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# Robust Estimator for Continuous Time Non-Linear Systems

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***Abstract* — In real life scenario, we are dealing with the non-linear dynamic systems which may be mechanical, electrical, and aerospace and many others systems including aerospace. Identification and parameter estimation of this non-linear dynamic system is challengeable and have high scope nowadays. The Kalman filter (KF), amongst other methods, has been rather successful in many such applications, however it requires a sort of complete model description of the dynamics of the system [1].The method of invariant embedding will be useful in determining accurate invariant model in two steps; one is, determining the model error and another step is fitting a model to the model error using Least square method. Still Invariant embedding method does not yield robust estimation of the model error. Here H-Infinity norms are used to introduce robustness in the estimation as H-∞ norms places bound on the energy gain from unknown deterministic input (error) to the filter error.**

***Index Terms – Least square method, Signal norms, Invariant embedding method, kalman filtering.***



## I. INTRODUCTION

In many real life situations we need accurate identification of nonlinear terms (parameters) in the model of a dynamic system. Generally, for all practical applications the parameter estimation methods is applied to nonlinear problems. As such, KF cannot determine the deficiency or discrepancy in the model of the system used in the filter, since it pre-supposes availability of accurate state-space model [1]. Consider a situation to find the state estimates for the given measurements from a nonlinear dynamic system. In this case, extended Kalman filter is used and the knowledge of the nonlinear functions is required. Any discrepancy in the model will cause model errors that will tend to create mismatch of the estimated states with the true state of the system. In KF, this is usually handled by including the process noise term Q. This method generally works, but it still could have some problems [3, 4]: i) deviation from the Gaussian assumption might degrade the performance of the algorithm, and ii) the filtering algorithm is dependent on the covariance matrix P of the state estimation error, since this is used for computation of Kalman gain K. Actually, the usage of process noise term in the filter doesn't improve the model since the model can be deficient, but it can help in getting good match of the states. The estimates depend more on the current measurements.

The foregoing limitations of the KF can be overcome largely by using the method based on principle of model error [3]. This approach not only estimates the states of the dynamic system from its noisy measurements, but also the model discrepancy as a time history. The known (deficient or linear) model is used in the state estimation procedure, and deterministic discrepancy of the model is determined, using the measurements in the model error estimation procedure. Once the discrepancy-time history is available, we can fit another model to it and estimate its parameters using regression method. Then adding the previously used model in the state estimation procedure and the new additional model which is developed would yield the accurate model of the underlying (nonlinear) dynamic system, which has generated the data. The main aim of the proposed work is to provide a link between estimation of deterministic model error by TPBVP/IE and HI norm to arrive at robust estimator. The structure of the HI filter has some similarity with the structure of the KF. The HI filter places a bound on the energy gain from the deterministic error-inputs to the filter error output. The approach is expected to produces accurate state trajectory, even in the presence of deficient/inaccurate model and additionally identifies the unknown model (form) as well as its parameters even with some uncertainties due to the property of robustness with H-infinity norms.

## II. BLOCK DIAGRAM

In the below block diagram figure 1, a non-linear dynamic system is considered as the true plant. Deficient model is the system with the errors. To minimize these errors invariant embedding (IE) algorithm and H-Infinity norms are used. Finally after finding the model discrepancy parameter estimation is done by using LS (Least Square).

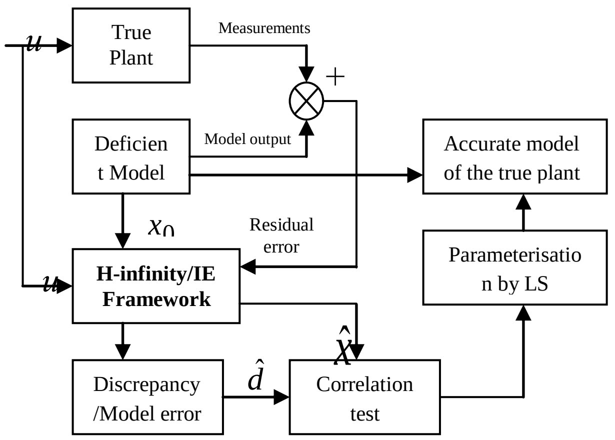


Figure 1: Block Diagram of the Model Error Estimation Algorithm with $\infty$ norms

## III. TWO POINT BOUNDARY VALUE PROBLEM (TPBVP)

Consider the dynamic system as:

$$\dot{x} = f(x(t), u(t), t) \quad ; \quad x(t_0) = x_0 \qquad (1)$$

Defining a composite performance index as:

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} \psi(x(\tau), u(\tau), \tau) d\tau \qquad (2)$$

