International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

Runge Kutta and Adams Multıstep Method for Second Kınd of Volterra Integral Equatıon

Nigar Mammadzada
Lecturer, Department of Cybersecurity, Azerbaijan Technical University, Baku, Azerbaijan.
PhD Candidate, Faculty of Mechanics and Mathematics, Baku State University, Baku, Azerbaijan.

Abstract

This research investigates the utilization of Runge–Kutta and Adams multistep numerical methods for the resolution of second-kind Volterra integral equations, which commonly occur in mathematical models of hereditary processes, population dynamics, viscoelasticity, and heat conduction with memory. A thorough theoretical review was undertaken, analyzing both classical and contemporary advancements concerning existence, uniqueness, resolvent kernels, weakly singular kernels, and nonlinear extensions. The numerical emphasis concentrated on the behavior of RK and Adams methods in the discretization of integral equations, either directly or by reformulating them as initial-value problems. Runge–Kutta methods were shown to be very flexible and strong because they have a one-step structure, well-developed error control systems, and are very stable when implicit schemes are used. Adams multistep techniques, on the other hand, showed high computational efficiency because they needed fewer function evaluations to get the same level of accuracy. Their predictor-corrector framework allows for effective long-term integration, but they have problems with stiffness handling and initialization requirements. The first part of the study will provide general information about both methods. The next part will examine the strengths and weaknesses of the methods in solving integral equations. In the second part of the methodology, the methods will be used in solving integral equations.The study concludes that the selection between RK and Adams methods should be determined by the particular characteristics of the Volterra equation being addressed, utilizing theoretical comparison, stability analysis, and demonstrative numerical examples. RK methods are better for problems with nonlinearity, stiffness, or quickly changing kernels. Adams methods are better for smooth, non-stiff, long-interval calculations. The results enhance comprehension of numerical methods for integral equations and offer practical direction for researchers and practitioners.

Keywords: Integral equations, equations, volterra integral equations, Runge-Kutta method, Adams Multistep method.

INTRODUCTİON

While integral equations are broadly characterized as equations where the unknown function appears under the integral sign, this description inadequately captures their true nature. It is unfeasible to construct a singular and comprehensive theory encompassing all integral equations based on that definition. Integral equations exhibit several forms, necessitating individual evaluation based on their structure, kernel function, boundary conditions, and application domain. This renders the subject a vast area of inquiry, necessitating an in-depth examination of diverse categories, particularly Volterra and Fredholm integral equations. Numerous issues commonly faced in physics and engineering are represented by equations where the unknown function appears beneath the integral sign. Such equations enable the formulation of models wherein the answer is encompassed by both the equation and the integral (Brunner, 2017). One of the primary advantages of integral equations over differential equations is their lack of necessity for supplementary boundary or initial conditions. Although numerous further criteria are required to resolve differential equations, integral equations inherently define the problem. This trait is regarded as a primary element that broadens their scope of applicability. The initial systematic investigations into integral equations emerged in the early 19th century. While the investigations conducted in the initial eras were largely random and unstructured, the latter half of the century yielded more methodical and theoretical outcomes in this domain. In 1823, Norwegian mathematician Niels Henrik Abel first encountered the integral equation while examining a mechanical problem, hence catalyzing advancements

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

in this subject. In 1826, Abel ascertained the general form of the integral equation with specific parameters and provided its solution. A specific variant of this equation pertains to the "toutochrone" problem, historically significant and initially examined by Huygens. The formal introduction of integral equations in scientific literature is linked to Du Bois Reymond's paper in 1888. In 1822, Fourier examined the issue of heat transmission through trigonometric series and ultimately identified links articulated as integral equations. These studies were the initial instances demonstrating that integral equations are significant in both theoretical and applied mathematics. The systematic examination of Volterra-type integral equations was conducted by Vita Volterra. This Italian mathematician, active from 1860 to 1940, conducted substantial study on equations involving variables at the upper limit of the integral, which subsequently became known as "Volterra integral equations" (Tang, et.al.,2008,p.830). These equations hold a significant position in the realm of application since they articulate the progression of physical processes in a time-dependent fashion. The significant category of Fredholm integral equations was initially examined in 1900 by the Swedish mathematician Erik Ivan Fredholm. In 1903, Fredholm introduced novel methodologies for addressing integral equations, emphasizing the characteristics of the kernel function and disseminated significant findings on the topic. These works, besides extending Volterra's equations, significantly contributed to the development of the general theory of integral equations (Mısır and Alsalih, 2025).

LİTERATURE REVİEW

Current status of research on Volterra integral equations of the second kind

A Volterra integral equation of the second kind has the general form

$$y(t) = f(t) + \int_a^t K(t,s)y(s)ds, t \in [a,b] \quad (1)$$

where f is a given function, K " is the kernel, and y is the unknown function. Such equations are "causal" because the value at t depends only on earlier values $s \leq t$ (Jung, S. M., 2007). They appear naturally in hereditary systems, population dynamics, viscoelasticity, heat conduction with memory, and many other fields.

