Second Order (b, F) – Convexity In Multiobjective Fractional Programming

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Abstract- second order duality theorems for multiobjective fractional programming problems under the assumption of second order (b, F) – convexity.

Key Words- Multiobjective fractional programming, ‘second order (b,F) – convexity’.

Introduction:

Many researchers have used proper efficiency and efficiency to establish optimality conditions and duality results for multiobjective programming problems under different assumptions of convexity. A second order dual for a non linear programming problem was introduced by Mangasarian and established duality results for non linear programming problems. MOND introduced the concept of second order convex functions and proved second order duality under the assumptions of second order convexity on the functions involved. MOND and Zhang established various duality results for multiobjective programming problems involving second order v-invex functions. Zhang and MOND

introduced second order F-convex functions as a generalization of F-convex functions in 1982 and obtained various second order duality results for multiobective nonlinear programming problems under the assumptions of second order F-convexity. In 2003 Suneja et al obtained duality results for multiobjective programming under the assumption of η-bonvexity and related functions. In 2004 Ahmed obtained optimality conditions and mixed duality results for non differentiable programming problems. In 2006 Ahmed and Hussain [OPSEARCH] obtained second order Mond – Weir type dual results for multiobjective programming problems under the assumption of the second order (F,α,p,d)- convexity on the function involved. The study of the second order duality is significant due to the computational advantage over first order duality as it provides tighten bounds for the value of the objective functions when approximations are used.

But, they not consider the recent developed concept like multiobjective fractional programming under second order (b, F) – convexity.

In this paper a new class of functions namely, second order (b, F) – convex functions which is as extension of (b, F) – convex functions in previous chapter and second order F-convex functions. Then, we derive sufficient – optimality conditions for proper efficiency and obtain second order duality theorems for multiobjective fractional programming problems under the assumption of second order (b, F) – convex

Preliminaries:

Let X be an open convex subset of R^n and R_+ denote the set of all positive real numbers. Let us assume that $h: x \to R, f: x \to R^k$ and g: $R^n \to R^m$ where $f = (f_1 .... f_k)$ and g= $(g_1............g_m)$ h=$(h_1, h_2........h_k)$ are differentiable functions on X f_i and $g_i : X \to R$ for all i=1,2,.........k and j=1,2,.........m

Let F be a function defined by

$F: X \times X \times R^n \to R$ and the functions $b_o(x, u)$ and C_o (x, u): $X \times X \to R_+$ and $b_i(x,u)$ and $C_j(x,u): X \times X \to R_+$ for all i=1,2............k, J = 1,2,..........m

Consider the following multiobjective fractional programming problem

(FP) minimize $\frac{f_i(x)}{h_i(x)}$, i=1,2,………..k

Subject to g(x) $\leq 0\ x \in x$

Where $f_i : x \to R\ \ i = 1, 2, \ldots\ldots\ldots.k$ and $g : x \to R^m$

Where g=(g_1………g_m) and h=(h_1, h_2, …………… h_k)

are differentiable function on X.

Let $p = \{x \in x;\ g_j(x) \leq 0, j = 1, 2\ldots.m\}$. That is, p is the set of all feasible solutions for the problems (F.P).

Definition:

A function $F : X \times X \times R^n \to R$ is said to be sublinear in its third argument if for each $x, u \in x$

$$F(x,u:a+b) \leq F(x,u;a) + F(x,u;b)$$

For all $a, b \in R^n$ and

$$F(x,u;\alpha a) = \alpha F(x,u:a) \text{ for all } \alpha \geq 0 \text{ in R and } a \in R^n$$

Definition:

A feasible point x^0 is said to be efficient for (FP) if there exits no other feasible point x in (FP) such that $\frac{f_i(x)}{h_i(x)} \leq \frac{f_i(x^0)}{h_i(x^0)}$, i=1,2,….K and

$\frac{f_i(x)}{h_i(x)} < \frac{f_i(x^0)}{h_i(x^0)}$ for some $r \in \{1,2,.....k\}$

Definition: A feasible point x^0 is said to be a properly efficient solution of (FP). It is efficient and if there exists a scalar M>0 such that, for each $i \in \{1,2,.....k\}$ and for all feasible x of (FP) satisfying $\frac{f_i(x)}{h_i(x)} < \frac{f_i(x^0)}{h_i(x^0)}$, have

$\frac{f_i(x^0)}{h_i(x^0)} - \frac{f_i(x)}{h_i(x)} \leq M \frac{f_i(x)}{h_i(x)} - \frac{f_i(x^0)}{h_i(x^0)}$ fore some r such that $f_r(x) > f_r(x^0)$

We need the following theorem for proving sufficient optimality conditions for proper efficiency and duality theorems which can be found in 1999.