The first term is the cost penalty on the final value of the state $x(t_f)$. The term $\psi(.)$ is the cost penalty governing the deviation of $x(t)$ and $u(t)$ from their desired time-histories. The aim is to determine the input $u(t)$, in the interval $t_0 \le t \le t_f$, such that the performance index J is minimized, subject to the constraint of eqn. (1), which states that the state should follow integration of eqn. (1) with the input thus determined [1]. Lagrange multiplier is used to handle the constraint within the functional J:

$$J_a = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [\psi(x(\tau), u(\tau), \tau) + \lambda^T(-f(x(\tau), u(\tau), \tau) + \dot{x})] d\tau \qquad (3)$$

Here, $\lambda$ is the Lagrange multiplier or co-state. That is, in the process of determination of $u(t)$ by minimization of $J_a$, the condition of eqn. (1) should not be violated. Eqn. (3) can be rewritten as:

$$J_a = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [H(x(\tau), u(\tau), \tau) - \dot{\lambda}^T(\tau) x(\tau)] d\tau$$

$$+ (\lambda^T x)_{t_f} - (\lambda^T x)_{t_0} \qquad (4)$$

Here, $H = \psi(x(\tau), u(\tau), \tau) - \lambda^T(\tau) f(x(\tau), u(\tau), \tau)$ (5)

H is called Hamiltonian. The term $\int_{t_0}^{t_f} \lambda^T \dot{x} d\tau$ of eqn.(4) is integrated by parts to obtain other terms in eqn.(5).

$$\delta J_a = 0 = \left(\frac{\partial \phi}{\partial x}\delta x\right)_{t_f} + \lambda^T \delta x\Big|_{t_f} - \lambda^T \delta x\Big|_{t_0} + \int_{t_0}^{t_f} \left[\left(\frac{\partial H}{\partial x} - \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right] d\tau \qquad (6)$$

From eqn.(6), the so-called Pontryagin's necessary conditions are obtained:

$$\lambda^T(t_f) = -\left.\frac{\partial \phi}{\partial x}\right|_{t_f} \qquad (7)$$

$$\frac{\partial H}{\partial x} = \dot{\lambda}^T \qquad (8)$$

and $\frac{\partial H}{\partial u} = 0$ (9)

Here, $\delta x(t_0) = 0$, assuming that the initial conditions $x(t_0)$ are independent of $u(t)$. The eqns. (1) and (7)–(9) define the TPBV problem: the boundary condition for state is specified at $t_0$ and for the co-state; $\lambda$ it is specified at $t_f$. From eqn.(7) and (9), we obtain:

$$\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T = -\left(\frac{\partial f}{\partial x}\right)^T \lambda + \left(\frac{\partial \psi}{\partial x}\right)^T \qquad (10)$$

$$\frac{\partial H}{\partial u} = 0 = -\left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial \psi}{\partial u}\right)^T \qquad (11)$$

One method to solve the TPBVP is to start with guesstimate on $\lambda(t_0)$ and use $x(t_0)$ to integrate forward to the final time $t_f$. Then verify the boundary condition $\lambda(t_f) = -\left.\frac{\partial \phi}{\partial x}\right|_{t_f}^T$. If the condition is not satisfied, then iterate once again with new $\lambda(t_0)$ and so on until the convergence of the algorithm is obtained. In the next section, we discuss the method of invariant embedding for solution of the TPBV problem.

## IV. INVARIANT EMBEDDING METHOD

Considering the resultant equations of TPBV problem,

$$\dot{x} = \Phi(x(t), \lambda(t), t) \qquad (12)$$

$$\dot{\lambda} = \Psi(x(t), \lambda(t), t) \qquad (13)$$

We see that $\Phi$ and $\Psi$ are dependent on $x(t)$ and $\lambda(t)$. Hence, here we have general two-point boundary value problem with associated boundary conditions as: $\lambda(0)=a$ and $\lambda(t_f)=b$. Now, though the terminal condition $\lambda(t_f)=b$ and time are fixed, we consider them as free variables. Therefore, this dependency can be represented as:

$$x(t_f)=r(c,t_f)=r(\lambda(t_f),t_f) \qquad (14)$$

with $t_f \to t_f+\Delta t$, we obtain by neglecting higher order terms:

$$\lambda(t_f+\Delta t)=\lambda(t_f)+\dot{\lambda}(t_f)\Delta t=c+\Delta c \qquad (15)$$

