpha = \alpha_0 f(y); \varepsilon = \varepsilon_0 f(y); \mu = \mu_0 f(y); \rho = \rho_o f(y) \end{aligned} \quad (4)$$

Here $f(y)$ is the inhomogeneity function which will be specified later.

Equations (1) and (2) can be considered as a set of first order six differential equations with six sought functions $\sigma_{yz}, \mathrm{D}_y, \mathrm{B}_y, W, \varphi, \phi$.

The functions $\sigma_{xz}, \mathrm{D}_x, \mathrm{B}_x$ can be defined from (3) via the sought functions.

Considering a wave with the circular frequency $\omega$ and wave number $k$ we present all functions in the form of plane harmonic wave travelling along the $x$ - direction

$$\{\sigma_{yz}, \mathrm{D}_y, \mathrm{B}_y, W, \varphi, \phi\}(x,y,t) = \\ = \{\sigma_{oyz}(y), \mathrm{D}_{oy}(y), \mathrm{B}_{oy}(y), W_0(y), \varphi_0(y), \phi_0(y)\} \cdot \\ \cdot \exp i(kx - \omega t)$$

Introducing vectors

$$\sigma(y) = (\sigma_{oyz}, \mathrm{D}_{oy}, \mathrm{B}_{oy})^T \\ U = (W_0, \varphi_0, \phi_0)^T$$

we can rewrite the anti-plane equations as the set of first order differential equations in a matrix form

$$\partial_y \sigma(y) = MU(y) \\ \partial_y U(y) = N\sigma(y) \quad (5)$$

Here $M, S$ are the following matrixes

$$M = \begin{pmatrix} G & \beta & e \\ \beta & -\mu & -\alpha \\ e & -\alpha & -\varepsilon \end{pmatrix}$$

$$N = T^{-1}$$

$$T = k^2 \begin{pmatrix} G - \dfrac{\rho\omega^2}{k^2} & \beta & e \\ \beta & -\mu & -\alpha \\ e & -\alpha & -\varepsilon \end{pmatrix}$$



Defining new functions $\tilde{\sigma}_{oyz}(y), \tilde{\mathrm{D}}_{oy}(y), \tilde{\mathrm{B}}_{oy}(y)$ and new auxiliary potentials $S(y), F(y)$ as

$$\sigma_{oyz}(y) = \tilde{\sigma}_{oyz}(y)\sqrt{f(y)} \\ \mathrm{D}_{oy}(y) = \tilde{\mathrm{D}}_{oy}(y)\sqrt{f(y)} \\ \mathrm{B}_{oy}(y) = \tilde{\mathrm{B}}_{oy}(y)\sqrt{f(y)} \\ W_0(y) = \tilde{W}_0(y)/\sqrt{f(y)} \\ \varphi_0(y) = \frac{-S(y)\mu_0 + W_0(y)\eta_0 + F(y)\alpha_0}{\gamma_0\sqrt{f(y)}} \\ \phi_0(y) = \frac{-F(y)\varepsilon_0 + S(y)\alpha_0 + W_0(y)\theta_0}{\gamma_0\sqrt{f(y)}} \quad (6)$$

where

$$\gamma_0 = \varepsilon_0\mu_0 - \alpha_0^2 \\ \theta_0 = \beta_0\varepsilon_0 - e_0\alpha_0 \quad (1) \\ \eta_0 = e_0\mu_0 - \alpha_0\beta_0$$

$\alpha + \beta = \chi$. (1)

and substituting (6) into (5) it is straightforward to derive the following matrix equations

$$U_0 = M_0 \hat{L}_p \sigma_0 \quad (7)$$

$$\sigma_0 = N_0 \hat{L}_q U_0 \quad (8)$$

with respect to the new unknown vector functions

$$\sigma_0(y) = (\tilde{\sigma}_{oyz}(y), \tilde{\mathrm{D}}_{oy}(y)\tilde{\mathrm{B}}_{oy}(y))^T \\ U_0(y) = (\tilde{W}_0(y), S(y)F(y))^T \quad (9)$$

