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Sharing of Borel Exceptional Values between Meromorphic Functions and Differential Polynomial involving Shift Function

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Abstract - In this paper we establish a first result for a transcendental meromorphic function of finite order sharing two Borel exceptional values under two cases. In the first case f and differential polynomial $\psi(f)$ share a non-zero complex number and ∞ as Borel exceptional values. In the second case they share 0 and ∞ as Borel exceptional values. We also prove a second result in which $(L(f^n))^{(k)}$ and $(L(g^n))^{(k)}$ share the value 1 counting multiplicities (CM), while $L(f^{(n)})$ and $L(g^{(n)})$ share ∞ ignoring multiplicities (IM).

Keywords: Uniqueness, Meromorphic function, linear, differential polynomial, shift function.

Mathematics Subject Classification (2020): 30D20, 30D30 30D35.

1 INTRODUCTION

Definition 1. Let us define a differential polynomial involving shift function

$$\psi(f) = \sum_{i=1}^{n} A_i(z) f^{(k_i)}(z) + \sum_{i=1}^{n} B_i(z) f^{(k_i)}(z + b_i), \tag{1.1}$$

where $A_i(z),\ B_i(z)$ are small functions of $f(z),\ k_i > 0 \in \mathbb{Z}^+$, b_i is a complex constant.

In 2013, Chen [3] proved the relationships between Picard values of entire functions $f(z)$ and their forward differences $\Delta^n f(z)$.

Theorem 1.1. [3] Let f be a transcendental entire function of finite order, let $c(\neq 0)$ be a constant, and let n be a positive integer. If $f \neq 0,\ \Delta_c^n f \neq 0$, then $f(z) = e^{az+b}$, where $a(\neq 0), b$ are constants.

In 2016, Chen et al., [2] proved difference analogue to theorem 1.1.

Theorem 1.2. [2] Let $a(\neq \infty),\ b$ be two distinct complex numbers (b may be ∞), let f be a transcendental meromorphic function of finite order with two Borel exceptional values a, b and c be a non zero constant such that $\Delta_c f \neq 0$. If f and $\Delta_c f$ share a, b CM, then $a = 0, b = \infty$ and $f(z) = e^{Az+B}$, where $A(\neq 0), B$ are constants.

In 2021, M. Fang and Y. Wang [7] worked for higher order difference operators.

Theorem 1.3. [7] Let $a(\neq \infty), b$ be two distinct complex numbers and $n \in \mathbb{Z}^+$, let f be a transcendental meromorphic function of finite order with two Borel exceptional values a, b and c is a non-zero constant such that $\Delta_c^n f \neq 0$. If f and $\Delta_c^n f$ share a, b CM, then $a = 0$,

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$b = \infty$ and $f(z) = e^{Az+B}$, where $A(\neq 0), B$ are constants.

In the year 1998, W. Yuefei and F. Mingliang[14] proved the criteria for normality of families of meromorphic functions.

Theorem 1.4. [14] Let $f(z)$ be a transcendental entire function, $n, k \in \mathbb{N}$ with $n \geq k + 1$. Then $(f^n)^{(k)} = 1$ has infinitely many solutions.

In 2002, M-L Fang [8] obtained the below result corresponding to unicity theorem.

Theorem 1.5. [8] Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k\, (c_1 c_2)^n\, (nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

J. Fan et al., [6] extended theorem 1.5 to prove the following.

Theorem 1.6. [6] Let f and g be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 8 - \Theta_{min}(k + 4)$, if $\Theta_{min} \geq \frac{2}{k+4}$, otherwise $n > 3k + 6$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k\, (c_1 c_2)^n\, (nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

2 LEMMAS

Lemma 2.1. [9, 4] Let f be a nonconstant meromorphic function of finite order, let c be a nonzero finite complex number. Then

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f)$$

and for any $\epsilon > 0$, we have

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\rho(f)+\epsilon-1}).$$

Lemma 2.2. [12, 7] Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ are meromorphic functions satisfying the following identity

$$\sum_{j=1}^{n} f_j(z) = 1.$$

If $f_n(z) \neq 0$ and

$$\sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + (n-1) \sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + O(1))T(r, f_k), \tag{2.1}$$

where I is a set of $r \in (0, \infty)$ with infinite linear measure, $r \in I, k = 1, 2, \cdots, n-1,\ \lambda < 1$, then $f_n \equiv 1$.

Lemma 2.3. [5] Let f be a meromorphic function of order $\rho(f) = \rho < 1$. Then for each given $\epsilon > 0$, and a positive integer n, there exists a set $E \subset (1, \infty)$ that depends on f, and it has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, we have

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$$\left|\frac{f\prime(z)}{f(z)}\right| \leq |z|^{\rho-1+\epsilon}.$$

Lemma 2.4. [7] Let α be a meromorphic function, let k be a positive integer, and let c be a nonzero finite complex number. If $\Delta_c^k \alpha \equiv 0$, then either $\rho(\alpha) \geq 1$ or α is a polynomial with $deg(\alpha) \leq k-1$.