Classical research focuses on existence and uniqueness of solutions. Under mild assumptions such as continuity of K and f on the triangle

$$\{(t,s): a \leq s \leq t \leq b\}, \quad (2)$$

and boundedness of K, the equation can be written as a fixed-point problem in a Banach space (typically $C[a,b]$):

$$(\mathcal{V}y)(t) = f(t) + \int_a^t K(t,s)y(s)ds \quad (3)$$

If $\mathcal{V}$ is a contraction, then Banach's fixed-point theorem ensures a unique solution y. Even when the operator is not a contraction on the full interval, local existence can be obtained and extended step by step (Vainikko, G., 2009). Another classical direction is the study of resolvent kernels. One seeks a kernel $R(t,s)$ such that the solution can be represented as

$$y(t) = f(t) + \int_a^t R(t,s)f(s)ds \quad (4)$$

where R satisfies its own integral equation involving K. This leads to series representations (Neumann series)

$$R(t,s) = K(t,s) + \int_a^t K(t,\tau)K(\tau,s)d\tau + \cdots \quad (5)$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

which are convergent under suitable smallness conditions on K.

Many recent studies on regularity and singular kernels investigate equations with weak singular kernels, for example:

$$K(t,s) \sim (t-s)^{-\alpha}, 0 < \alpha < 1 \quad (6)$$

In this case, which is important in fractional calculus and anomalous diffusion, the integral operator approximates a fractional integral, and elementary analysis uses tools from fractional Sobolev spaces and fractional derivatives (Riemann-Liouville, Caputo) (Aktash, 2025,p.105). The regularity of the solution, its stability against data perturbations, and asymptotic behavior are active areas of research.

The Nonlinear Volterra integral equations are as follows:

$$y(t) = f(t) + \int_0^t K(t,s,p(s))da \quad (7)$$

It is also frequently pinned. The length of K can depend on g(s). The amplitude and uncertainty are handled by variants of the feedpoint theorems (Earach, Schaudar) and Monctane operator theory. Applications of the arts in nonlinear models of cell proliferation, branched chemical reactions, and mammalian neuronal processes.

From a numerical stand point, the current research is rich and clweres

Product-hategration and quadrature methods -The Integral is approximated using numerical quadrature nulse. For a uniform gid $t_n = a + nh$,

$$\int_a^{t_n} K(t_n,a)y(a)ds = s\sum_{j=0}^{n} \omega_n, K\big(t_n,t_j\big)b\big(t_j\big), \quad (8)$$

where MnJ are quadrature weights [ag, baeed on trepenaldal or Srrpeon fermulas]. Thes lasds to Iswer triang der eyderne that can be schexd affidently step by step.

Collocation method

$$y_N(t) = \sum_{k=1}^{N} c_x\phi_k(t) \quad (9)$$

and applies the integral equation at selected occlusion points. Recent research studies convergence in various norms, superconvergence at nodes, and adaptive call placement strategies.

Runge-Kutta and multistep type methods

Volterra integral equations can be transformed into equivalent initial-value problems (IVPs) or directly discretized using Runge-Kutta (RK) or Adams-type methods designed specifically for integral equations (Lichae, et.al.,2018).

Spectral and wavelet methods

High-order and spectral convergence can be achieved by expanding y and K in orthogonal polynomials (e.g., Chebyshev, Legendre) or wavelets. Research focuses on handling non-smooth kernels, adaptivity, and fast transforms to reduce computational cost (Wen, et.al.,2024,p.42).

Inverse and parameter identification problems

A modern line of work considers inverse Volterra problems, where one seeks to determine K or f from partial observations of y. This is important in system identification and control. Such problems are typically ill-posed; regularization methods (Tikhonov, iterative regularization, sparsity-promoting penalties) play a central role.

Stochastic Volterra equations

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

When random perturbations are present, one obtains stochastic Volterra integral equations, often written as

$$y(t) = f(t) + \int_a^t K(t,s,y(s))ds + \int_a^t G(t,s,y(s))dW(s), \quad (10)$$

with W a Wiener process. This is an evolving research area combining stochastic analysis and integral equations.

Theoretical basis of the importance of the Runge-Kutta method equations of the form

$$v'(t) = f(t,t(t)), v(t_n) = t_n. \quad (11)$$

The bask idea is to approdimate the saltion at $\mathrm{t_{n+1}} = \mathrm{f_s} + \mathrm{h}$ using exwal avaluetiens of f wittin a single step sultably comblined to achiwe a chrown ordar of acouracy (Qatanani, 2014).