Here

$$M_0 = \frac{1}{E_0 p^2 k^2}\begin{pmatrix} 1 & \eta_0\gamma_0^{-1} & \theta_0\gamma_0^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$N_0 = \begin{pmatrix} E_0 & -\eta_0\gamma_0^{-1} & -\theta_0\gamma_0^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{L}_p = \frac{d}{dy} + P(y) \\ \hat{L}_q = \frac{d}{dy} - P(y) \\ P(y) = \frac{1}{2f}\frac{df}{dy}$$

$$p = \sqrt{1 - \zeta^2} \\ E_0 = G_0 + (e_0\eta_0 + \beta_0\theta_0)\gamma_0^{-1} \\ \zeta = \sqrt{\omega^2\rho_0 / E_0 k^2}$$

Now by substituting the vector $U_0$ from (7) into (8) and the vector $\sigma_0$ from (8) into (7) we come to the two decoupled sets of equations

$$\begin{pmatrix} p^{-2} & \eta_0\gamma_0^{-1}\left(p^{-2}-1\right) & \theta_0\gamma_0^{-1}\left(p^{-2}-1\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{L}_q\hat{L}_p\tilde{\sigma}_{oyz} \\ \hat{L}_q\hat{L}_p\tilde{\mathrm{D}}_{oy} \\ \hat{L}_q\hat{L}_p\tilde{\mathrm{B}}_{oy} \end{pmatrix} = k^2\begin{pmatrix} \tilde{\sigma}_{oyz} \\ \tilde{\mathrm{D}}_{oy} \\ \tilde{\mathrm{B}}_{oy} \end{pmatrix} \quad (10)$$

$$\begin{pmatrix} p^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{L}_p\hat{L}_q\tilde{W}_0 \\ \hat{L}_p\hat{L}_q S \\ \hat{L}_p\hat{L}_q F \end{pmatrix} = k^2\begin{pmatrix} \tilde{W}_0 \\ S \\ F \end{pmatrix} \quad (11)$$

Noting that

$$\hat{L}_p\hat{L}_q = \frac{d^2}{dy^2} - \left(\frac{dP}{dy} + P^2\right)$$

$$\hat{L}_q\hat{L}_p = \frac{d^2 y}{dy^2} - \left(P^2 - \frac{dP}{dy}\right)$$

we come to the following two ways of finding the exact solutions if we assume that

$$\left(\frac{dP}{dy} - P^2\right) = b \quad (12)$$

or

$$\left(\frac{dP}{dy} + P^2\right) = b \quad (13)$$

where $b$ is a constant.

Since

$$\left(\frac{dP}{dy} - P^2\right) \equiv \frac{1}{f^{-1/2}}\frac{d^2 f^{-1/2}}{dx^2}$$

$$\left(P^2 + \frac{dP}{dy}\right) \equiv \frac{1}{\sqrt{f}}\frac{d^2\sqrt{f}}{dx^2}$$

we get the two types of inhomogeneity functions admitting the exact solutions of (10) and (11)

Case (i)

$$f(y) = \left[A\cosh\left(\sqrt{b}\right) + \frac{B}{\sqrt{b}}\sinh\left(\sqrt{b}y\right)\right]^{-2} \quad (14)$$

or

Case (ii)

$$f(y) = \left[A\cosh\left(\sqrt{b}\right) + \frac{B}{\sqrt{b}}\sinh\left(\sqrt{b}y\right)\right]^{2} \quad (15)$$

$\alpha + \beta = \chi.$ (1) (1)

Depending on whether $b$ is positive, negative, or equal to zero several kind of functions can be found. Assumptions (14,15) are obviously somewhat artificial and narrow the class of functions characterizing the inhomogeneity. Nevertheless it allows solving the problem for functionally graded MEE materials and estimating the effect of inhomogeneity on the wave dispersion relations of waves. Let us note that these inhomogeneity functions profiles do not depend on material properties and can be used for both piezomagnetic, piezoelectric and pure elastic materials. For piezoelectric materials these inhomogeneity functions (14,15) were derived also in [18]. Some types of these functions have been used also in [17, 25, 26, 28] where shear wave prorogation in elastic and piezoelectric media was studied.

