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Shape Design of Cantilever Springs

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***Abstract*—Cantilever springs are simple flat springs in which one end is fixed and the other end is loaded. There are various applications for cantilever springs that include automobiles, medical devices and consumer products. Cantilever springs commonly have straight, slender and uniform beam configurations and are designed for small deflections that are perpendicular to the beam axis. Because of the small transverse deflection, the longitudinal deflection is usually ignored, the spring stiffness is regarded as constant and the maximum stress is considered to be proportional to the deflection. With the increase of the transverse deflection of a cantilever spring, the longitudinal deflection gradually becomes large that cannot be ignored, the spring stiffness can no longer be regarded as constant, the maximum stress does not have linear relationship with the transverse deflection. It is not trivial to design nonlinear cantilever beam springs. In this paper, the longitudinal deflection is derived for a nonlinear cantilever spring. The shape of a cantilever spring is designed to reduce its longitudinal deflection. The results of the paper provides a roadmap for designing nonlinear cantilever beam springs.**

Keywords*—Cantilever Spring; Shape Design; Deflection; Analysis.

I. INTRODUCTION

Flat springs usually refers to springs that are made of sheet, strip or plate. Although term flat is used, the shape of a flat spring is not necessarily flat. It may contain bends and other complicated forms [1]. The main purpose of using term flat is to distinguish the shapes of flat springs from those of helical, spiral, washer or power springs. Cantilever springs are simple flat springs in which one end is fixed and the other free end is loaded. They are designed to generate desired force and deflection relationships. There are various applications for cantilever springs that include automobiles, medical devices and consumer products [2].

Cantilever springs commonly have straight, slender and uniform beam configurations and are designed for small deflections that are perpendicular to the beam axis. Because of the small transverse deflection, the longitudinal deflection is usually ignored, the spring stiffness is regarded as constant and the maximum stress is considered to be proportional to the deflection.

Figure 1 shows a cantilever beam with uniform rectangular cross section. The in-plane thickness and out-of-plane width of the cantilever beam are t and b, respectively. The beam has length of L. When the transverse deflection (δ_x) at the loading end is small, it can be calculated as follows [3].

$$\delta_x = \frac{4FL^3}{Ebt^3} \tag{1}$$

In Equation (1), E is Young's modulus of the beam material. F is the force that is perpendicular to the beam axis and applied at the free end of the cantilever beam.

When the longitudinal deflection at the loading end of the beam is ignored, the bending stress (σ) along the beam axis (y) can be calculated by the following formula [4].

$$\sigma(y) = \frac{6Fy}{bt^2} \tag{2}$$

The maximum bending stress ($\sigma_{\max} = 6FL/bt^2$) occurs at the fixed end of the beam. Bending stress is tensile on one side of the beam and compressive on the other side.

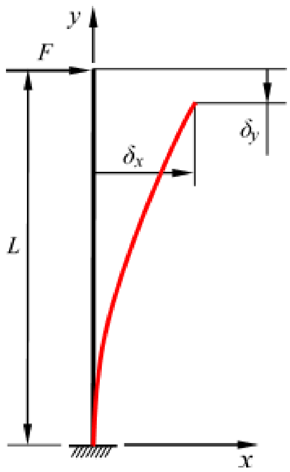

Fig. 1 A cantilever beam with straight and uniform undeformed shape.

The material of the cantilever beam is considered as homogeneous and isotropic in the paper. The slender beam is assumed to be inextensible. The strain (ε) of the beam remains small and is within its linear elastic range. The cantilever beam in the paper is an Euler-Bernoulli beam. Plane hypothesis holds for the cantilever beam, i.e., a plane cross section that is perpendicular to the neutral axis of the beam before deformation remains plane and perpendicular to the neutral axis after deformation. The bending moment (M) of an Euler-Bernoulli beam is proportional to its curvature (κ). The relationship can be written as follows [5].

$$\kappa(s) = \frac{1}{\rho(s)} = \frac{d\theta}{ds} = \frac{M(s)}{EI} \tag{3}$$

In Equation (3), s is the arc length along the deflection curve, ρ is the radius of curvature, θ is the slope of the deflection curve, I is the moment of inertia of the cross section of the beam.