As a general formulation, we can write the RK method as follows:

$$\begin{aligned} k_i &= f\left(t_n + e_1 h_1 w_n + h\sum_{j=1}^{n} a_1 k_j\right), i = 1, \dots, a_1 \\ w_{n+1} &= w_n + h\sum_{k=1}^{n} b_1 k_1. \end{aligned} \quad (12)$$

The coefficients a_{ij}, b_i, c_i are typically collected in a Butcher tableau

c_1	a_{11}	$\cdots$	a_{1s}
$\vdots$	$\vdots$		$\vdots$
c_s	a_{s1}	$\cdots$	a_{ss}
	b_1	$\cdots$	b_s

If the matrix $\boldsymbol{A} = \left(a_{ij}\right)$ is strictly lower triangular, the method is explicit; otherwise it is implicit. The theoretical foundation of RK methods is based on matching the Taylor expansion of the numerical solution with that of the exact solution. The local truncation error $\boldsymbol{T_{n+1}}$ behaves like

$$\tau_{n+1} = \frac{C_{p+1}}{(p+1)!} h^{p+1} + O(h^{p+2}), \quad (13)$$

where $\boldsymbol{p}$ is the order of the method. To obtain order $\boldsymbol{p}$, the coefficients must satisfy a system of algebraic order conditions derived from expanding both the exact solution and the RK update in powers of h. A compact and systematic way to derive and study these conditions uses Butcher's rooted trees. Each rooted tree corresponds to a specific combination of derivatives of f, and the method's coefficients must satisfy certain identities corresponding to trees up to a given order. This combinatorial structure is a key theoretical aspect of RK methods (Capobianco, et.al.,2007). The stability of RK methods is analyzed by applying them to the test equation

$$y' = \lambda y, \lambda \in \mathbb{C}, \quad (14)$$

to obtain a stability function $R(z)$, with $z = h\lambda$. For an RK method, applying it to the test problem yields,

$$y_{n+1} = R(z)y_n. \quad (15)$$

The stability region is

$$\mathcal{S} = \{z \in \mathbb{C} \cdot |R(z)| \leq 1\}. \quad (16)$$

For stiff equations, one desires A-stable methods, where the entire left half-plane $\{z: \Re(z) < 0\}$ is contained in $\mathcal{S}$. Some implicit RK methods, such as Gauss-Legendre collocation methods, are A-stable and even L-stable, making them highly suitable for stiff problems.

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

Based on a review of various scientific literature, the importance of RK methods is confirmed by a number of theoretical and practical arguments. Being one-step methods, RK schemes do not require storing multiple past steps. This simplifies analysi implementation, and error control, and makes them suitable for problems with variable step size. Classical methods such as the 4th-order Runge-Kutta (RK4) provide a good balance between accuracy and computational effort. More sophisticated high-order explicit and implicit RK methods can be constructed for specialized applications. RK methods do not rely on specific structure of the ODE; they can be applied to nonautonomous systems, nonlinear systems, and systems arising from PDE discretizations (Rabiei, et.al.,2023). The Butcher theory of rooted trees, B-series, and geometric integration extends the RK framework to structure-preserving methods (symplectic RK, partitioned RK). These preserve invariants or geometric properties (e.g., symplectic structure in Hamiltonian systems), which is critical in long-time simulations. Initial value problems and certain Volterra integral equations can be discretized through RK ideas, making the RK framework a bridge between differential and integral equations. Thus, the Runge-Kutta method is central in numerical analysis due to its rigorous order theory, flexible design, stability properties, and wide applicability to both stiff and non-stiff problems (Pandey and Singh, 2021).

Theoretical basis of the importance of the Adams multistep method

Adams methods belong to the broader family of linear multistep methods (LMMs), which approximate the solution of

$$y'(t) = f(t, y(t)), y(t_0) = y_0, \qquad (17)$$

by combining information from several previous time steps. In contrast to one-step methods, they exploit past values to achieve higher accuracy with fewer function evaluations per step (Mohamed,2016).

Linear multistep formulation a general k-step LMM has the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, f_{n+j} = f\left(t_{n+j}, y_{n+j}\right), \qquad (18)$$

where the coefficients α_j, β_j define the method. For Adams methods, $\alpha_k = 1, \alpha_0 \neq 0$, and the β_j are constructed using interpolating polynomials for f.

An Adams-Bashforth method is an explicit multistep method: $\beta_k \neq 0$ and β_k multiplies a known f_n. while $\beta_k = 0$ for y_{n+k} in the derivative part. The standard k-step Adams-Bashforth formula is (Ibrahimov and Imanova, 2021)

$$y_{n+1} = y_n + h \sum_{j=0}^{k-1} \beta_j^{(AB)} f_{n-j}, \qquad (19)$$

where the coefficients $\beta_j^{(AB)}$ are chosen to integrate the interpolating polynomial for f through the points $t_n, t_{n=1}, \dots, t_{n=k+1}$.

An Adams-Moulton method is implicit

$$y_{n+1} = y_n + h \sum_{j=0}^{k} \beta_j^{(AM)} f_{n+1-j}, \qquad (20)$$

including the (unknown) $f_{n+1} = f(t_{n+1}, y_{n+1})$. These methods often serve as correctors in predictorcorrector pairs.