## III. Governing Equations and Solutions

The equations for the first type of inhomogeneity functions can be cast as:

Case (i):

$$\frac{d^2\tilde{\sigma}_{oyz}}{dy^2} - r^2\tilde{\sigma}_{oyz} + \left(1-p^2\right)\left[\frac{\eta_0}{\gamma_0}\left(\frac{d^2\tilde{D}_{oy}}{dy^2} - b\tilde{D}_{oy}\right) + \frac{\theta_0}{\gamma_0}\left(\frac{d^2\tilde{\mathrm{B}}_{oy}}{dy^2} - b\tilde{\mathrm{B}}_{oy}\right)\right] = 0$$

$$\frac{d^2\tilde{\mathrm{D}}_{oy}}{dy^2} - q^2\tilde{\mathrm{D}}_{oy} = 0; \frac{d^2\tilde{\mathrm{B}}_{oy}}{dy^2} - q^2\tilde{\mathrm{B}}_{oy} = 0 \quad (16)$$

$$r = \sqrt{b + p^2k^2}; q = \sqrt{b + k^2}$$

Now the solutions of this system of second order differential equations with constants coefficient can be easily found:

$$\tilde{D}_{oy}(y) = C_1\exp(qy) + \mathrm{A}_1\exp(-qy)$$

$$\tilde{B}_{oy}(y) = C_2\exp(qy) + \mathrm{A}_2\exp(-qy)$$

$$\tilde{\sigma}_0(y) = -\frac{\eta_0\left(C_1\exp(qy) + \mathrm{A}_1\exp(-qy)\right)}{\gamma_0} - \frac{\theta_0\left(C_2\exp(qy) + \mathrm{A}_2\exp(-qy)\right)}{\gamma_0} + C_3\exp(ry) + C_4\exp(-ry) \quad (17)$$

Solutions for $\tilde{W}_0(y), \varphi_0(y), \phi_0(y)$ follow from (6, 7)

$$\tilde{W}_0(y) = \frac{e^{ry} C_3 \left[ r + P(y) \right]}{E_0 k^2 p^2} + \frac{e^{-ry} A_3 \left[ -r + P(y) \right]}{E_0 k^2 p^2}$$

$$\varphi_0(y) =$$

$$= -\frac{e^{qy} E_0 p^2 (\mu_0 C_1 - \alpha_0 C_2)\left[ q + P(y) \right] - e^{ry} \eta_0 C_3 \left[ r + P(y) \right]}{E_0 k^2 p^2 \gamma_0} -$$

$$- \frac{e^{-qy} E_0 p^2 (\mu_0 A_1 - \alpha_0 A_2)\left[ -q + P(y) \right] - e^{-ry} \eta_0 A_3 \left[ -r + P(y) \right]}{E_0 k^2 p^2 \gamma_0} \quad (18)$$

$$\phi_0(y) =$$

$$= \frac{e^{qy} E_0 p^2 (\alpha_0 C_1 - \varepsilon_0 C_2)\left( q + P(y) \right) + e^{ry} \theta_0 C_3 \left( r + P(y) \right)}{E_0 k^2 p^2 \gamma_0} +$$

$$+ \frac{e^{-qy} E_0 p^2 (\alpha_0 A_1 - \varepsilon_0 A_2)\left( -q + P(y) \right) + e^{-ry} \theta_0 A_3 \left( -r + P(y) \right)}{E_0 k^2 p^2 \gamma_0}$$

Here $C_1, C_2, C_3, A_1, A_2, A_3$ are arbitrary constants.