We also get, using eqn.(13) in (15)

$$c+\Delta c=c+\Psi(x(t_f),\lambda(t_f),t_f)\Delta t \qquad (16)$$

and therefore, we get

$$\Delta c=\Psi(r,c,t_f)\Delta t \qquad (17)$$

In addition, we get, like eqn. (15):

$$x(t_f+\Delta t)=x(t_f)+\dot{x}(t_f)\Delta t=r(c+\Delta c,t_f+\Delta t) \qquad (18)$$

and hence, using eqn.(12) in (18), we get:

$$r(c+\Delta c,t_f+\Delta t)=r(c,t_f)+\Phi(x(t_f),\lambda(t_f),t_f)\Delta t$$
$$=r(c,t_f)+\Phi(r,c,t_f)\Delta t \qquad (19)$$

Using Taylor's series, we get:

$$r(c+\Delta c,t_f+\Delta t)=r(c,t_f)+\frac{\partial r}{\partial c}\Delta c+\frac{\partial r}{\partial t_f}\Delta t \qquad (20)$$

Comparing eqn. (18) and (19), we get:

$$\frac{\partial r}{\partial t_f}\Delta t+\frac{\partial r}{\partial c}\Delta c=\Phi(r,c,t_f)\Delta t \qquad (21)$$

or, using eqn.(20) in (21), we obtain:

$$\frac{\partial r}{\partial t_f}\Delta t+\frac{\partial r}{\partial c}\Psi(r,c,t_f)\Delta t=\Phi(r,c,t_f)\Delta t \qquad (22)$$

The above equation simplifies to

$$\frac{\partial r}{\partial t_f}+\frac{\partial r}{\partial c}\Psi(r,c,t_f)=\Phi(r,c,t_f) \qquad (23)$$

The eqn. (23) links the variation of the terminal condition $x(t_f)=r(c,t_f)$ to the state and co-state differential functions, see eqn. (12) and (13). Now in order to find an optimal estimate $\hat{x}(t_f)$, we need to determine $r(b,t_f)$:

$$\hat{x}(t_f)=r(b,t_f) \qquad (24)$$

The eqn. (21) can be transformed to an initial value problem by using approximation:

$$r(c,t_f)=S(t_f)c+\hat{x}(t_f) \qquad (25)$$

Substituting eqn. (23) in eqn.(25), we get:

$$\frac{dS(t_f)}{dt_f}c+\frac{d\hat{x}(t_f)}{dt_f}+S(t_f)\Psi(r,c,t_f)=\Phi(r,c,t_f) \qquad (26)$$

Next, expanding $\Phi$ and $\Psi$ about $\Phi(\hat{x},b,t_f)$ and $\Psi(\hat{x},b,t_f)$, we obtain:

$$\Phi(r,c,t_f)=\Phi(\hat{x},b,t_f)+\Phi_{\hat{x}}(\hat{x},b,t_f)(r(c,t_f)-\hat{x}(t_f))$$
$$=\Phi(\hat{x},b,t_f)+\Phi_{\hat{x}}(\hat{x},b,t_f)S(t_f)c \qquad (27)$$

and

$$\Psi(r,c,t_f)=\Psi(\hat{x},b,t_f)+\Psi_{\hat{x}}(\hat{x},b,t_f)S(t_f)c \qquad (28)$$

Utilizing expressions of eqn.(27)and (28), in eqn.(26), we obtain:

$$\frac{dS(t_f)}{dt_f}c+\frac{d\hat{x}(t_f)}{dt_f}+S(t_f)[\Psi(\hat{x},b,t_f)+\Psi_{\hat{x}}(\hat{x},b,t_f)S(t_f)c]$$
$$=\Phi(\hat{x},b,t_f)+\Phi_{\hat{x}}(\hat{x},b,t_f)S(t_f)c \qquad (29)$$

The eqn. (29) is in essence a sequential state estimation algorithm but a composite one involving $\hat{x}$ and $S(t_f)$. The above equation can be separated by substituting the specific expressions for $\Phi$ and $\Psi$ in eqn. (29).



### *A. Continuous-time algorithm*

Let the dynamic system be represented by:

$$\dot{x}=f(x(t),t)+d(t) \qquad (30)$$

$$z(t)=Hx(t)+v(t) \qquad (31)$$

The invariant embedding method is solved for continuous time system resulting in the following equations,

$$\dot{\hat{x}}=f(x(t),t)+2S(t)H^TR^{-1}(z(t)-Hx(t)) \qquad (32)$$

$$\dot{S}(t)\lambda=S(t)f_{\hat{x}}^T\lambda+f_{\hat{x}}S(t)\lambda-2S(t)H^TR^{-1}HS(t)\lambda+\frac{1}{2}Q^{-1}\lambda+S(t)\frac{\delta}{\delta\hat{x}}(\lambda^T f_{\hat{x}})S(t)\lambda \qquad (33)$$