The theoretical importance of Adams methods is grounded in the general convergence theorem for LMMs-a linear multistep method is convergent if and only if it is consistent and zero-stable (Shafiyeva,et.al.,2024).

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

Consistency-Requires that the local truncation error τ_{n+1} tends to zero as $h \to 0$ and that the method reproduces exact solutions for simple test problems (e.g., constant and linear functions). In terms of coefficients, consistency is encoded in the conditions

$$\sum_{j=0}^{k} \alpha_j = 0, \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j. \quad (21)$$

Zero-stability-The characteristic polynomial of the method

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^f \quad (22)$$

must satisfy the root condition:

- All roots ξ_1 satisfy $|\xi_1| \leq 1$;
- Any root with $|\xi_1| = 1$ is simple.

For Adams methods, the characteristic polynomial has a simple root at $\xi = 1$ and all other roots inside the unit disk, ensuring zero-stability for all standard orders.

Convergence-If the LMM is consistent and zero-stable, and if f satisfies a Lipschitz condition, then the numerical solution converges to the exact solution as $h \to 0$. For Adams methods of order p, the global error behaves like $O(h^{\mathrm{p}})$.

Efficiency and high order-Adams methods are particularly important because they achieve high order with relatively few function evaluations per step. For example, a 4-step explicit Adams-Bashforth method has order 4 but requires only one new function evaluation per step. In contrast, a 4th-order explicit RK method uses four function evaluations per step. This makes multistep methods attractive for large-scale problems where each function evaluation is expensive (e.g. systems derived from PDEs).

The combination of a k-step Adams-Bashforth method (predictor) and a k-step Adams-Moulton method (corrector) provides a powerful predictor-corrector scheme:

1. Predictor

$$\bar{y}_{n+1} = y_n + h \sum_{j=0}^{k-1} \beta_j^{(AB)} f_{n-j}. \quad (23)$$

2. Corrector

$$y_{n+1} = y_n + h\left(\beta_0^{(AM)} f(t_{n+1}, \bar{y}_{n+1}) + \sum_{j=1}^{k} \beta_j^{(AM)} f_{n+1-j}\right). \quad (24)$$

This framework allows for error estimation and adaptive step-size control, since the difference between predictor and corrector offers a cheap local error indicator.

The theoretical and practical importance of Adams' multi-step methods is already significant. We have already discussed this issue extensively above. In short, we can note that their high efficiency for non-hard problems requiring long integration intervals, their clear stability and convergence theory using LMM concepts, their role in adaptive stepwise algorithms and predictor-corrector schemes, and their adaptability to systems coupled with Volterra integral equations when reformulated as integral terms, delay equations, and even ODE systems are among the main advantages (Bulatov and Budnikova,2025).

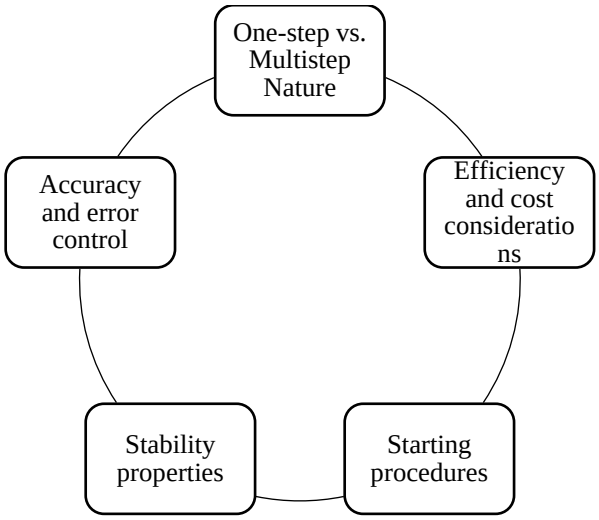
Comparison of Runge-Kutta and Adams multistep methods: advantages and disadvantages

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

Both Runge-Kutta and Adams multistep methods are fundamental for numerically solving ODEs and related integral equations, but they have different characteristics.

The Runge-Kutta (RK) family and the Adams multistep methods represent two major classes of numerical techniques for solving ordinary differential equations. Although both aim to approximate the solution of initial value problems, they follow fundamentally different principles. We will analyze and compare their differences on several dimensions, such as step structure, efficiency, initial requirements, stability, accuracy, and implementation complexity, below.

Scheme 1. Analysis and comparison of differences in various dimensions



Source: Prepared by the author himself as a result of research

One-step vs. multistep Nature

Runge-Kutta methods are one-step schemes, meaning that the computation of y_{n+1} depends solely on the most recent value y_n. As a result, RK methods do not require storing long histories of earlier approximations, which makes them particularly convenient for problems in which adaptive step-size control is necessary. The absence of dependence on multiple prior steps also simplifies the structural complexity of the algorithm. In contrast, Adams methods belong to the class of multistep schemes. A k-step Adams method computes y_{n+1} using previous values $y_n, y_{n-1}, \dots, y_{n-k+1}$ along with their corresponding derivative evaluations. This structure requires maintaining a record of past data, and implementing variable step sizes becomes more complicated because coefficients must be recomputed whenever the steps change.