Lemma 2.5. [12, 6] Let f be a meromorphic function such that $f(k) \not\equiv 0$, and let k be a positive integer. Then

$$T(r, f^{(k)}) \leq T(r, f) + k\bar{N}(r, f) + S(r, f)$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$

Lemma 2.6. [12, 11, 6] Let f be a nonconstant meromorphic function, let k be a positive integer, and let c be a nonzero finite complex number. Then

$$T(r, f) \leq \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}-c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f)$$

$$\leq \bar{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f)$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)}(z) = 0$, but $f(z)(f^{(k)}(z) - c) \neq 0$.

Lemma 2.7. [10, 12, 11, 6] If f is a meromorphic function, $k \in N$. And then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 2.8. [10, 1] Let f(z) be a meromorphic function and a be a finite complex number. Then

$(i)\ T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1)$

$(ii)\ m\left(r, \frac{f^{(k)}}{f^{(l)}}\right) = S(r, f),$ for $k > l \geq 0$

$(iii)\ T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f-a_1(z)}\right) + \bar{N}\left(r, \frac{1}{f-a_2(z)}\right) + S(r, f)$

where $a_1(z),\ a_2(z)$ are two meromorphic functions such that $T(ra_i) = S(r, f),\ (i = 1,2)$.

Lemma 2.9. [6] Let f be a nonconstant entire function, and let $k(\geq 2)$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a(\neq 0), b$ are two constants.

Lemma 2.10. [13, 6] Let f and g be two nonconstant entire functions, and let $n(\geq 1)$ be a positive integer. If $f^n f' g^n g' \equiv 1$, then $f(z) = c_1 e^{cz},\ g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.

3 MAIN RESULTS

Theorem 3.1. Let $a_1(\neq \infty),\ a_2$ be two distinct complex numbers and $n \in \mathbb{Z}^+$, let f be a transcendental meromorphic function of finite order with two Borel exceptional values a_1,

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a_2 and c is a non-zero constant such that $\psi(f) \not\equiv 0$. If f and $\psi(f)$ share a_1, a_2 CM, then $a_1 = 0,\ a_2 = \infty$.

Proof. Case 1. a_1 is a nonzero finite complex number, $a_2 = \infty$. Since a_1, ∞ are two distinct Borel exceptional values of f and f is of finite order, by Hadamard's factorization theorem, we have

$$f(z) = a_1 + \alpha(z)e^{p(z)}, \tag{3.1}$$

where $\alpha(\neq 0, \infty)$ is a meromorphic function such that $\rho(\alpha) < \rho(f)$ and p is a non constant polynomial with $deg(p) = \rho(f)$. Hence we have

$$T(r,\alpha) = S(r, e^p), \qquad T(r,f) = T(r,e^p) + S(r,f). \tag{3.2}$$

Thus, we have

$$f'(z) = \alpha e^{p(z)}p'(z) + \alpha'(z)e^{p(z)}$$
$$= e^{p(z)}[\alpha(z)p'(z) + \alpha'(z)]$$
$$= e^{p(z)}T_1(z),$$

where $T_1(z) = \alpha(z)p'(z) + \alpha'(z)$.

$$f''(z) = e^{p(z)}T'_1(z) + T_1(z)e^{p(z)}p'(z)$$
$$= e^{p(z)}[T_1(z)p'(z) + T'_1(z)]$$
$$= e^{p(z)}T_2(z),$$

where $T_2(z) = T_1(z)p'(z) + T'_1(z)$.

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$$f^{(k)}(z) = e^{p(z)}T_k(z),$$

where $T_k(z) = T_{k-1}(z)p'(z) + T'_{k-1}(z)$.

And

$f^{(k)}(z + b_i) = e^{p(z+b_i)}T_k(z + b_i)$,

where $T_k(z + b_i) = T_{k-1}(z + b_i)p'(z + b_i) + T'_{k-1}(z + b_i)$.

Hence (3.1) becomes

$$\psi(f) = \sum_{i=1}^{n} A_i(z)T_{k_i}(z)e^{p(z)} + \sum_{i=1}^{n} B_i(z)T_{k_i}(z + b_i)e^{p(z+b_i)},$$
$$= \left[\sum_{i=1}^{n} A_i(z)T_{k_i}(z) + \sum_{i=1}^{n} B_i(z)T_{k_i}(z + b_i)e^{p(z+b_i)-p(z)}\right]e^{p(z)}$$

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$$= H(z)e^{p(z)}, \tag{3.3}$$

where $H(z) = \sum_{i=1}^{n} A_i(z)T_{k_i}(z) + \sum_{i=1}^{n} B_i(z)T_{k_i}(z + b_i)e^{p(z+b_i)-p(z)}$. Since $\psi(z) \not\equiv 0$, it follows that $H(z) \not\equiv 0$. Thus $H(\not\equiv 0)$ is a meromorphic function with $\rho(H) < \rho(e^p)$.