The equations and solutions for the second type of inhomogeneity functions: Case (ii):

$$\frac{d^2 \tilde{W}_0}{dy^2} - r^2 \tilde{W}_0 = 0; \quad \frac{d^2 S}{dy^2} - q^2 S = 0$$
$$\frac{d^2 F}{dy^2} - q^2 F = 0 \quad (19)$$

$$\tilde{W}_0(y) = c_1 \exp(ry) + a_1 \exp(-ry)$$
$$S(y) = c_2 \exp(qy) + a_2 \exp(-qy) \quad (20)$$
$$F(y) = c_3 \exp(qy) + a_3 \exp(-qy)$$

Solutions for $\tilde{\sigma}_{oyz}(y), \tilde{D}_{oy}(y), \tilde{B}_{oy}(y), \varphi_0(y), \phi_0(y)$ follow from (6, 8)

$$\tilde{D}_{oy}(y) = c_2 e^{qy} \left[ q - P(y) \right] - a_2 e^{-qy} \left[ q + P(y) \right]$$
$$\tilde{B}_{oy}(y) = c_3 e^{qy} \left[ q - P(y) \right] - a_3 e^{-qy} \left[ q + P(y) \right]$$
$$\tilde{\sigma}_0(y) = -e^{qy} (c_1 \eta_0 + c_2 \theta_0) \gamma_0^{-1} \left[ q - P(y) \right] +$$
$$+ c_3 e^{ry} E_0 \left[ r - P(y) \right] +$$
$$+ e^{-qy} (a_1 \eta_0 + a_2 \theta_0) \gamma_0^{-1} \left[ q + P(y) \right] -$$
$$- a_3 e^{-ry} E_0 \left[ r + P(y) \right] \quad (21)$$
$$\phi_0(y) = \gamma_0^{-1} \left[ e^{ry} \theta_0 c_3 - e^{qy} (-\alpha_0 c_1 + \varepsilon_0 c_2) \right] +$$
$$+ \gamma_0^{-1} \left[ e^{-ry} \theta_0 a_3 - e^{-qy} (-\alpha_0 a_1 + \varepsilon_0 a_2) \right]$$
$$\varphi_0(y) = \gamma_0^{-1} \left[ e^{ry} \eta_0 c_3 - e^{qy} (-\alpha_0 c_2 + \mu_0 c_1) \right] +$$
$$+ \gamma_0^{-1} \left[ e^{-ry} \eta_0 a_3 - e^{-qy} (-\alpha_0 a_2 + \mu_0 a_1) \right]$$

Here $c_1, c_2, c_3, a_1, a_2, a_3$ are arbitrary constants.

All these solutions can be useful in tackling the problems of shear surface wave propagation over the semi-infinite space surface, interfacial wave in multilayer and periodic structures, Love type waves in a layer overlying a half-space, guided waves in waveguides , etc.

## IV. SHEAR LOCALIZED WAVES IN WAVEGUIDE

Here we shall limit ourselves by considering the localized wave propagation in waveguide $0 < y < h, -\infty < x < \infty$, when $\zeta < 1 \ (p > 0)$.

The quadratic and the inverse quadratic inhomogeneity profiles will be considered according to case $b = 0$ (14, 15), $a > 0$

Case (i): $f(y) = (1 + ay/h)^{-2}$

Case (ii): $f(y) = (1 + ay/h)^2$

The following magneto-electro-elastic contact conditions are within our interest:

Symmetric conditions

$$\sigma_{yz} = 0; \varphi = 0; \phi = 0; y = 0; y = h \quad (22)$$

$$\sigma_{yz} = 0; \phi = 0; D_y = 0; y = 0; y = h \quad (23)$$

$$\sigma_{yz} = 0; \varphi = 0; B_y = 0; y = 0; y = h \quad (24)$$

Asymmetric conditions

$$\begin{cases} W = 0; \varphi = 0; \phi = 0; y = 0 \\ \sigma_{yz} = 0; \varphi = 0; \phi = 0; y = h \end{cases} \quad (25)$$

Substituting the solutions (17, 18, 20, 21) into the boundary conditions (22-25), the homogeneous systems of equations with respect to the constants will be obtained. Equating the determinants of simultaneous sets of equations to zero we can obtain dispersion equations.