Efficiency and cost considerations

Explicit RK methods typically require several function evaluations per step. For an order $\boldsymbol{p}$ RK method, the number of stages is usually $s \approx p$, making each computational step relatively costly. However, the efficiency of RK methods is enhanced by the availability of embedded pairs-such as the Dormand-Prince RK45 scheme-which provide reliable error estimation at almost no additional cost. This makes RK schemes highly attractive for practical applications involving automatic step-size adjustment (Al-Shimmary, et.al.,2021). In contrast, Adams methods especially Adams—Bashforth (explicit) schemes are computationally efficient because they require only one new function evaluation per step while achieving high order. A k-step AdamsBashforth method reaches order k but still evaluates the function only once per

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

iteration. This feature is advantageous for large-scale systems or for problems in which evaluating $f(t, y)$ is expensive.

Starting procedures

A major operational difference lies in how the two methods begin. Runge-Kutta methods are self-starting, meaning they require only the initial value (t_0, y_0) to begin the computation. This simplicity is a major strength. Adams methods, however, are not self-starting. A k-step Adams method requires the initial sequence $y_1, y_2, \ldots, y_{k-1}$, which must be generated using another numerical method-typically an RK scheme. In practice, many modern solvers use RK for the initial steps and switch to Adams for long-term integration (Wang, and Li,2011).

Stability properties

Stability behavior varies significantly between the families. Explicit RK methods have stability regions that depend on the number of stages and the underlying coefficients. They perform well on non-stiff problems, but very small step sizes are required for stiff equations. Implicit RK methods offer stronger stability but at higher computational cost. Explicit Adams-Bashforth methods generally exhibit smaller stability regions compared to explicit RungeKutta methods and are therefore limited to non-stiff problems (Rabiei, et.al.,2019). The implicit Adams-Moulton schemes provide enhanced stability; some lower-order members of this family are even A-stable. As a result, AdamsMoulton correctors are often paired with Adams-Bashforth predictors in predictor-corrector frameworks. Nevertheless, for highly stiff systems, backward differentiation formulas (BDF) or implicit RK methods are typically preferred.

Accuracy and error control

Runge-Kutta methods excel in error control due to embedded method pairs that compute two solutions of different orders simultaneously. This enables efficient and reliable adaptive step-size selection, contributing significantly to their dominance in general-purpose ODE solvers. Adams methods rely on predictor-corrector schemes for error estimation: the difference between the Adams-Bashforth predictor and the Adams-Moulton corrector provides a natural local error indicator. However, adjusting step sizes is more complicated because past values must be recomputed or reinitialized, making adaptive strategies less straightforward than for RK methods.

Table 1. Comparison of Runge-Kutta and Adams Multistep methods

Criterion	Runge-Kutta Methods	Adams Multistep Methods
Type of method	One-step method: y_{n+1} depends only on y_n	Multistep method: y_{n+1} depends on several previous values $y_n, y_{n-1}, \ldots, y_{n-k+1}$
Memory requirements	No history storage needed	Requires storing past solutions and derivative values
Variable step size	Very easy to implement	Possible but requires recomputing coefficients; more complex
Function evaluations per step	Typically $s \approx p$ evaluations for order p	Only one new evaluation per step (Adams-Bashforth)
Efficiency	Less efficient per step but highly efficient with adaptive control	Very efficient for long, smooth, non-stiff problems
Starting procedure	Self-starting	Not self-starting; needs $k-1$ additional starting values (often computed using RK)

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

Stability (explicit)	Good for non-stiff problems; poor for stiff problems	Smaller stability region; restricted to non-stiff problems
Stability (ımplicit)	Implicit RK very stable but costly	Adams-Moulton more stable; some A-stable at low orders
Error control	Excellent via embedded RK pairs (e.g, RK45)	Predictor-corrector difference used; more complicated with variable step sizes
Implementation complexity	Simple, well-structured; based on Butcher tableau	More complex; requires history management and start-up procedures
Best use cases	General-purpose solvers, stiff or non-stiff ODEs, adaptive integration	Long-term integration of smooth, non-stiff problems where function evaluation is expensive
Main advantages	Self-starting, strong error control, easy adaptivity	High-order accuracy with low cost per step
Main disadvantages	More function evaluations per step	Requires initialization, more delicate stability, harder adaptivity

The table shows the use of both methods in Volterra integral equations of the second kind and the results of this use. In addition, the table shows the positive and negative aspects of the use of the methods.

Source: Prepared by the author himself as a result of research

For a better understanding of the comparative part, let's look at Table 1 above, which shows the different characteristics of the method, as well as its advantages and disadvantages.