Second Order (b, F) – Convex Functions Definition:

The function h is said to be second order (b_o, F) – Convex at $u \in x$ with respect to $b_o(x,u)$ is for all $x \in X$ and $p \in R^n$

$$b_o(x,u)\left[\frac{f_i(x)}{h_i(x)}-\frac{f_i(u)}{h_i(u)}+\frac{1}{2}P^T\nabla^2\frac{f_i(u)}{h_i(u)}p\right]\geq F\left(x,u:\nabla\frac{f_i(u)}{h_i(u)}+\nabla^2\frac{f_i(u)}{h_i(u)}p\right)$$

The function h is said to be strictly second order (b_o, F)-convex at $u\in x$ with respect $b_o(x, u)$ if for all $x\in X$, $x\neq u$ and $p\in R^n$

$$b_o(x,u)\left[\frac{f_i(x)}{h_i(x)}-\frac{f_i(u)}{h_i(u)}+\frac{1}{2}p^T\nabla^2\frac{f_i(u)}{h_i(u)}p\right]>F\left[\left(x,u;\frac{\nabla f_i(u)}{h_i(u)}\right)+\nabla^2\frac{f_i(u)}{h_i(u)}p\right]$$

Necessary Optimality Conditions:

Assume that x^0 is an efficient solution for (FP) at which a constraint qualification is satisfied for each $Fp_r\left(x^0\right), r\in\{1,2,.....k\}$

where $F_{p_r}\left(x^0\right)$, minimize $f_r(x)$

subject to $\frac{f_i(x)}{h_i(x)}\leq\frac{f_i\left(x^0\right)}{h_i\left(x^0\right)}$ for all $i\neq r$ $x\in p$

then $y^0\geq 0$ in R^m such that

$$\sum_{i=1}^{K}\nabla\frac{f_i\left(x^0\right)}{h_i\left(x^0\right)}+\sum_{j=1}^{m}y_j^0\nabla g_j\left(x^0\right)=0$$

$$y_j^0 g_j\left(x^0\right)=0, j=1,2,....m$$

Sufficient optimality conditions :

Let x^0 be feasible for (FP) and there $y_j \geq 0$ in R, $j \in I\left(x^0\right)$ and $p^0 \in R^n$ such that

$$\sum_{i=1}^{k} \nabla \frac{f_i\left(x^0\right)}{h_i\left(x^0\right)} + \sum_{J \in I\left(X^0\right)} y_j^0 \nabla g_i\left(x^0\right) + \left[\sum_{i=1}^{k} \nabla^2 \frac{f_i\left(x^0\right)}{h_i\left(x^0\right)}\right] +$$

$$\sum_{j \in I\left(x^0\right)} \left(y_j^0 . \nabla^2 g_j\left(x^0\right)\right) p^0 = 0$$

Where $I(x^0) = \left\{j : g_j\left(x^0\right) = 0\right\}$. If f is second order ($b_0$, F) – convex at x^0 with respect to $b_0(x, x^0)$ with $b_o(x, x^0)>0$ and each $g_j, j \in I\left(x^0\right)$ is second order $\left(C_j, F\right)$- convex at x^0 with respect $C_j\left(x, x^0\right)$, then x^0 is a properly efficient solution for FP provided that

$P^{0^T}\left(\nabla^2\left(x^0\right)\right)p^0 \leq 0$ and $P^{0^T}\left(\nabla^2 g_j\left(x^0\right)\right)p^0 \leq 0$ for all $j \in I\left(x^0\right)$