Introducing dimensionless parameters

$$K = \frac{K_\beta + K_e - 2\gamma \sqrt{K_e K_\beta}}{(1 - \gamma^2)}$$
$$K_e = \frac{e_0^2}{G_0 \varepsilon_0}; K_\beta = \frac{\beta_0^2}{G_0 \mu_0}$$
$$\gamma = \frac{\alpha_0}{\sqrt{\varepsilon_0 \mu_0}}; d = kh$$

the dispersion equations according to a symmetric boundary condition (22, 23) can be written as

Case (i):

$$\begin{aligned}&2d^2p^3(1+a)K(1+K)\left[1-\operatorname{sech}(d)\operatorname{sech}(dp)\right]+\\&+a^2dp\left[p^2-K\left(1-p^2\right)\right]\left[K\tanh(d)-(1+K)p\tanh(dp)\right]-\\&-\left[d^2p^2(1+a)\left(K^2+(1+K)^2p^2\right)-a^2\left(p^2-K\left(1-p^2\right)\right)^2\right]\cdot\\&\cdot\tanh(dp)\tanh(d)=0\end{aligned} \tag{26}$$

Case (ii):

$$\begin{aligned}&2d^2p(1+a)K(1+K)\left[1-\operatorname{sech}(d)\operatorname{sech}(dp)\right]-\\&-a^2d\left[(1+K)p\tanh(d)-K\tanh(dp)\right]-\\&-\left[d^2(1+a)\left((1+K)^2p^2+K^2\right)-a^2\right]\tanh(dp)\tanh(d)=0\end{aligned} \tag{27}$$

The dispersion equations according to the boundary condition (24) for Case (ii) have the form

$$\begin{aligned}&2(1+a)pd^2(1+K)\left(K-K_\beta\right)\left[\operatorname{sech}(d)\operatorname{sech}(dp)-1\right]+\\&+a^2d\left(1+K_\beta\right)\left[(1+K)p\tanh(d)-\left(K-K_\beta\right)\tanh(dp)\right]-\\&-\left[a^2\left(1+K_\beta\right)^2-d^2(1+a)\left((1+K)^2p^2+\left(K-K_\beta\right)^2\right)\right]\cdot\\&\cdot\tanh(dp)\tanh(d)=0\end{aligned} \tag{28}$$

The dispersion equation according to the boundary condition (24) can be obtained by replacing $K_\beta \to K_e$ in (28).

The dispersion equations according to an asymmetric boundary condition (25) can be written as

Case (i):

$$\begin{aligned}&a^2d^2p\left[-K+p^2(1+K)\right]+\\&+dp\left[\left(-a^2+d^2+ad^2\right)p^2(1+K)-a^3K\right]\tanh(d)+\\&+d\left[a^3p^2+\left(a^2+\left(a^3-(1+a)d^2\right)p^2\right)K\right]\tanh(dp)+\\&+a\left[\left(-a^2+(1+a)d^2\right)p^2-a^2\left(-1+p^2\right)K\right]\cdot\\&\cdot\tanh(dp)\tanh(d)=0\end{aligned} \tag{29}$$

Case (ii):

$$\begin{aligned}&(1+a)d\left[p(1+K)\tanh(d)-K\tanh(dp)\right]-\\&-a\tanh(dp)\tanh(d)=0\end{aligned} \tag{30}$$

## V. NUMERICAL ANALYSIS OF DISPERSION EQUATIONS AND DISCUSSION OF RESULTS

### A. *Symmetric boundary conditions*

Dispersion equations (26-30) impose a relationship between dimensionless phase speed of localized wave $\zeta < 1$ $\left(p = \sqrt{1-\zeta^2} > 0\right)$ and wave number $k$ $(d = kh)$.

All numerical calculations will be carried out for the MEE crystal $BaTiO_3$–$CoFe_2O_4$ with electro-magneto-elastic coupling coefficient $K = 0.612$ [16].

For fixed $h$ in the long wave approximation $d << 1$ from (26) and (27) it follows that $\zeta_1 = (1+K)^{-1/2} = 0.790$, in the short wave approximation $d >> 1$ we have $\zeta_2 = \sqrt{(1+2K)}/(1+K) = 0.927$.