Methodology

In the context of the study of Volterra integral equations of the second kind and the comparison of Runge-Kutta and Adams multistage methods, we will structure a typical research methodology as follows:

We consider a Volterra integral equation of the second kind

$$y(t) = f(t) + \int_a^t K(t,s)y(s)ds, t \in [a,b] \quad (25)$$

with assumed regularity conditions:

- $f \in C[a,b]$;
- $K \in C(\Delta)$, with $\Delta = \{(t,s) \mid a \le s \le t \le b\}$;
- possibly additional structure (e.g., weakly singular kernel).

We want to find a numerical approximation for the solution y(t) and compare the Runge-Kutta and Adams methods. We can do this by either changing the problem into an initial value ODE system or directly discretizing the integral equation using RK-type quadrature ideas and multistep schemes that work with Volterra equations.

Discretization

Choose a partition of $[a,b]$:

$$a = t_0 < t_1 < \cdots < t_N = b, h_n = t_{n+1} - t_n, \quad (26)$$

with either uniform step h or variable steps h_n –

Runge-Kutta-based discretization

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

One approach is to differentiate the Volterra equation with respect to t when permissible, yielding an ODE with a memory term. Alternatively, one may treat the integral as part of the right-hand side and approximate it using quadrature within each RK stage.

A general RK discretization for the IVP

$$y'(t) = F(t, y(t)), y(a) = y_0 \tag{27}$$

would be

$$\begin{aligned} k_i &= F\left(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij}k_j\right), \\ y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i. \end{aligned} \tag{28}$$

If we define

$$F(t, y) = \frac{\partial}{\partial t}\left(f(t) + \int_a^t K(t, s)y(s)ds\right) \tag{29}$$

suitable numerical approximation of the derivative and integral yields a computable scheme.

Adams-type discretization

For a reformulated IVP, we apply a k-step Adams-Bashforth or Adams-Moulton method. Assuming we have approximations $y_n, y_{n-1}, \ldots, y_{n-k+1}$, we compute:

Predictor (Adams-Bashforth)

$$\tilde{y}_{n+1} = y_n + h\sum_{j=0}^{k-1} \beta_j^{(AB)} F_{n-j}, \tag{30}$$

Corrector (Adams-Moulton)

$$y_{n+1} = y_n + h\left(\beta_0^{(AM)} F(t_{n+1}, \bar{y}_{n+1}) + \sum_{j=1}^{k} \beta_j^{(AM)} F_{n+1-j}\right). \tag{31}$$

When you apply the integral term directly to Volterra equations, you get discrete convolution-like sums that look like multistep methods.

Error analysis

The methodology should include:

- Derivation of the local truncation error for each scheme.
- Determination of the global convergence order by using standard tools (Gronwall inequalities, stability estimates).
- For Volterra equations, use of integral inequalities to bound the error:

$$|e(t)| \le C\left(h^p + \int_a^t |e(s)|ds\right), \tag{32}$$

which yields, via Gronwall's lemma,

$$|e(t)| \le Ch^{\mathrm{P}} e^{C(t-a)}. \tag{33}$$

Stability analysis

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

Apply both methods to test problems with known stability properties (e.g., linear Volterra equations or equivalent ODEs) and determine the stability regions of the step sizes for each method, the behavior for hard kernels or rapidly changing kernels, and the sensitivity to perturbations in the initial data or kernel.

Numerical experiments

A set of test equations will be designed with known or high-accuracy reference solutions to investigate. Convergence rates (log-log plots of error versus step size), accuracy comparisons with CPU time between the RK and Adams methods, and the impact and performance of step size adaptation in the presence of single kernels or nonlinearities are important.

This systematic methodology forms the basis of a rigorous comparative study. Runge-Kutta and Adams multi-step methods will be used to solve the equations.

Example with Runge-Kutta (classical RK4)

Consider the IVP

$$y' = y - t^2 + 1, y(0) = 0.5$$

which has the exact solution

$$y(t) = (t+1)^2 - \frac{1}{2}e^t.$$

Let $h > 0, t_n = nh$. The dassical 4th-order Runge-Kutta method (RK4) is

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right) \\ k_4 &= f(t_n + h, y_n + hk_3) \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \tag{34}$$

For our equation, $f(t, y) = y - t^2 + 1$. For example, with $h = 0.1$ and $n = 0$:

- $t_0 = 0, y_0 = 0.5.$
- $k_1 = f(0{,}0.5) = 0.5 - 0 + 1 = 1.5.$
- $k_2 = f(0.05{,}0.5 + 0.05 \cdot 1.5) = f(0.05{,}0.575) = 0.575 - 0.05^2 + 1.$
- $k_{zz} = f(0.05{,}0.5 + 0.05 \cdot k2)$, and so on.

Iterating this yields y_n approximations which converge to the exact solution as $h \to 0$ with global error $O(h^4)$.