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In the rectangular coordinate system as shown in Figure 1, the curvature of the deflection curve of the cantilever beam can be written as:

$$\kappa(s)=\frac{1}{\rho(s)}=\frac{\dfrac{d^2x}{dy^2}}{\left[1+\left(\dfrac{dx}{dy}\right)^2\right]^{3/2}}=\frac{M(y)}{EI} \tag{4}$$

Equation (4) is a second order nonlinear differential equation. It is difficult to have an analytical closed-form solution to the equation. When beam deflection is small, dx/dy is small and $(dx/dy)^2$ approaches zero. When the denominator of Equation (4) is approximated as one, the nonlinear differential equation is simplified as follows.

$$\frac{d^2x}{dy^2}=\frac{M(y)}{EI} \tag{5}$$

$M(y)$ in Equation (4) can be derived from Figure 1 as:

$$M(y)=F\left(L-\delta_y-y\right) \tag{6}$$

When δ_y is ignored as zero for small deflection, $M(y)$ can be approximately simplified as $M(y)=F\left(L-y\right)$. Equation (5) becomes a second order linear differential equation with the simplified $M(y)$. Its analytical closed-form solution can be solved as:

$$x(y)=\frac{Fy^2}{6EI}\left(3L-y\right) \tag{7}$$

When beam deflection is not small, the denominator in Equation (4) cannot be simplified as one, and the longitudinal deflection δ_y in $M(y)$ can no longer be ignored. Although it is difficult to have an analytical closed-form solution to the second order nonlinear differential equation of Equation (4), many different numerical approaches have been proposed and published [6], which include elliptic integral approach [7], power series approach [8], Runge-Kutta approach [9], finite element approach [10], equivalent system approach [11], and others [12]. In most existing approaches, the problem is focused on solving the large deflection curve of a cantilever beam or a frame under a given loading such as concentrated, distributed or combined. However, we are more interested in the reverse problem for cantilever beam springs, i.e., solving the needed reaction force (F) and corresponding longitudinal deflection (δ_y) under the given transverse deflection (δ_x). For a cantilever beam spring, δ_y is usually an undesired axial deviation that can be reduced by designing the shape of the cantilever beam. The authors of the paper are motivated by the challenges facing cantilever beam springs. The research objective of the paper is to provide a guideline and systematic approach for the analysis and shape design of cantilever beam springs.

The remainder of the paper is organized as follows. The beam deflection analysis is presented in section II. The analysis on cantilever beam springs is provided in section III. Section IV is on the shape design of cantilever beam springs. Conclusions are drawn in section V.

II. BEAM DEFLECTION ANALYSIS

For the cantilever beam shown in Figure 1, we assume δ_x is given together with known beam's cross sectional sizes (t and b) and Young's modus (E) of the beam material. We are trying to find out δ_y and F.

Rearranging Equation (3) yields the following equation.

$$EI\frac{d\theta}{ds}=M(s) \tag{8}$$

Differentiating both sides of Equation (8) with respect to s, we have

$$EI\frac{d^2\theta}{ds^2}=\frac{dM(s)}{ds} \tag{9}$$

$M(s)$ is calculated by Equation (6) as $F\left(L-\delta_y-y\right)$. We have $dM(s)/ds=-F\,dy/ds$. Since $dy/ds=\cos\theta$, we have $dM(s)/ds=-F\cos\theta$. Substituting the expression of $dM(s)/ds$ into Equation (9) and moving the term on the right hand side to the left yields the following equation.

$$EI\frac{d^2\theta}{ds^2}+F\cos\theta=0 \tag{10}$$

From Equation (10), we have

$$\frac{d}{ds}\left[\frac{1}{2}EI\left(\frac{d\theta}{ds}\right)^2+F\sin\theta\right]=0 \tag{11}$$

Equation (11) can be solved with its solution as

$$\frac{1}{2}EI\left(\frac{d\theta}{ds}\right)^2+F\sin\theta=C \tag{12}$$

C in Equation (12) is an arbitrary constant. It can be decided by the boundary condition of the deflected cantilever beam. At the loading end of the beam, we have $s=L$, $M(L)=0$, and $\left.\frac{d\theta}{ds}\right|_{s=L}=0$. Assume the slope of the beam at the loading end is θ_m. Substituting the expressions of $d\theta/ds$ and θ at the loading end into Equation (12) yields the following equation.

$$F\sin\theta_m=C \tag{13}$$

Substituting Equation (13) into Equation (12), we have

$$\left(\frac{d\theta}{ds}\right)^2=\frac{2F}{EI}(\sin\theta_m-\sin\theta) \tag{14}$$

Taking square root on both sides of Equation (14) yields the following equation.

$$\frac{d\theta}{ds}=\sqrt{\frac{2F}{EI}}\sqrt{\sin\theta_m-\sin\theta} \tag{15}$$