In short wave approximation the localized wave speed $\zeta$ coincides with the speed of the Bleustein–Gulyaev surface wave for electrically shorted and traction free interface of MEE half-space [5]. Let us note that the inhomogeneity does not affect the phase speeds of both short and long waves and that for the pure elastic layer $(K_\beta = K_e = 0)$ all dispersion equations (26-30) have no solutions corresponding to localized wave.



For any value of the inhomogeneity parameter $a$, equation (26) has one root in the interval $d < d_0$ and two roots in the interval $d \geq d_0$ (see Tab.1). The corresponding dispersion curves $\zeta(d)$ diagrammatically shown on Fig.1.

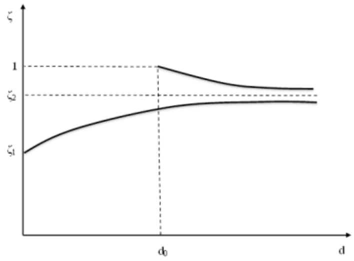


Fig.1 Structure of dispersion curves

| $a$ | Case(i) $d_0$ | Case(ii) $d_0$ |
|---|---|---|
| 0 | 5.27 | 5.27 |
| 0.5 | 5.35 | 5.93 |
| 1 | 5.47 | 7.22 |
| 2 | 568 | 9.62 |
| 5 | 6.12 | 14.22 |

Tabl.1

In Tabl.1 the data for the function $d_0(a)$ are presented for several values of inhomogeneity parameter. As it follows from the data of Tabl.1 the inhomogeneity in the case of inverse quadratic inhomogeneity profile sufficiently affects at the location structure of the roots and extends the interval where the one localised wave exist. In the case of quadratic inhomogeneity profile the effect of inhomogeneity factor is very weak.

### B. *Asymmetrical boundary conditions*

Contrary to the results of symmetrical boundary conditions, the results corresponding to asymmetrical boundary conditions are qualitatively different in Case (i) and Case (ii).

For short waves, equations (29, 30) have no solutions corresponding to localized waves and may have only one solution corresponding to localized waves for certain values of parameters $d, a$.

In Fig.2 and Fig.3 in the phase plane of parameters $d, a$ the curves of functions $d_0(a)$ are presented defining the regions where equations (29, 30) have no solutions corresponding to localized waves. In Fig. 2, 3 shaded regions correspond to the regions of parameters $d, a$ where the equations have no solutions (a localized wave does not exist). Outside of shaded regions the solutions corresponding to localized waves do exist for any values of $d, a$ including the points of curves $\mathrm{d}_0(a)$.

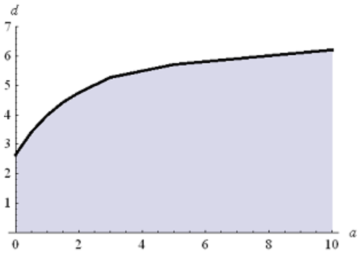


Fig.2 Localized wave existence region for Case (i )

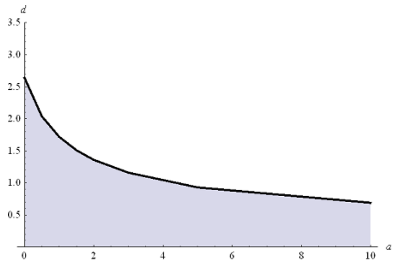


Fig.3 Localized wave existence region for Case (ii)

As it follows from the analysis of Fig.2 and Fig.3 the inhomogeneity factor plays an important role in the localized wave propagation behavior. Increasing the parameter $a$ in Case(i) leads to the elimination of localized shear wave for any value of $d \in (2.64, 6.31)$, while in Case(ii) results in the appearance of localized wave for any value of $d \in (0.79, 2.64)$.

## VI. Conclusions

Two classes of inhomogeneity functions are defined providing the exact solution of shear wave propagation in 6mm symmetry magneto-electro elastic functionally graded media. Solutions of the wave field are derived, which can be used in the problems of shear surface wave propagation over the semi-infinite space surface, interfacial wave in multilayer and periodic structures, Love type waves in a layer overlying a half-space, guided waves in waveguides, etc. The quadratic and inverse quadratic inhomogeneity profiles were considered in the several boundary problems of shear guided localized wave propagation in MEE waveguide. The dispersion equations are deduced analytically and by means of the numerical analysis for the BaTiO3–CoFe2O4 MEE crystal the effects of the inhomogeneity are discussed in detail.