Example with Adams-Bashforth and Adams-Moulton (predictor-corrector)

Use the same IVP and consider a 2-step Adams-Bashforth predictor and 2-step Adams-Moulton corrector:

2-step Adams-Bashforth (AB2):

$$y_{n+1}^{(p)} = y_n + \frac{h}{2}(3f_n - f_{n-1}). \tag{35}$$

2-step Adams-Moulton (AM2):

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

$$y_{n+1} = y_n + \frac{h}{12}\left(5f_{n+1}^{(p)} + 8f_n - f_{n-1}\right). \quad (36)$$

Here, $f_n = f(t_n, y_n)$ and $f_{n+1}^{(p)} = f\left(t_{n+1}, y_{n+1}^{(p)}\right)$.

Because this is a 2 -step method, we need y_0 and y_1. We can compute y_1 using RK4 or another one-step method. Once $\boldsymbol{y_1}$ is available, the procedure is:

1. For each step $n \geq 1$, compute predictor $y_{n+1}^{(p)}$ using AB2.
2. Evaluate $f_{n+1}^{(p)} = f\left(t_{n+1}, y_{n+1}^{(p)}\right)$.
3. Compute corrected value y_{n+1} using AM2.

This predictor-corrector scheme is of order 2, and in practice it can be more efficient than an RK2 method because after the starting phase only one new function evaluation is required per step.

Example related to Volterra integral equation

Consider the Volterra integral equation

$$y(t) = 1 + \int_0^t (t-s)y(s)ds \quad (37)$$

Differentiating with respect to t gives

$$\begin{aligned} \frac{d}{dt}y(t) &= 0 + \int_0^t \frac{\partial}{\partial t}[(t-s)y(s)]ds + (t-t)y(t) \\ &= \int_0^t y(s)ds \end{aligned} \quad (38)$$

Let

$$z(t) = \int_0^t y(s)ds, z'(t) = y(t) \quad (39)$$

then the system becomes

$$\begin{cases} y'(t) = z(t), \\ z'(t) = y(t), \end{cases} \quad (40)$$

with initial conditions $y(0) = 1$ and $z(0) = 0$. This leads to the linear system

$$\mathbf{Y}'(t) = A\mathbf{Y}(t), \mathbf{Y}(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (41)$$

We can now apply either Runge-Kutta or Adams methods to this system:

Using RK4, at each step we treat $\mathbf{Y}' = A\mathbf{Y}$ as the ODE and compute stages

$$K_i = A\left(\mathbf{Y}_n + h\sum_{j=1}^{s} a_{ij}K_j\right), \quad (42)$$

then

$$\mathrm{Y}_{n+1} = \mathrm{Y}_n + h\sum_{i=1}^{n} b_i K_i. \quad (43)$$

Using an Adams-Bashforth method, we update

$$\mathrm{Y}_{n+1} = \sum_{j=0}^{k-1} \alpha_j \mathrm{Y}_{n-j} + h\sum_{j=0}^{k-1} \beta_j A\mathrm{Y}_{n-j}, \quad (44)$$

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

where α_j, β_j correspond to the chosen 1 mm . We can test how well the RK and Adams methods work for equations that come from Volterra integral forms by comparing the numerical results to the exact solution (which can be found analytically in this linear case).

CONCLUSİON AND DİSCUSSİON

The comparative analysis of Runge–Kutta and Adams multistep methods in addressing Volterra integral equations of the second kind underscores several critical theoretical and computational differences. The research illustrated that both categories of numerical algorithms serve as robust instruments for addressing integral equations in physics and engineering , however, their efficacy is significantly influenced by the structural characteristics of the specific problem. Runge-Kutta methods are very flexible when it comes to dealing with variable step sizes, nonlinearity, and complex dynamical behavior because they only need one step. Their strong theoretical foundation, especially in terms of order conditions, stability theory, and Butcher tree formulations, lets them work well in a wide range of situations. This means that RK schemes are especially good for problems where being able to adapt and being strong are very important. Adams multistep methods are very efficient when it comes to calculations, especially when it comes to long-term integration or large-scale systems. One big benefit is that they can reach high order with only a few function evaluations per step. However, this efficiency has some drawbacks: multistep schemes can't start on their own, have more limited stability properties, and don't work as well for stiff or rapidly changing problems unless implicit variants are used. When it comes to Volterra equations with smooth kernels, Adams–Bashforth and Adams–Moulton combinations can be cheaper to compute than RK methods. However, when it comes to equations with singular kernels or strong memory effects, Runge–Kutta-based discretizations tend to give more stable and accurate results.

RESULT

This study's computational and theoretical analysis produces several significant findings. First, the Runge–Kutta method showed that it was always accurate for a wide range of Volterra integral equations, especially those that were nonlinear or changed quickly. The method was able to stay accurate even when step sizes were fairly large because of its stability, especially when implicit variants were used. The RK4 scheme, in particular, gave fourth-order convergence and error behavior that could be predicted. When used on smooth, non-stiff Volterra problems, the Adams multistep methods made very good numerical approximations. The Adams–Bashforth predictor and the Adams–Moulton corrector worked together to get accurate results with a lot fewer function evaluations than RK methods. However, performance got worse when the kernel showed singularities or when stiffness went up, which was what was expected.