Proof: Let x be a feasible solution of (FP) Now, since x^0 is feasible for (FP) and $y_j \geq 0$ and $P^{0^T}\left(\nabla^2 g_j\left(x^0\right)\right)p^0 \leq 0$, for all $j \in I\left(x^0\right)$ and by the second order (C_j F) – convexity of g_i at x^0, for all $j \in I\left(x^0\right)$ and since $y_j^0 \geq 0, j \in I\left(x^0\right)$ and F is sub linear, we have

$$F\left(x_i x^0; \nabla\left(\sum_{j \in I\left(x^0\right)} y_j^0 g_j\left(x^0\right) + \nabla^2\left(\sum_{j = I\left(x^0\right)} y_j^0 g_j\left(x^0\right) p^0\right)\right)\right) \leq 0 \qquad \textbf{(9.4)}$$

Now, by the sub linearity of F and from (9.3)

We have

$$F\left(x, x^0 : \sum_{i=1}^{k} \nabla \frac{f_i\left(x^0\right)}{h_i\left(x^0\right)} + \sum_{i=1}^{k} \nabla^2 \frac{f_i\left(x^0\right)}{h_i\left(x^0\right)} p^0 \right) \geq -F\left(x, x^0 : \sum y_j^0 \nabla g_j\left(x^0\right)\right) + \sum_{j \in I\left(x^0\right)} y_j^0 \nabla^2 g_j\left(x^0\right) p^0)$$

Now from (9.4) and (9.5) by the second order (b_0, F) – convexity since

$$p^{0^T} \left(\nabla^2 \frac{f_i\left(x^0\right)}{h_i\left(x^0\right)} p^0 \leq 0 \right. \text{ and } b_0(x, x^0) > 0$$

We can conclude that

$$\frac{f_i(x)}{h_i(x)} \geq \frac{f_i\left(x^0\right)}{h_i\left(x^0\right)}$$

Thus x^0 is an optimal solution (FP). Hence X^0 is a properly efficient solution for (FP)

Hence the theorem.

Duality Theorems:

Let j, be a subset of $M = \{1, 2, ...m\}$ and J_2=M/J1 consider the following second order dual for (FP).

$$(FD)\text{Maximize}\frac{f_i(u)}{h_i(u)}+y_jg_j(u)-\frac{1}{2}P^T\nabla^2\left[\frac{f_i(u)}{h_i(u)}+y_jg_j(u)\right]p$$

Subject to

$$\nabla\frac{f_i(u)}{h_i(u)}+\left(\nabla^2\frac{f_i(u)}{g_i(u)}p+\nabla y_jg_j(u)+\nabla^2y_ig_i(u)\right)p=0$$

$$y_jg_j(u)-\frac{1}{2}p^T\nabla^2y_ig_i(u)\rho\geq 0$$

$$y_j\geq 0 \quad j=1,2\ldots\ldots..m$$

Weak Duality Theorem:

Let x be feasible for (FP) and (u, y, p) be feasible for (FD)

If $\frac{f_i}{h_i}+y_jg_j$ is second order (b_o, F) – convex at u with respect to b_0(x, u) with b_o (x, u)>0 and y_jg_j is second order (C_oF) – convex at u with respect to C_o(x, u) then

$$\frac{f_i(x)}{h_i(x)}\geq\frac{f_i(u)}{h_i(u)}+y_jg_j(u)-\frac{1}{2}P^T\nabla^2\left[\frac{f_i(u)}{h_i(u)}+y_jg_j(u)\right]p$$

Proof:

Now since $\frac{f_i}{h_i}+y_ig_i$ is second order (b_o, F) – convex at u. We have

$$b_o(x,u)\left[\frac{f_i(x)}{h_i(x)}+y_j g_j(x)-\frac{f_i(u)}{h_i(u)}-y_i g_i(u)+\frac{1}{2}p^T\nabla^2\left(\frac{f_i(u)}{h_i(u)}+y_j g_j(u)p\right]\right.$$

$$\geq F\left(x,u;\nabla\left(\frac{f_i(u)}{h_i(u)}+y_j g_j(u)+\nabla^2\frac{f_i(u)}{h_i(u)}+y_j g_j(u)(p)\right)\right)$$

Now, since x is feasible and (u, y, p) is feasible for (FD) and since $y_j g_j$ is second order (C_o, F) – convex at u with respect to $C_0(x, u)$ we have.