We divide eqn.(33) by $\lambda$ and for $\lambda\to 0$, we get:

$$\dot{S}(t)=S(t)f_{\hat{x}}^T+f_{\hat{x}}S(t)-2S(t)H^TR^{-1}HS(t)+\frac{1}{2}Q^{-1} \qquad (34)$$

$$\hat{d}(t)=2S(t)H^TR^{-1}(z(t)-Hx(t)) \qquad (35)$$

The equations (33)-(35) give invariant embedding based model error estimation algorithm for continuous time system of eqn. (30) and (31), in a recursive form. The eqn. (34) is often called matrix Riccati equation, and eqn. (35) is the direct estimation or the determination of deterministic discrepancy or the so called model error.

## V. H-INFINITY NORMS/FILTERING

Consider the system shown in the figure below:

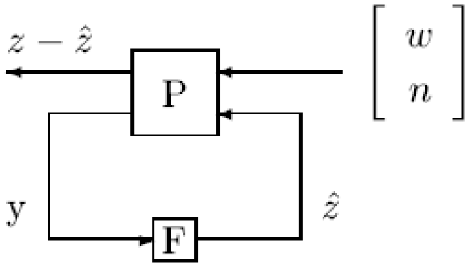


Figure 2: Filter Configuration.

Where the system and variables are defined as follows:

P Plant; F Filter

y Plant output, input to filter ( measurements)

w System disturbances

n Output disturbances

$\hat{z}$ Performance estimates

$\tilde{z} = z - \hat{z}$ Estimate error (output performance)

The goal is to determine a filter F, which will operate in the plant output, y, and produce an estimate, $\hat{z}$, that is some linear combination of plant state estimates, and satisfies a strategically developed performance criterion. The performance criterion is generally a minimization or restriction on a norm of some system parameters.

Performance criteria:

For the system, the ∞ norm can be interpreted as the peak system gain (squared). For $H_\infty$ criteria, the transfer function, from the input disturbances to the estimate error, $G_{\tilde{z}d}$, shall be required to have a system gain that conforms to an upper bound, i.e. it is confined below the bound:

$$||G_{\tilde{z}d}||_\infty^2 < \gamma^2 \quad (36)$$

This criteria represents a family of solutions where the peak energy gain of the transfer function from the input disturbances to the estimate error are less that an upper bound, γ. The performance criteria can also be rewritten as:

$$\sup_{\left\|\begin{matrix} w \\ n \end{matrix}\right\|_2 \neq 0} \left(\|z-\hat{z}\|_2^2\right) - \gamma^2 \left\|\begin{matrix} w \\ n \end{matrix}\right\|_2^2 < 0 \quad (37)$$

The below figure shows the peak gain for $H_\infty$ criteria:

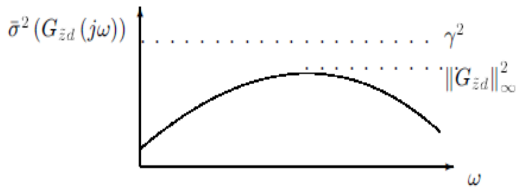


Figure 3: Peak gain of $H_\infty$ filter

A method to get proper weighting of these elements is to normalize the inputs as follows:

$$w = q_\infty \overline{w}; n = r_\infty \overline{n}\ ; (q_\infty, r_\infty > 0) \quad (38)$$

Where,

$$\|w\|_2^2 < q_\infty^2;\ \|n\|_2^2 < r_\infty^2 \quad (39)$$

The performance bound can be written as,

$$sup_w \frac{\|z-\hat{z}\|_2^2}{\left\|\frac{\overline{w}}{\overline{n}}\right\|_2^2} < \gamma^2 \quad (40)$$

Or, can be re-written as,

$$sup_w\ \|z-\hat{z}\|_2^2 - \gamma^2 \left\|\frac{\overline{w}}{\overline{n}}\right\|_2^2 < 0 \quad (41)$$

Because the system disturbances and sensor noise will produce an estimate error, there is a lower bound on the value of γ which is designated $\gamma_o$. The cost function for the variational approach is written as:

$$min : J = \|z-\hat{z}\|_2^2 - \gamma^2 \left\|\frac{\overline{w}}{\overline{n}}\right\|_2^2 \quad (42)$$

Thus, considering any linear, time-invariant dynamic system, the model error can be determined as the robust model error using $H_\infty$ norms.