Rearranging Equation (15) yields the following equation.

$$ds = \sqrt{\frac{EI}{2F}} \frac{d\theta}{\sqrt{\sin\theta_m - \sin\theta}} \qquad (16)$$

Integrating Equation (16) from the fixed end to the loading end of the cantilever beam yields the following equation.

$$L = \sqrt{\frac{EI}{2F}} \int_0^{\theta_m} \frac{d\theta}{\sqrt{\sin\theta_m - \sin\theta}} \qquad (17)$$

Substituting Equation (16) into $dx = \cos\theta ds$ yields the following equation.

$$dx = \sqrt{\frac{EI}{2F}} \frac{\cos\theta d\theta}{\sqrt{\sin\theta_m - \sin\theta}} \qquad (18)$$

Integrating Equation (18) from the fixed end to the loading end of the cantilever beam yields the following equation.

$$\delta_x = \sqrt{\frac{EI}{2F}} \int_0^{\theta_m} \frac{\cos\theta d\theta}{\sqrt{\sin\theta_m - \sin\theta}} \qquad (19)$$

θ_m and F are unknowns now. They can be solved by Equations (17) and (19). Combining Equations (17) and (19) and Eliminating F from them yields the following equation.

$$\delta_x \int_0^{\theta_m} \frac{d\theta}{\sqrt{\sin\theta_m - \sin\theta}} = L \int_0^{\theta_m} \frac{\cos\theta d\theta}{\sqrt{\sin\theta_m - \sin\theta}} \qquad (20)$$

θ_m is the only unknown in Equation (20). It can be solved numerically.

After θ_m is solved through Equation (20), F can then be solved from either Equation (17) or (19).

Substituting Equation (16) into $dy = \sin\theta ds$ yields the following equation.

$$dy = \sqrt{\frac{EI}{2F}} \frac{\sin\theta d\theta}{\sqrt{\sin\theta_m - \sin\theta}} \qquad (21)$$

Integrating Equation (21) from the fixed end to the loading end of the cantilever beam yields the following equation.

$$\delta_y = \sqrt{\frac{EI}{2F}} \int_0^{\theta_m} \frac{\sin\theta d\theta}{\sqrt{\sin\theta_m - \sin\theta}} \qquad (22)$$

Equation (22) leads to the solution of δ_y.

III. CANTILEVER SPRING ANALYSIS

The deflection and reaction force of a cantilever beam spring can be directly analyzed by finite element analysis software ANSYS [13-15]. The stress of the deflected beam can also be directly obtained during the analysis process.

Figure 2 shows the solid model of an initially straight and uniform cantilever beam spring. The height of the beam is 150 mm. The thickness (t) and width (b) of the beam are 0.25 mm and 10 mm, respectively. The material of the cantilever spring is structural steel with Young's modules (E) of MPa, Poisson's ratio (ν) of 0.3, yield strength (σ_y) of 250 MPa. The Design Modeler [16] of ANSYS is used to create the solid model. ANSYS Design Modeler is an ANSYS Workbench application that provides modeling tool for the creation and modification of geometries.

Fig. 2 The solid model of a cantilever beam spring with straight undeformed shape.

The solid model created in ANSYS Design Modeler is then analysed in ANSYS Mechanical [17] that is also an application of ANSYS Workbench. The lower end of the beam is fixed and its upper end is for loading. A δ_x of 50 mm is applied at the upper end of the beam. The deformed and undeformed shapes of the beam are shown in Figure 3.

The deformation numbers and their corresponding colors shown in Figure 3 are for the transverse deformation of the cantilever beam that is directional deformation in ANSYS along x axis. Because of the large deformation of the beam, its loading end of the beam has a significant longitudinal deformation, which is shown in Figure 4. The longitudinal deformation is directional deformation in ANSYS along y axis. The δ_y of the cantilever beam is 10.41 mm, which is large that cannot be ignored. In Figure 4, the longitudinal deformation has negative sign. That is because it is downward and the positive y axis direction is upward.

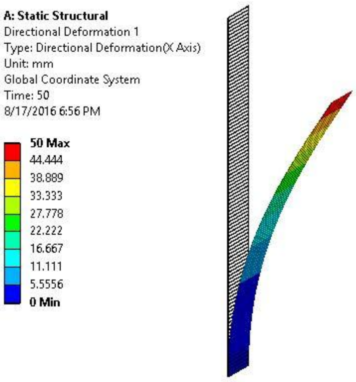

Fig. 3 The transverse deformation of the cantilever beam spring with straight undeformed shape.