Hence H is a small function of e^p. By second fundamental theorem and (3.3) we have

$$\begin{aligned} T(r, e^p) &\le T(r, He^p) + T\left(r, \frac{1}{H}\right) + O(1) \\ &\le T(r, He^p) + S(r, e^p) \\ &\le N(r, He^p) + N\left(r, \frac{1}{He^p}\right) + N\left(r, \frac{1}{He^p - a_1}\right) + S(r, e^p) \\ &\le N\left(r, \frac{1}{He^p - a_1}\right) + S(r, e^p) \\ &= N\left(r, \frac{1}{\psi(f) - a_1}\right) + S(r, e^p). \end{aligned} \tag{3.4}$$

Since f and $\psi(f)$ share a_1CM, it follows that

$$N\left(r, \frac{1}{f - a_1}\right) \ge T(r, e^p) + S(r, e^p). \tag{3.5}$$

Thus, we deduce from (3.2) and (3.5) that $\lambda(f - a_1) = \rho(f)$, this contradicts that a_1 is a Borel exceptional value of f. Hence this is absurd.

Case 2. $a_1 = 0,\ a_2 = \infty$. Since $0, \infty$ are two distinct Borel exceptional values of f and f is of finite order, by Hadamard's factorization theorem we have

$$f(z) = \alpha(z)e^{p(z)}, \tag{3.6}$$

where $\alpha(\not\equiv 0, \infty)$ is a meromorphic function such that $\rho(\alpha) < \rho(f)$ and p is a non constant polynomial with $deg(p) = \rho(f) \ge 1$. Hence we have

$$T(r, \alpha) = S(r, e^p), \qquad T(r, f) = T(r, e^p) + S(r, f). \tag{3.7}$$

Thus, we have

$$\psi(f) = H(z)e^{p(z)},$$

where $H(z) = \sum_{i=1}^{n} A_i(z)T_{k_i}(z) + \sum_{i=1}^{n} B_i(z)T_{k_i}(z + b_i)e^{p(z+b_i)-p(z)}$. (3.8)

Since f and $\psi(f)$ share $0, \infty$ CM, there exists a polynomial q satisfying

$$\sum_{i=1}^{n} \frac{A_i(z)T_{k_i}(z)}{\alpha(z)} + \sum_{i=1}^{n} \frac{B_i(z)T_{k_i}(z+b_i)}{\alpha(z)} e^{p(z+b_i)-p(z)} = e^{q(z)}. \tag{3.9}$$

It follows from (3.9) and Lemma (2.1) that

$$\rho\left(\frac{B_i(z)T_{k_i}(z+b_i)}{\alpha(z)}\right) < deg(p) - 1, \quad deg(q) \le deg(p) - 1. \tag{3.10}$$

We consider two subcases.

Case 2.1. $deg(p) \ge 2$. Here again we have two subcases.

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Case 2.1.1 $1 \le deg(q) \le deg(p) - 1$.

Thus by (3.9) we obtain

$$\frac{A_1(z)T_{k_1}(z)}{\alpha(z)} + \frac{B_1(z)T_{k_1}(z+b_1)}{\alpha(z)}e^{p(z+b_1)-p(z)} + \frac{A_2(z)T_{k_2}(z)}{\alpha(z)}$$

$$+\frac{B_2(z)T_{k_i}(z+b_2)}{\alpha(z)}e^{p(z+b_2)-p(z)} +\cdots+ \frac{A_n(z)T_{k_n}(z)}{\alpha(z)}$$

$$+\frac{B_n(z)T_{k_n}(z+b_n)}{\alpha(z)}e^{p(z+b_n)-p(z)} = e^{q(z)}.$$

$$\frac{1}{\alpha(z)}[A_1(z)T_{k_1}(z) + B_1(z)T_{k_1}(z+b_1)e^{p(z+b_1)-p(z)} + A_2(z)T_{k_2}(z)$$

$$+ B_2(z)T_{k_2}(z+b_2)e^{p(z+b_2)-p(z)} + \cdots + A_1(z)T_{k_1}(z)$$

$$+ B_n(z)T_{k_n}(z+b_n)e^{p(z+b_n)-p(z)}] - e^{q(z)} = 1. \qquad (3.11)$$

Set

$$f_i(z) = \frac{A_i(z)T_{k_i}(z)}{\alpha(z)} + \frac{B_i(z)T_{k_i}(z+b_i)}{\alpha(z)}e^{p(z+b_i)-p(z)} = e^{q(z)}, \qquad (3.12)$$