$$F(x,u:\nabla)\left(y_j g_j(u)+\nabla^2 y_j g_j(u)p\right)\leq 0$$

Now, from and since (u, y, p) is feasible for (FD) and F is sublinear

We have

$$F(x)u:\nabla\left(\frac{f_i(u)}{h_i(u)}+y_j g_j(u)+\nabla^2\left(\left(\frac{f_i(u)}{h_i(u)}+y_j g_j(u)p\right)\geq 0\right.\right.$$

$b_o(x,u)>0$ and $y_j g_j(x)\leq 0$ we have

$$\frac{f_i(x)}{h_i(x)}\geq\frac{f_i(x)}{h_i(u)}+y_j g_j(u)-\frac{1}{2}P^T\nabla^2\left[\frac{f_i(u)}{h_i(u)}+y_j g_j(u)\right]p.$$

Strong Duality Theorem:

Assume that x^0 an efficient solution for (FP) at which a constraint qualification is satisfied for each $Fp_r(x^0)$ $x\in\{1,2,......k\}$. Then, there exists $y^0\in R^m$ such that $(x^0,y^0,p^0=0)$ is feasible solution for (FP) and the corresponding objective functions values of (FP) and (FD) are equal. If the conditions of weak duality Theorem holds then (x^0, y^0, p^0) = 0 is properly efficient solution for (FD).

Proof:

By the theorem necessary optimal condition $y^0 \geq 0$ in R^m such that (x^0, y^0) satisfies (1) and (2). Therefore (x^0, y^0, p^0 =0) is feasible for FD and the objective value of the problem FP at x^0 and the objective value of (FD) at (x^0, y^0, p^0 = 0) are equal.

Suppose that (x^0, y^0, p^0 =0) is not efficient for (FD) then there exists a feasible (u, y, p) for (FD) such that

$$\frac{f_i(x^0)}{h_i(x^0)} \leq \frac{f_i(u)}{h_i(u)} + y_j g_{j(u)} - \frac{1}{2} p^T \nabla^2 \left[\frac{f_i(u)}{h_i(u)} + y_j g_j(u) \right] p$$

$$\frac{f_i(x^0)}{h_i(x^0)} < \frac{f_i(u)}{h_i(u)} + y_j g_j(u) - \frac{1}{2} p^T \nabla^2 \left[\frac{f_i(u)}{h_i(u)} + y_j g_j(u) \right] p$$

Which contridcts the theorem weak duality theorem. Thus (x^0, y^0, p^0 =0) is an efficient solution for (FD). Suppose that (x^0, y^0, p^0 =0) is not properly efficient for FD. Then for every M>0, there exists a feasible solution (u, y, p) of FD and an index i such that

$$\frac{f_i(u)}{h_i(u)} + y_j g_j(u) - \frac{f_i(x^0)}{h_i(x^0)} > M\left[f_r(x^0) - f_r(u) - y_j g_j \right]$$

For all r satisfying $\frac{f_r(x^0)}{h_r(x^0)} - \frac{f_r(u)}{h_r(u)} - y_i g_j(u) > 0$ whenever

$$\frac{f_i(u)}{h_i(u)} + y_j g_j(u) - f_i(x^0) > 0$$. This means hat

$\frac{f_i(u)}{h_i(u)} + y_j g_j(u) - \frac{f_i(x^0)}{h_i(x^0)}$ can be made arbitary large. $\frac{f_i}{h_i} + y_j g_i$ is second order (b_o, F) convex at u with respect to b_o (x, u) with b_0(x, u) > 0. Since (u, y, p) is feasible for FD and F is sublinear we can conclude that.

$$F\left(x, u; \nabla\left(y_j g_j(u) + \nabla^2 y_j g_j(u) p\right) > 0\right.$$

Now since x^0 is feasible for (FP) and (u, y, p) is feasible for (FD) and by the second order (C_o,F) – convexity of y_j, g_j at u we have.

$$F\left(x^0, u : \nabla\left(y_j g_j(u)\right) + \nabla^2\left(y_j g_j(u) p\right) \le 0\right.$$

Which contradicts Thus (x^0, y^0, p^0=0) is properly efficient solution for FD.

Hence the theorem

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