## VI. ILLUSTRATION

Let us consider a non-linear continuous time system as an example:

$$\dot{X}_1(t) = 2.5\cos(t) - 0.68X_1(t) - X_2(t) - 0.0195X_2^3(t)$$
$$\dot{X}_2(t) = X_1(t)$$

Step 1: $\dot{X}_1(t)$ & $\dot{X}_2(t)$ are the state equations after integrating both consider it as true plant.

Step 2: Now by removing one or two co-efficient from the equation introduce error in it.

Step 3: By the combination of invariant embedding method and HI norms find out the model discrepancy.

Step 4: Fit a model i.e., parameter estimation using Least square method.

Here two cases are considered

Case i) Removing the co-efficient of $X_2^3(t)$ (a3)

Caseii)Removing the co-efficient of $X_1(t)$ (a1), $X_2(t)$ (a2)& $X_2^3(t)$ (a3)

## VII. SIMULATION RESULTS

Simulation results of above mentioned cases are as follows:

Figure 4 and Figure 5 shows the estimated state $X_1$ & $X_2$ respectively. Figure 6 & 7 is the simulated results of the model discrepancies for case(i) & case(ii) respectively.

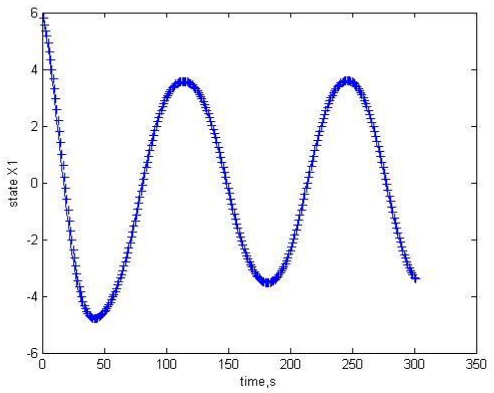


Figure 4: State $X_1$

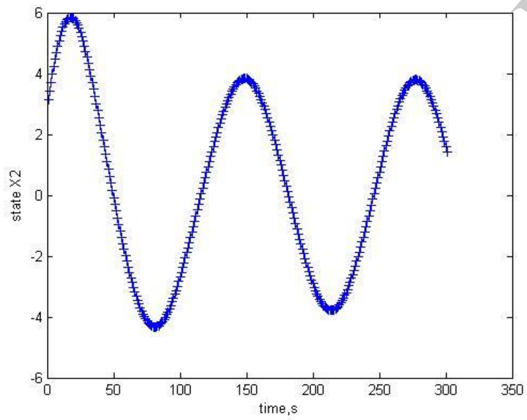


Figure 5: State $X_2$

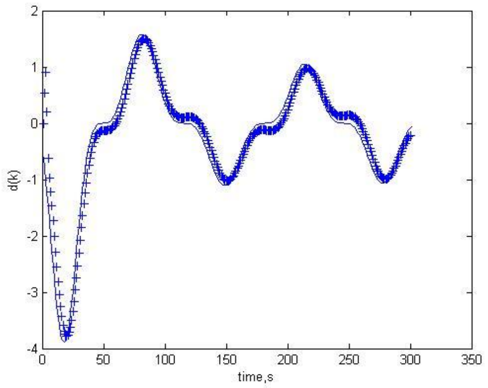


Figure 6: time history match of model discrepancy (case (i))

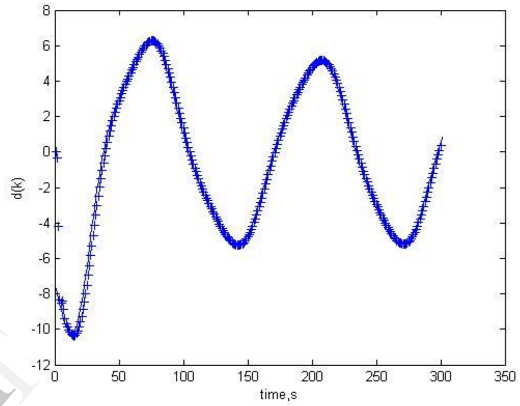


Figure 7: Time history match of model discrepancies (Case (ii))



TABLE 1: Non-linear parameter estimation results.

| parameters | True values | Estimated values | |
|---|---|---|---|
| | | Case(i) | Case(ii) |
| a1 | 0.68 | - | 0.5576 |
| a2 | 1 | - | 0.9647 |
| a3 | 0.0195 | 0.0182 | 0.198 |

TABLE 2: Cost function for case (i) and case (ii)

| No. Of cases | Without ∞norms | With ∞norms |
|---|---|---|
| Case(i) | 0.207 | 0.109 |
| Case (ii) | 0.187 | 0.052 |

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