$i = 1,2,\ldots,n.$ $\qquad f_{n+1} = 1 - e^{q(z)}. \qquad (3.13)$

Then by (3.11) we have

$$f_1(z) + f_2(z) + \cdots + f_n(z) + f_{n+1}(z) \equiv 1. \qquad (3.14)$$

If n = 1, then by (3.12) - (3.14) we obtain

$$T\left(r, e^{p(z+b_1)-p(z)}\right) \le T\left(r, \frac{B_1(z)T_{k_1}(z+b_1)}{\alpha(z)}e^{p(z+b_1)-p(z)}\right) + T\left(r, \frac{\alpha(z)}{B_1(z)T_{k_1}(z+b_1)})\right)$$

$$\le N\left(r, \frac{B_1(z)T_{k_1}(z+b_1)}{\alpha(z)}e^{p(z+b_1)-p(z)}\right) + N\left(r, \frac{1}{\frac{B_1(z)T_{k_1}(z+b_1)}{\alpha(z)}e^{p(z+b_1)-p(z)}}\right)$$

$$+ N\left(r, \frac{1}{\frac{B_1(z)T_{k_1}(z+b_1)}{\alpha(z)}e^{p(z+b_1)-p(z)} - 1}\right) + S(r, e^{p(z+b_1)-p(z)})$$

$$\le N\left(r, \frac{1}{\frac{B_1(z)T_{k_1}(z+b_1)}{\alpha(z)}e^{p(z+b_1)-p(z)} - 1}\right) + S(r, e^{p(z+b_1)-p(z)})$$

$$\le N\left(r, \frac{1}{e^{q(z)}}\right) + S(r, e^{p(z+b_1)-p(z)})$$

$$\le S(r, e^{p(z+b_1)-p(z)}),$$

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a contradiction.

If $n \geq 2$ then by (3.12) - (3.14) we know that $f_1, f_2, \cdots, f_n$ are nonconstant. $f_{n+1} \not\equiv 0$ and (2.1) is valid, thus by Lemma (2.2) we obtain that $f_{n+1} \equiv 1$ a contradiction.

Case 2.1.2. $deg(q) = 0$. If $1 - e^{q(z)} \neq 0$, then by using the same argument as used in case 2.1.1 we obtain a contradiction.

If $1 - e^{q(z)} = 0$, then by (3.11) we have

$$\frac{A_1(z)T_{k_1}(z)}{\alpha(z)} + \frac{B_1(z)T_{k_1}(z+b_1)}{\alpha(z)} e^{p(z+b_1)-p(z)} + \frac{A_2(z)T_{k_2}(z)}{\alpha(z)}$$

$$+ \frac{B_2(z)T_{k_i}(z+b_2)}{\alpha(z)} e^{p(z+b_2)-p(z)} + \cdots + \frac{A_n(z)T_{k_n}(z)}{\alpha(z)}$$

$$+ \frac{B_n(z)T_{k_n}(z+b_n)}{\alpha(z)} e^{p(z+b_n)-p(z)} \equiv 1.$$

By second fundamental theorem and Lemma (2.2) and using the same argument as used in case 2.1.1 we obtain a contradiction.

Case 2.2. $deg(p) = 1$. Thus by (3.6) we have

$$f(z) = \alpha(z)e^{A(z)}, \tag{3.15}$$

where $\alpha(\not\equiv 0, \infty)$ is a meromorphic function such that $\rho(\alpha) < 1$ and A is a nonzero constant. By (3.9) and (3.15) we obtain

$$\sum_{i=1}^{n} \frac{A_i(z)\alpha_{k_i}(z)}{\alpha(z)} + \sum_{i=1}^{n} \frac{B_i(z)\alpha_{k_i}(z+b_i)}{\alpha(z)} e^{Ab_i} = B \tag{3.16}$$

where A, B are two nonzero constants. We now write equation (3.16) in the form

$$(e^{Ab_i})^n \frac{\psi(\alpha_{k_n}(z))}{\alpha(z)} + B_{n-1}\frac{\psi(\alpha_{k_{n-1}}(z))}{\alpha(z)} + \cdots + B_1\frac{\psi(\alpha_{k_1}(z))}{\alpha(z)} = B \tag{3.17}$$

where $B_n = (e^{Ab_i})^n, B_{n-1}, \cdots, B_1$ are constants.

We choose ϵ such that $0 < \epsilon < 1 - \rho(\alpha)$. Lemma (2.3) asserts that there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure, such that for all $|z| = r \notin E \cup [0,1]$

$$\frac{\psi(\alpha_{k_n}(z))}{\alpha(z)} = o(1), \quad \text{for } 1 \leq j \leq n. \tag{3.18}$$

Let $|z| = r \notin E \cup [0,1]$ and $|z| \to \infty$, then it follows from (3.17) and (3.18) that $B = 0$.