The maximum stress within the deformed cantilever beam is 175.51 MPa, which is below the yield strength of the beam material. The stress distribution in the deformed beam is shown in Figure 5. To have the transverse deformation of 50 mm at the loading end of the cantilever beam, an input force of 0.1315 N is needed. The input force is called reaction force in ANSYS Mechanical. The reaction force is shown in Figure 6. The input force is small because the cantilever beam is slim.

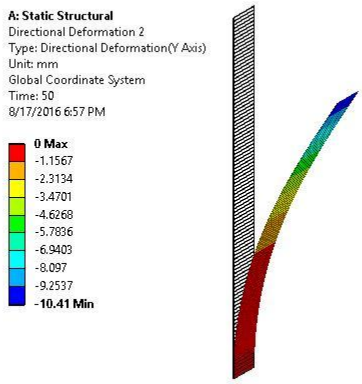

Fig. 4 The longitudinal deformation of the cantilever beam spring with straight undeformed shape.

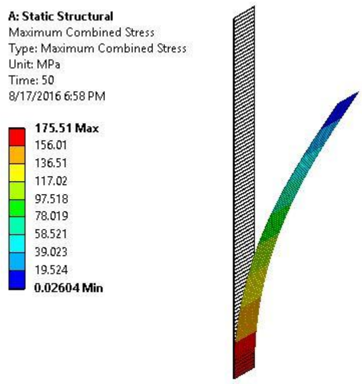

Fig. 5 The stress of the cantilever beam spring with straight undeformed shape.

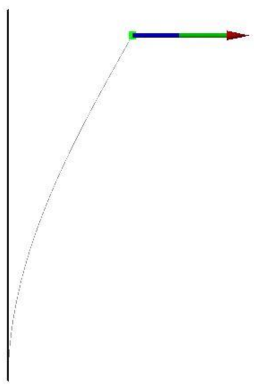

Fig. 6 The input force of the cantilever beam spring with straight undeformed shape.

IV. SHAPE DESIGN OF CANTILEVER SPRINGS

Shape design of a cantilever beam spring is to improve its performance and better meet its needs and requirements by changing its shape. The vertical cantilever beam spring analyzed in the preceding section has uniform cross section. Its height, width and thickness are 150 mm, 10 mm and 0.25 mm, respectively. In this section, we use the same spring material, beam cross section, beam height along *y* axis, but change the beam shape to see the performance difference under the same δ_x of 50 mm.

The first shape change is to make the vertically straight undeformed shape become slantingly straight undeformed shape. Figure 7 shows the solid model of the cantilever spring. The slanted beam has vertical height of 150 mm that is along *y* axis. Its lower end is fixed. Its upper end is now away from its lower end horizontally by 50 mm. When a δ_x of 50 mm is applied at the free loading end of the slanted cantilever spring, its horizontal and vertical deformations along the beam are shown in Figures 8 and 9, respectively. The δ_y of the slanted cantilever beam is now 6.4863 mm, which is smaller than that of 10.41 mm from the vertical cantilever beam. As shown in Figure 9, the vertical deformation has positive sign, which means its direction is upward along positive *y* axis.

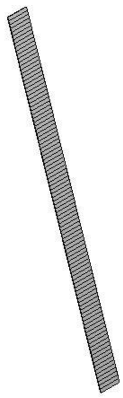

Fig. 7 The solid model of a cantilever beam spring with slanted undeformed shape.

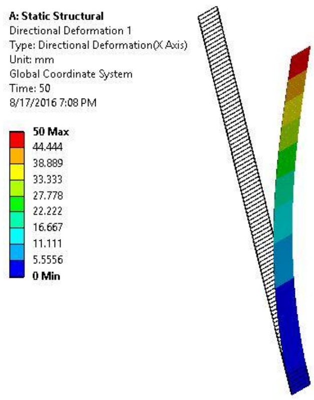

Fig. 8 The horizontal deformation of the cantilever beam spring with slanted undeformed shape.

The stress distribution in the deformed cantilever beam spring is shown in Figure 10. The maximum stress is 152.01 MPa, which is below that of 175.51 MPa from the vertical cantilever beam. To have the horizontal deflection of 50 mm at the loading end of the slanted cantilever beam, an input horizontal force of 0.1013 N is needed, which is smaller than that of 0.1315 N from the vertical cantilever beam.