## References

[1] J.van Suchtelen, Product properties: a new application of composite materials. Philips Res Rep, vol. 27, 1972, pp. 28–37.

[2] C.W. Nan, Magnetoelectric effect in composites of piezoelectric and piezomagnetic phases, Physical Review B, vol. 50, 1994, pp. 6082-6088.

[3] C.W. Nan, M.I. Bichurin, S.X. Dong, D. Viehland and G. Srinivasan, “Multiferroic magnetoelectric composites: historical perspective, status, and future directions”, Journal of Applied Physics, 2008., pp. 0131101/1-34,

[4] A.Srinivas, M.M. Raja, D. Sivaprahasam, P. Saravanan, Enhanced ferroelectricity and magnetoelectricity in 0.75BaTiO3-0.25BaFe12O19 by spark plasma sintering, Processing and Application of Ceramics, vol. 7, 2013, pp. 29–35.

[5] B.L. Wangab, Y.W. Maib, O.P. Niraulaa, “A horizontal shear surface wave in magnetoelectroelastic materials”, Philosophical Magazine Letters, vol. 87, 2007, pp. 53–58.

[6] A.A. Zakharenko, “On propagation problems of new surface wave in cubic pizeoelectomagnetics”, Open Journal of Acoustics, vol. 2, , 2012,pp. 104-114.

[7] J.K. Du, X.Y. Jin, J. Wang, Love wave propagation in layered magnetoelectro-elastic structures, Science in China Series G: Physics, Mechanics & Astronomy, vol. 51,2008 , pp. 617-631.

[8] W.J. Feng, E. Pan, X. Wang, J. Jin, Rayleigh waves in magneto-electro-elastic half planes, Acta Mech., 2008,vol. 202, pp. 127–134.

[9] A. Melkumyan, “Twelve shear surface waves guided by clamped/free boundaries in magneto-electro-elastic materials”, Journal of Solids and Structures,vol.44,2007,pp. 3594-3599.

[10] D. Piliposyan, Shear surface waves at the interface of two magneto-electro-elastic media, Multidisciplinе Modelling in Materials and Structures, vol. 8, 2012, pp. 417–426.

[11] H. Calas, J.A. Otero, R. Rodrı´guez-Ramos, G. Monsivais, C. Stern, “Dispersion relations for SH wave in magneto-electro-elastic heterostructures”, International Journal of Solids and Structures, vol. 45, 2008,pp. 5356–5367,.

[12] J. Chen, E. Pan, H. Chen, “Wave propagation in magneto-electro-elastic multilayered plates”, International Journal of Solids and Structures, vol. 44, pp. 1073–1085, February 2007.

[13] A. Alaimo, A. Milazzo, C. Orlando, Forced Vibration Analysis of Magneto-Electro-Elastic beam, XX Congresso Nazionale AIDAA – Milano, 2009, pp. 23-27.

[14] A Milazzo, C. Orlando, A. Alaimo, On the shear influence on the free bending vibration behavior of magneto-electro-elastic beam, International Conference on Computational & Experimental Engineering and Sciences, January 2009, pp. 55-62.

[15] P. Kondaiah, K. Shankar, N. Ganesan, Pyroelectric and pyromagnetic effects on behavior of magneto-electro-elastic plate, Coupled Systems Mechanics, vol. 2, 2013, pp. 1-22.

[16] M.F. Liu, “An exact deformation analysis for the magneto-electro-elastic fiber-reinforced thin plate”, Applied Mathematical Modelling, vol. 35, 2011, pp. 2443–2461,.

[17] E. Pan, “Exact Solution for Simply Supported and Multilayered Magneto-Electro-Elastic Plates”, Journal of Applied Mechanics, vol. 68, 2001, pp. 608-618.