Thus, we have

$$(e^{Ab_i})^n\psi(\alpha_{k_n}(z)) + B_{n-1}\psi(\alpha_{k_{n-1}}(z)) + \cdots + B_1\psi(\alpha_{k_1}(z)) = 0. \tag{3.19}$$

If $\psi(\alpha_{k_1}(z)) = 0$ then by Lemma (2.4) we know that α is a nonzero constant and $f(z) = e^{Az+B}$. If $\psi(\alpha_{k_1}(z)) \neq 0$ then it follows from $\rho(\psi(\alpha_{k_1}(z))) \leq \rho(\alpha) < 1$, (3.19) and Lemma (2.3) that $B_1 = 0$. Now suppose that $B_l \neq 0, B_{l-1} = 0, \cdots, B_1 = 0,\ 2 \leq l \leq n$. Thus, we have

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$$(e^{Ab_i})^n\psi(\alpha_{k_n}(z)) + B_{n-1}\psi(\alpha_{k_{n-1}}(z)) + \cdots + B_l\psi(\alpha_{k_l}(z)) = 0, \; B_l \neq 0. \tag{3.20}$$

We claim that $\psi(\alpha_{k_1}(z)) = 0$. Otherwise, we have

$$(e^{Ab_i})^n \frac{\psi(\alpha_{k_n}(z))}{\psi(\alpha_{k_l}(z))} + B_{n-1}\frac{\psi(\alpha_{k_{n-1}}(z))}{\psi(\alpha_{k_l}(z))} + \cdots + B_1\frac{\psi(\alpha_{k_{l+1}}(z))}{\psi(\alpha_{k_l}(z))} = -B_l\,. \tag{3.21}$$

By Lemma (2.3) and (3.21) we deduce that $B_l = 0$ a contradiction. Thus, we prove that $\psi(\alpha_{k_1}(z)) = 0$. Hence, we have from (3.17) that

$$\sum_{i=1}^{l} B_{1,l-i}\frac{\psi(\alpha_{k_{l-i}})(z)}{\alpha(z)} = B, \tag{3.22}$$

where $B_{1,l-1}, \cdots, B_{1,1}, B_{1,0}, B$ are constants and $B \neq 0$, $\psi(\alpha_{k_0}(z)) = \alpha$.

Now using the same argument as for proving $\psi(\alpha_{k_l}(z)) = 0$ we obtain that $\psi(\alpha_{k_{l_1}}(z)) = 0$, $1 \leq l_1 \leq l-1$.

By taking $l_1 = 1$ we have $\psi(\alpha_{k_1}(z)) = 0$ and by Lemma (2.4) we deduce that $\alpha_k(z)$ is a nonzero constant. Hence the theorem is proved. ■

Theorem 3.2. Let f and g be two meromorphic functions and n, k be two positive integers with $n > 3k + 8 - \Theta_{min}(k+4)$, if $\Theta_{min} \geq \frac{2}{k+4}$, otherwise $n > 3k + 6$. If $(L(f^n))^{(k)}$ and $(L(g^n))^{(k)}$ share 1 CM; $L(f^n)$ and $L(g^n)$ share ∞ IM and

$$H = [(k+2)\theta(\infty, f) + 2\theta(\infty, g) + \theta(0, f) + \theta(0, g) + n\delta_{k+1}(0, f) + n\delta_{k+1}(0, g)] > n + k + 6 \tag{3.23}$$

then either $L(f^n) = c_1e^{cz}$, $L(g^n) = c_2e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k\,(c_1c_2)^n\,(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

Proof. Set $F = [L(f^n)]^{(k)}$, $G = [L(g^n)]^{(k)}$.

Since $[L(f^n)]^{(k)}$ and $[L(g^n)]^{(k)}$ share 1 C M then F and G share 1 C M. By Lemma (2.5) we obtain

$$\begin{aligned} T(r, F) &= T\big(r, [L(f^n)]^{(k)}\big) \\ &\leq T(r, L(f^n)) + k\overline{N}(r, f) + S(r, f) \\ &\leq (n+k)T(r, f) + S(r, f). \end{aligned}$$

It follows $S(r, F) = S(r, f)$. Similarly, we get $S(r, G) = S(r, g)$.

Set $$\zeta = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{3.24}$$

Next we consider two cases.

Case 1. $\zeta = 0$ then by (3.24)

$$\frac{F-1}{F} = c\,\frac{G-1}{G} \tag{3.25}$$

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where c is a finite complex constant.

In the following we consider two subcases.

Case 1.1 $c = 1$. It follows from (3.25) that $F = G$, that is $[L(f^n)]^{(k)} = [L(g^n)]^{(k)}$. Which implies $L(f^n) = L(g^n) + P$, where P is a polynomial with $degree(P) \leq k - 1$.