The undeformed beam shape does not have to be straight. It can be curved. The solid model shown in Figure 11 has a circular undeformed shape. Both the fixed and loading ends of the cantilever beam are on the *y* axis. The arc length of the beam is set as the same as the length of the slanted straight beam shown in Figure 7, which is 158.11 mm.

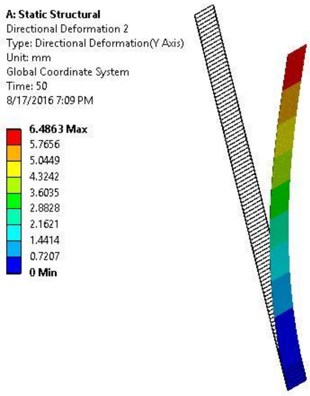

Fig. 9 The vertical deformation of the cantilever beam spring with slanted undeformed shape.

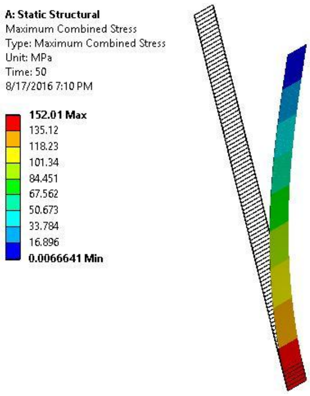

Fig. 10 The stress of the cantilever beam spring with slanted undeformed shape.

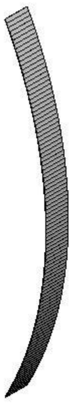

Fig. 11 The solid model of a cantilever beam spring with right circular undeformed shape.

When a δ_x of 50 mm is applied at the free loading end of the circular cantilever spring, its horizontal and vertical deformations along the beam are shown in Figures 12 and 13, respectively. The δ_y of the right circular cantilever beam is now 6.0895 mm, which is close to that of 6.4863 mm from the slanted cantilever beam. As shown in Figure 13, the vertical deflection is downward along negative *y* axis.

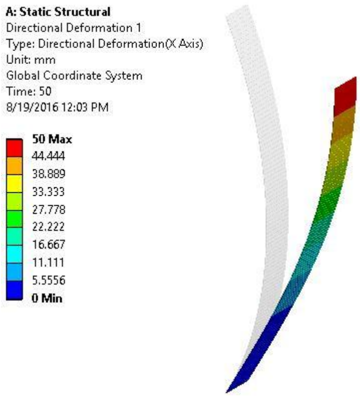

Fig. 12 The horizontal deformation of the cantilever beam spring with right circular undeformed shape.

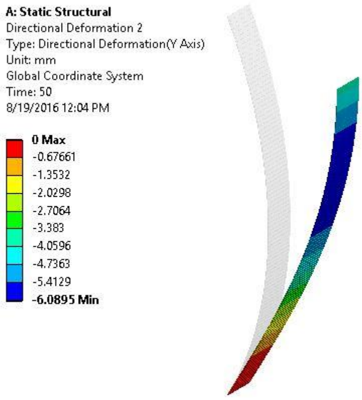

Fig. 13 The vertical deformation of the cantilever beam spring with right circular undeformed shape.

The stress distribution in the deformed cantilever beam spring is shown in Figure 14. The maximum stress is 155.35 MPa, which is close to that of 152.01 MPa from the slanted cantilever beam. To have the horizontal deflection of 50 mm at the loading end of the right circular cantilever beam, an input horizontal force of 0.1107 N is needed, which is slightly below that of 0.1013 N from the slanted cantilever beam.

If the right circular cantilever beam is flipped with respect to *y* axis, the cantilever beam becomes a left circular beam. Its solid model is shown in Figure 15. When a δ_x of 50 mm is applied at the free loading end of the left circular cantilever spring, its horizontal and vertical deformations along the beam are shown in Figures 16 and 17, respectively. The δ_y of the left circular cantilever beam is now 16.995 mm, which is well above that of 6.0895 mm from the right circular cantilever beam. As shown in Figure 17, the vertical deformation is downward along negative *y* axis. The stress distribution in the deformed cantilever beam spring is shown in Figure 18. The maximum stress is 181.07 MPa, which is larger than that of 155.35 MPa from the right circular cantilever beam. To have the horizontal deflection of 50 mm at the loading end of the left circular cantilever beam, an input horizontal force of 0.1429 N is needed, which is above that of 0.1107 N from the right circular cantilever beam.