[18] B. Collet, M. Destrade, G.A. Maugin, “Bleustein–Gulyaev waves in some functionally graded materials”, European Journal of Mechanics A/Solids, vol. 25, pp. 695–706, February 2006.

[19] V. Birman, L.W. Byrd, “Modeling and Analysis of Functionally Graded Materials and Structures”, Applied Mechanics Reviews, vol. 60, 2007, pp. 195-215.

[20] R.M. Mahamood, E.T. Akinlabi, M. Shukla, S. Pityana, Functionally Graded Material: An Overview,Proceedings of the World Congress on Engineering, London, UK, 2012, vol. 3, pp. 35-41.

[21] S.N. Bhattacharya, “Exact solution of SH wave equation in transversely isotropic inhomogeneous elastic media”, Pure and Appl. Geophysics, vol. 93, 1972, pp. 19-35.

[22] G.A. Maugin, Elastic surface waves with transverse horizontal polarization, Adv. Appl. Mech., vol. 23, 1983, pp. 373–434.

[23] M.B. Белубекян, “Об условиях существования волн Лява с неоднородным слоем”, Изв. АН Армении. Механика, Т. 44, №3, 1991 с. 7-10, (In Russian).

[24] M.B. Белубекян, А.Р. Мухсихачоян, “Сдвиговая поверхностная волна в слабонеоднородных упругих телах”, Акуст. Ж., Т.42, №2, 1996, с.179-182, (In Russian).

[25] З.Н. Даноян, Л.А. Атоян, С.Л. Саакян, Н.З. Даноян, “Волны типа Лява в слоистой структуре, состоящей из двух однородных полупространств с неоднородными слоями, соединённых однородным слоем”, Известия НАН Армении, Механика, Т.67, №3, 2014,с. 49-55.

[26] K.B. Ghazaryan, D.G. Piliposyan, “Love Waves in a Structure with an Inhomogeneous layer”, NAS RA, vol. 111, , 2011, pp. 138-147.

[27] А.С. Аветисян, А.А. Камалян, “О распространении электроупругого сигнала в неоднородном пьезодиэлектрическом слое класса 6мм”, Докл.НАН Армении, Т. 114, №2, 2014, с. 108-115, (In Russian.)

[28] А.А. Камалян, “О распространении и поведении электроупругой волны в пьезодиэлектрике в зависомости от неоднородности”, Изв. НАН Армении, Механика, Т.106, №4, 2014, с. 38-48, (In Russian)

[29] А.С. Аветисян, “Поверхностные электроупругие волны Лява в случае неоднородного пьезоэлектрического слоя”, Изв. АН Армении, Механика, Т.40, № 1, 1987 с. 24-29, (In Russian).

[30] X. Cao, J. Shi, F. Jin, “Lamb wave propagation in the functionally graded piezoelectric–piezomagnetic material plate”, Acta Mechanica, vol. 223, 2012, pp. 1081-1091.

[31] P. Lia, Feng Jina, Z. Qianb, “Propagation of the Bleustein–Gulyaev waves in a functionally graded transversely isotropic electro-magneto-elastic half-space”, European Journal of Mechanics - A/Solids, vol. 37, 2013,pp. 17–23.

[32] B.A. Auld, Acoustic Fields and Waves in Solids, vol. 1, 1973, p. 431.

[33] M.B. Белубекян, “Экранированная поверхностная сдвиговая волна в пьезоактивном полупространстве гексогональной симметрии. Сб: Проблемы динамики взаимодействия деформируемых сред”, Ереван, Институт механики НАН РА, с.125-130, 2008. (In Russian)

[34] D.G. Piliposyan, K.B. Ghazaryan, G.T. Piliposian, “Internal resonances in a periodic magneto-electro-elastic structure”, Journal of Applied Physics, vol. 116, 2014, pp. 044107/1-7.

[35] Д.К. Гаспарян, К.Б. Казарян, “Связанные волны в электро-магнито-упругой пьезоактивной среде”, Доклады НАН Армении, Т.214, №1, 2014.,с. 25-32, (In Russian).