If $P \not\equiv 0$, then we have

$$\frac{L(f^n)}{P} - \frac{L(g^n)}{P} = 1. \tag{3.26}$$

Since f and g are two nonconstant meromorphic functions, then

$$T(r,f) \geq logr + O(1), \quad T(r,g) \geq logr + O(1). \tag{3.27}$$

By second fundamental theorem and (3.27) we obtain

$$\begin{aligned} T\left(r,\frac{L(f^n)}{P}\right) &\leq T(r,L(f^n)) + T(r,P) + O(1) \\ &\leq nT(r,L(f)) + (k-1)logr + O(1) \\ &\leq (n+k-1)T(r,L(f)) + O(1). \end{aligned}$$

Hence, we get

$$S\left(r,\frac{L(f^n)}{P}\right) = S(r,f). \tag{3.28}$$

By $n > 2k + 4$, Nevanlinna second fundamental theorem and (3.26) - (3.28) we have

$$\begin{aligned} nT(r,L(f)) = T(r,L(f^n)) &\leq T\left(r,\frac{L(f^n)}{P}\right) + T(r,P) \\ &\leq \bar{N}\left(r,\frac{L(f^n)}{P}\right) + \bar{N}\left(r,\frac{P}{L(f^n)}\right) + \bar{N}\left(r,\frac{1}{\frac{L(f^n)}{P}-1}\right) + (k-1)logr + O(1) \\ &\leq \bar{N}(r,L(f)) + \bar{N}\left(r,\frac{1}{L(f)}\right) + \bar{N}\left(r,\frac{1}{L(g)}\right) + 2(k-1)logr + S(r,f) \\ &\leq 2kT(r,L(f)) + \bar{N}\left(r,\frac{1}{L(g)}\right) + S(r,f). \end{aligned} \tag{3.29}$$

Which implies

$$(n-2k)T(r,L(f)) \leq \bar{N}\left(r,\frac{1}{L(g)}\right) + S(r,f). \tag{3.30}$$

Similarly

$$(n-2k)T(r,L(g)) \leq \bar{N}\left(r,\frac{1}{L(f)}\right) + S(r,g). \tag{3.31}$$

By either $n > 3k + 6$ or $n > 3k + 8 - \Theta_{min}(k+4) \geq 2k + 4$ we get

$$T(r,L(f)) + T(r,L(g)) \leq S(r,f) + S(r,g)$$

a contradiction.

Hence $P \equiv 0$. It follows $f = tg$ where t is a constant such that $t^n = 1$.

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Case 1.2 $c \neq 1$. Then by (3.25) we obtain

$$\frac{1}{F} - \frac{c}{G} = 1 - c. \tag{3.32}$$

Since f and g share ∞ IM, it follows from (3.32) that $F \neq \infty$ and $G \neq \infty$. Hence $\frac{1}{F} \neq 0$ and then by (3.32) we deduce that $G \neq \frac{c}{c-1}$. By Lemma (2.6) we obtain

$$nT(r, L(g)) = T(r, L(g^n))$$

$$\leq \overline{N}(r, L(g^n)) + N\left(r, \frac{1}{L(g^n)}\right) + N\left(r, \frac{1}{(L(g^n))^{(k)} - \frac{c}{c-1}}\right)$$

$$-N\left(r, \frac{1}{(L(g^n))^{(k+1)}}\right) + S(r, g)$$

$$\leq (k+1)\overline{N}\left(r, \frac{1}{L(g)}\right) + S(r, g).$$

It follows from either $n > 3k + 6$ or $n > 3k + 8 - \Theta_{min}(k+4) \geq 2k + 4$ that

$T(r, g) \leq S(r, g)$ a contradiction.

Case 2. $\zeta \not\equiv 0$. Let z_0 be a pole of $[L(f^n)]$ with multiplicity l_1. Then by $[L(f^n)]$and $[L(g^n)]$share ∞ IM we know that z_0 is a pole of $[L(g^n)]$ with multiplicity l_2.

Set $l = min\{l_1, l_2\}$ by (3.24) we deduce that z_0 is a zero of ζ with multiplicity $\geq nl + k - 1$. Hence by Lemma (2.7) we have

$$\overline{N}(r, L(f^n)) = \overline{N}(r, L(g^n)) \leq \frac{1}{n+k-1} N\left(r, \frac{1}{\zeta}\right)$$

$$\leq \frac{1}{n+k-1} T(r, \zeta) + O(1)$$

$$\leq \frac{1}{n+k-1} m(r, \zeta) + \frac{1}{n+k-1} N(r, \zeta) + O(1)$$

$$\leq \frac{1}{n+k-1}\left[\overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right)\right] + S(r, f) + S(r, g). \tag{3.33}$$

It follows from Lemma (2.5) that

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{(L(f^n))^{(k)}}\right)$$

$$= N\left(r, \frac{1}{(L(f^n))^{(k)}}\right) - \left[N\left(r, \frac{1}{(L(f^n))^{(k)}}\right) - \overline{N}\left(r, \frac{1}{(L(f^n))^{(k)}}\right)\right]$$

$$\leq N\left(r, \frac{1}{L(f^n)}\right) + k\overline{N}(r, L(f)) - \left[N\left(r, \frac{1}{(L(f^n))^{(k)}}\right) - \overline{N}\left(r, \frac{1}{(L(f^n))^{(k)}}\right)\right] + \mathrm{S(r,f)}$$

$$\leq (k+1)\overline{N}\left(r, \frac{1}{L(f^n)}\right) + k\overline{N}(r, L(f^n)) + S(r, f)$$

$$\leq (2k+1)T(r, f) + S(r, f). \tag{3.34}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{G}\right) \leq (2k+1)T(r, g) + S(r, f). \tag{3.35}$$