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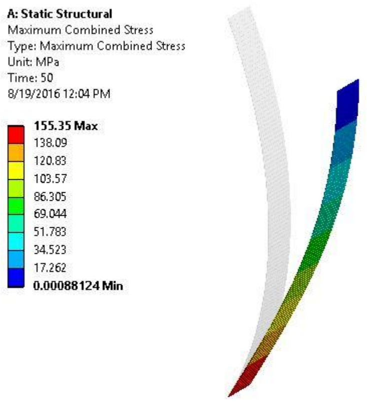

Fig. 14 The stress of the cantilever beam spring with right circular undeformed shape.

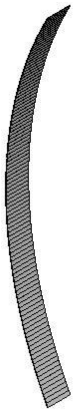

Fig. 15 The solid model of a cantilever beam spring with left circular undeformed shape.

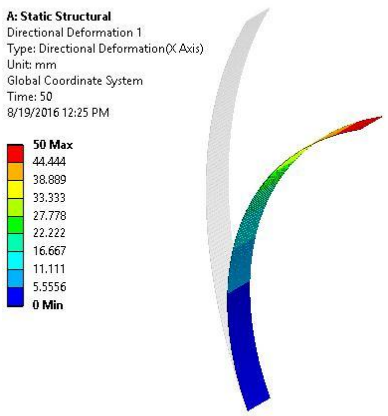

Fig. 16 The horizontal deformation of the cantilever beam spring with left circular undeformed shape.

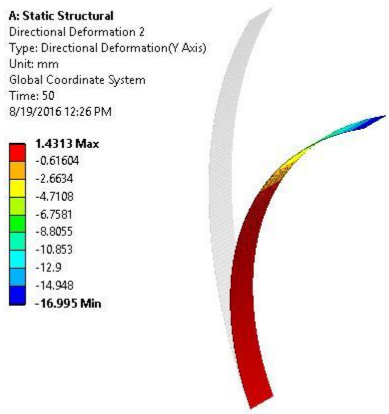

Fig. 17 The vertical deformation of the cantilever beam spring with left circular undeformed shape.

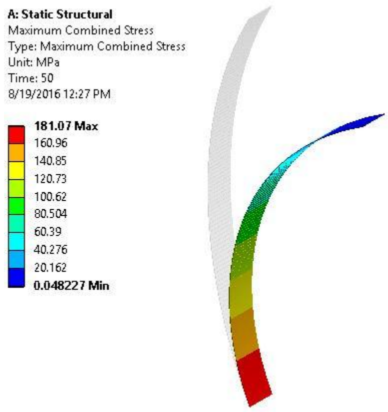

Fig. 18 The stress of the cantilever beam spring with left circular undeformed shape.

The shape design results are summarized in the following table. The cantilever beam spring in each case has the same material, same cross section, and same horizontal input deflection (50 mm).

Table 1 Shape design results of the cantilever spring

Cantilever Beam Spring Shape	Vertical Deflection	Maximum Stress
Vertical Straight	-10.41 mm	175.51 MPa
Slanted Straight	6.4863 mm	152.01 MPa
Right Circular	-6.0895 mm	155.35 MPa
Left Circular	-16.995 mm	181.07 MPa

As shown in Table 1, the cantilever beam spring with right circular shape has the smallest vertical deflection. The slanted straight spring's vertical deflection is slightly above that of the right circular spring. Their vertical deflections have opposite directions. The lowest maximum stress comes from the spring with slanted straight shape. The right circular spring's maximum stress is a little above that of the slanted straight spring. Cantilever beam springs with vertical straight or left circular shape have larger vertical deflection and higher maximum stress than cantilever beam springs with any of the other two shapes.

V. CONCLUSIONS

A cantilever beam spring usually has straight shape and is designed for small deflection that is perpendicular to the spring's straight line. When the transverse (horizontal) deflection at the free end of the cantilever beam is large, the longitudinal (vertical) deflection is also not small. To reduce the vertical deflection, the spring's straight line can be slanted (which is no longer perpendicular to the horizontal deflection). A cantilever beam spring with slanted straight shape has smaller vertical deflection and lower maximum stress than its corresponding vertical straight spring. Besides slanting a vertical straight spring, the vertical straight shape of the spring can be changed to be circular to reduce its vertical deflection. There are two symmetric circular shapes among which one (called right circular in the paper) decreases vertical deflection and another (left circular) increases vertical deflection. Cantilever beam springs with slanted straight and right circular shapes have similar effects on reducing vertical deflection and lowering the maximum stress. Their vertical deflections have opposite directions.

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