REFERENCES

[1] Aktaş, F. B. (2025). İntegral ve İntegro Tipi Denklemlerin Çözüm Yöntemleri.
[2] Al-Shimmary, A. F., Hussain, A. K., & Radhi, S. K. (2021, March). Numerical solution of Volterra integro–differential equation using 6th order Runge-Kutta method. In *Journal of Physics: Conference Series* (Vol. 1818, No. 1, p. 012183). IOP Publishing.
[3] Brunner, H. (2017). *Volterra integral equations: an introduction to theory and applications* (Vol. 30). Cambridge University Press.
[4] Bulatov, M. V., & Budnikova, O. S. (2025). Extrapolation Multistep Methods for Numerical Solution of Linear Second-Order Differential Algebraic Equations. *Numerical Analysis and Applications*, *18*(2), 101-118.
[5] Capobianco, G., Conte, D., Del Prete, I., & Russo, E. (2007). Fast Runge–Kutta methods for nonlinear convolution systems of Volterra integral equations. *BIT Numerical Mathematics*, *47*(2), 259-275.
[6] Ibrahimov, P., & Imanova, M. (2021). Multistep methods of the hybrid type and their application to solve the second kind volterra integral equation. Symmetry. 2021; 13: 1087. *Integral Equations*, 17.
[7] Ibrahimov, V., & Imanova, M. (2021). Multistep methods of the hybrid type and their application to solve the second kind Volterra integral equation. *Symmetry*, *13*(6), 1087.
[8] Jung, S. M. (2007). A fixed point approach to the stability of a Volterra integral equation. *Fixed Point Theory and Applications*, *2007*(1), 057064.

International Journal of Engineering Research & Technology (IJERT)
ISSN: 2278-0181
Vol. 15 Issue 05 , May - 2026

[9] Qatanani, N. (2014). *Analytical and Numerical Solutions of Volterra Integral Equation of the Second Kind* (Doctoral dissertation, Faculty of Graduate Studies Analytical and Numerical Solutions of Volterra Integral Equation of the Second Kind By Feda'Abdel Aziz Mustafa Salameh Supervisor Prof. Naji Qatanani This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Computational Mathematics, Faculty of Graduate Studies, An-Najah National University).

[10] Lichae, B. H., Biazar, J., & Ayati, Z. (2018). A class of Runge–Kutta methods for nonlinear Volterra integral equations of the second kind with singular kernels. *Advances in Difference Equations*, *2018*(1), 349.

[11] Mısır, A., & Alsalih, İ. (2025). Volterra integral denklemlerinin ve Volterra integro-diferensiyel denklemlerinin $\boldsymbol{Gr}$-dönüşümü kullanılarak çözümü. *Gazi Üniversitesi Fen Fakültesi dergisi (Online)*, *6*(1).

[12] Mohamed, N. A. B. (2016). *Multistep Block Methods for Solving Volterra Integro-Differential Equations of Second Kind* (Doctoral dissertation, Master's thesis, School of Graduate Studies, Universiti Putra Malaysia).

[13] Pandey, R. K., & Singh, H. (2021). An Efficient Numerical Algorithm to Solve Volterra Integral Equation of Second Kind. In *Topics in Integral and Integro-Differential Equations: Theory and Applications* (pp. 215-228). Cham: Springer International Publishing.

[14] Rabiei, F., Abd Hamid, F., Md Lazim, N., Ismail, F., & Abdul Majid, Z. (2019). Numerical solution of Volterra integro-differential equations using improved Runge-Kutta methods. *Applied Mechanics and Materials*, *892*, 193-199.

[15] Rabiei, F., Abd Hamid, F., Rashidi, M. M., Ali, Z., Shah, K., Hosseini, K., & Khodadadi, T. (2023). Numerical Simulation of Fuzzy Volterra Integro-differential Equation using Improved Runge-Kutta Method. *Journal of applied and computational mechanics*, *9*(1), 72-82.

[16] Shafiyeva, G. K., Ibrahimov, V. R., & Juraev, D. A. (2024). On Some Comparison of Adam's Methods with Multistep Methods and Their Application to Solve Initial-Value Problems for First-Order ODEs. *Karshi Multidisciplinary International Scientific Journal*, *1*(2), 181-188.

[17] Tang, T., Xu, X., & Cheng, J. (2008). On spectral methods for Volterra integral equations and the convergence analysis. *Journal of Computational Mathematics*, 825-837.

[18] Vainikko, G. (2009). Cordial Volterra integral equations 1. *Numerical Functional Analysis and Optimization*, *30*(9-10), 1145-1172.

[19] Wang, W., & Li, D. (2011). Stability analysis of Runge-Kutta methods for nonlinear neutral Volterra delay-integro-differential equations. *Numerical Mathematics: Theory, Methods and Applications*, *4*(4), 537-561.

[20] Wen, J., Huang, C., & Guan, H. (2024). Two-step Runge–Kutta methods for Volterra integro-differential equations. *International Journal of Computer Mathematics*, *101*(1), 37-55.