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By (3.33) - (3.35) we get

$$\overline{N}(r, L(f^n)) = \overline{N}(r, L(g^n)) \leq \frac{2k+1}{n+k-1}[T(r,f) + T(r,g)] + S(r,f) + S(r,g). \quad (3.36)$$

Set $$\zeta_1 = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}. \quad (3.37)$$

Suppose $\zeta_1 \not\equiv 0$. Let z_0 be a common simple zero of $F(z) - 1$ and $G(z) - 1$, by a simple computation we see that $\zeta_1(z_0) = 0$. Thus by first fundamental theorem and Lemma (2.7) we have

$$\begin{aligned} N_{1)}\left(r, \frac{1}{F-1}\right) &= N_{1)}\left(r, \frac{1}{G-1}\right) \\ &\leq N\left(r, \frac{1}{\zeta_1}\right) \leq T(r, \zeta_1) + O(1) \\ &\leq N(r, \zeta_1) + S(r,F) + S(r,G), \end{aligned} \quad (3.38)$$

where $N_{1)}\left(r, \frac{1}{F-1}\right)$ is the counting function of simple zeros of $F(z) - 1$. It follows from F and G share 1 CM and (3.37) that

$$\begin{aligned} N(r, \zeta_1) \leq \overline{N}(r, L(f^n)) + \overline{N}(r, L(g^n)) + \overline{N}\left(r, \frac{1}{L(f^n)}\right) \\ + \overline{N}\left(r, \frac{1}{L(g^n)}\right) + N_0\left(r, \frac{1}{(L(f^n))^{(k+1)}}\right) + N_0\left(r, \frac{1}{(L(g^n))^{(k+1)}}\right), \end{aligned} \quad (3.39)$$

where $N_0\left(r, \frac{1}{(L(f^n))^{(k+1)}}\right)$ is the counting function for which $(L(f^n))^{(k+1)} = 0$ and $L(f^n)[F(z) - 1] \neq 0$. Since F and G share 1 CM, then we get

$$\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) = 2\overline{N}\left(r, \frac{1}{F-1}\right) \leq N_{1)}\left(r, \frac{1}{F-1}\right) + N\left(r, \frac{1}{F-1}\right). \quad (3.40)$$

By Lemma (2.6) we have

$$T(r, L(f^n)) \leq \overline{N}(r, L(f)) + N_{k+1}\left(r, \frac{1}{L(f^n)}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) - \overline{N}_0\left(r, \frac{1}{F'}\right) + S(r,f). \quad (3.41)$$

$$T(r, L(g^n)) \leq \overline{N}(r, L(g)) + N_{k+1}\left(r, \frac{1}{L(g^n)}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r,g). \quad (3.42)$$

It follows from (3.38) - (3.42) that

$$\begin{aligned} T(r, L(f^n)) + T(r, L(g^n)) \leq 2\overline{N}(r, L(f)) + 2\overline{N}(r, L(g)) + N_{k+1}\left(r, \frac{1}{L(f^n)}\right) \\ + N_{k+1}\left(r, \frac{1}{L(g^n)}\right) + \overline{N}\left(r, \frac{1}{L(f)}\right) + \overline{N}\left(r, \frac{1}{L(g)}\right) \\ + N\left(r, \frac{1}{(L(f^n))^{(k)} - 1}\right) + S(r,f) + S(r,g). \end{aligned} \quad (3.43)$$

Since, $N\left(r, \frac{1}{(L(f^n))^{(k)} - 1}\right) \leq T(r, L(f^n)) + k\overline{N}(r, L(f)) + S(r,f)$.

We obtain from (3.43) that

$$T(r, L(g^n)) \leq (2+k)\overline{N}(r, L(f)) + 2\overline{N}(r, L(g)) + nN_{k+1}\left(r, \frac{1}{L(f)}\right)$$

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$$+nN_{k+1}\left(r,\frac{1}{L(g)}\right)+\overline{N}\left(r,\frac{1}{L(f)}\right)+\overline{N}\left(r,\frac{1}{L(g)}\right)+S(r,f)+S(r,g).$$

Without loss of generality, we suppose that there exist a set I with infinite measure such that $T(r,f)\le T(r,g)$ for $r\in I$.

Hence

$$nT(r,L(g))\le\{6+2n+k-[(k+2)\Theta(\infty,f)+2\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g) \\ +n\delta_{k+1}(0,f)+n\delta_{k+1}(0,g)]+\epsilon\}T(r,g)+S(r,g) \qquad (3.44)$$

for $r\in I$ and $0<\epsilon<H-(6+n+k)$ that is

$$[H-(6+n+k)-\epsilon]T(r,g)\le S(r,g).$$

That is

$$H-(6+n+k)\le 0.$$

Which implies

$$H\le 6+n+k$$

which is a contradiction to our hypothesis $H>6+n+k$. Hence we get $\zeta_1\equiv 0$.

That is

$$\frac{F''}{F'}-2\frac{F'}{F-1}-\frac{G''}{G'}+2\frac{G'}{G-1}. \qquad (3.45)$$

Integrating this equation

$$\frac{1}{F-1}=\frac{a}{G-1}+b \qquad (3.46)$$

where $a(\neq 0),b$ are two finite complex numbers.

Next, we consider two subcases.

Case 2.1 $b\neq 0$. Since $L(f^n)$ and $L(g^n)$ share ∞ IM. We know that F and G share ∞ IM. It follows from (3.46) that $F\neq\infty, G\neq\infty$. Hence $\frac{1}{F-1}\neq 0$ thus by (3.46) we deduce $G\neq\frac{b-a}{a}$

Now we consider two subcases.

Case 2.1.1 $b=a$. It follows from $\frac{a}{G-1}\neq 0$ and (3.46) that $F\neq 1+\frac{1}{b}$.

In the following, we consider two subcases.

Case 2.1.1.1 $b\neq -1$. Then we have $1+\frac{1}{b}\neq 0$. By Lemma (2.6) we obtain

$$nT(r,L(g))\le\overline{N}(r,L(g))+nN_{k+1}\left(r,\frac{1}{L(g)}\right)+\overline{N}\left(r,\frac{1}{G-\left(1+\frac{1}{b}\right)}\right)+S(r,g). \qquad (3.47)$$

From (3.46) we can write

$$\overline{N}\left(r,\frac{1}{G-\left(1+\frac{1}{b}\right)}\right)\le\overline{N}\left(r,\frac{G}{G-\left(1+\frac{1}{b}\right)}\right)=\overline{N}\left(r,\frac{1}{F}\right)$$

By Lemma (2.8) we obtain the following inequality

$$\overline{N}\left(r,\frac{1}{F}\right)\le(k+1)\overline{N}(r,L(f))+\overline{N}\left(r,\frac{1}{F}\right)+S(r,f).$$

Hence

$$\overline{N}\left(r,\frac{1}{G-\left(1+\frac{1}{b}\right)}\right)\le\overline{N}\left(r,\frac{1}{F}\right)\le(k+1)\overline{N}(r,L(f))+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).$$

Therefore (3.47) becomes

$$nT(r,L(g))\le[3+k+n-\Theta(\infty,g)-\Theta(0,f)-n\delta_{k+1}(0,g)-(k+)\Theta(\infty,f)]T(r,L(g)) \\ +S(r,f)+S(r,g). \qquad (3.48)$$

Hence by (3.23) and (3.48) we deduce that

$T(r,L(g))\le S(r,g)$ a contradiction.

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Case 2.1.1.2 $b = -1$. Thus $a = -1$ by (3.46) we deduce that $FG \equiv 1$. That is

$$(L(f^n))^{(k)}(L(g^n))^{(k)} \equiv 1. \tag{3.49}$$

Since $L(f^n)$ and $L(g^n)$ share ∞ IM then by (3.49) we deduce that $L(f^n) \neq \infty$, $L(g^n) \neq \infty$. It follows from (3.49) that $(L(f^n))^{(k)} \neq 0$, $(L(g^n))^{(k)} \neq 0$, $L(f^n) \neq 0$, $L(f^n) \neq 0$. If $k \geq 2$, then by Lemma (2.9) we get $L(f^n) = c_1 e^{cz}$, $L(g^n) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k\, (c_1c_2)^n\, (nc)^{2k} = 1$. If $k = 1$ then by Lemma (2.10) we get $L(f^n) = c_1 e^{cz}$, $L(g^n) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(c_1c_2)^{n+1} = -1$.

Case 2.1.2 $b \neq a$. Hence we have $\frac{b-a}{b} \neq 0$, $G - \frac{b-a}{b} \neq 0$. In this case by using the same argument as in 2.1.1.1 we get a contradiction.

Case 2.2 $b = 0$ then by (3.46)

$$F = \frac{1}{a} G + \frac{a-1}{a}. \tag{3.50}$$

If $a = 1$ then by (3.50) we have $F \equiv G$. That is $(L(f^n))^{(k)} = (L(g^n))^{(k)}$, by using the same argument as in case 1.1, we get $f \equiv tg$, where t is a constant such that $t^n = 1$. If $a \neq 1$ then by (3.50) we get $a(L(f^n))^{(k)} \equiv (L(g^n))^{(k)} + a - 1$. That is $a(L(f^n))^{(k)} - \left(L(g^n)\right)^{(k)} = a - 1$. Thus, we obtain $L(f^n) = \frac{1}{a} L(g^n) + P$ where P is a polynomial of degree k. Then by using the same argument as in case 1.1 we get a contradiction. Hence the proof. ■

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