Reliability Estimation for the Distribution of a k-Unit Parallel System with Rayleigh Distribution as the Component Life Distribution

K. G. Potdar
Department of Statistics, Ajara Mahavidyalaya, Ajara, Dist Kolhapur, India-416505,

D. T. Shirke
Department of Statistics, Shivaji University, Kolhapur, India-416004,

Abstract- A k-unit parallel system having component lifetime distribution to be Rayleigh is considered. Based on the progressively Type-II censored sample, the maximum likelihood estimator (MLE) of the scale parameter of the Rayleigh distribution is derived and is used to estimate reliability function. EM algorithm is used to obtain MLE. Asymptotic confidence interval for the scale parameter and reliability function is constructed. Confidence interval based on the log-transformed MLE is also constructed. Simulation study is conducted to investigate performance of estimates and confidence intervals. An example with real data is presented for illustration.

Index Terms- Progressively Type-II censoring, Rayleigh distribution, EM algorithm, MLE, Reliability, confidence interval, coverage probability.
AMS subject classification– 62N01, 62N02, 62N05.

ACRONYMS

CI	Confidence Interval
CDF	Cumulative Density Function
EM	Expectation Maximization
MLE	Maximum Likelihood Estimate
MSE	Mean Square Error
PDF	Probability Density Function

NOTATIONS

n	Sample size.
m	Number of observed failures in a Type-II censored sample.
λ	Scale parameter of the Rayleigh distribution.
$\hat{\lambda}$	MLE of λ.
g(y)	PDF of the Rayleigh distribution.
G(y)	CDF of the Rayleigh distribution.
f(x)	PDF of the life distribution of k unit parallel system.
F(x)	CDF of the life distribution of k unit parallel system.
$X_{(i)}$	Order Statistics from a progressively censored sample of size m.
R_i	Number of surviving units withdrawn from the experiment after i^{th} failure.
L	Log-likelihood function of all n observations.
$L(\lambda)$	Likelihood function of observed data.
L_c	Log-likelihood function of complete data.
$I_x(\lambda)$	Observed Fisher information.
$I_w(\lambda)$	Complete Fisher information.
$I_{w\|x}(\lambda)$	Missing information.
R(t)	Reliability of system at time t.

1. Introduction

In a life testing experiment, censoring is common practice because of various restrictions on data collection such as time limit, cost etc. In such a situation we remove some observations in the experiment. Such data is called censored data. Censoring is broadly classified into two types; Type-I and Type-II censoring.

Type-I censoring is related with time. In this type, an experiment continues up to a pre-determined time T. Units having failure time after time T are not observed. Here, lifetime will be known exactly only if it is less than T. For example, suppose ‘n’ units are put on test, but decision is made to terminate the test after time T. In this experiment life times will be known exactly only for those units that fail before time T. In Type-I censoring, the number of exact life times observed is random.

Type-II censoring is related with number of failures, that is, experiment continues up to the pre-determined number of failures. For example, in life testing experiment, suppose ‘n’ units are put on test, but instead of continuing until all ‘n’ units have failed; the test is terminated at the time of the m^{th} ($m \leq n$) unit failure. In case of Type-II censoring, the number of exact life times observed is fixed.

Based on the time epochs for removals of units from the experiment, the censoring is further classified in to two types, such as conventional censoring and progressive censoring. In conventional Type-I and Type-II censoring, units may not be removed before terminal point. In progressive censoring scheme, units may be removed at different stages rather than at terminal point. Progressive censoring scheme is applied in both Type-I and Type-II censoring schemes.

In progressive Type-I censoring scheme, m censoring times $T_1, T_2, \ldots, T_m$ and $R_1, R_2, \ldots, R_m$ are fixed in advance. Since $\sum_{i=1}^{m} R_i = n - m$. At time T_1 remove R_1 units, at time T_2 remove R_2 units and so on. The experiment terminates at time T_m with R_m units still surviving. In progressive Type-II censoring scheme, suppose 'n' units are put on test. The number 'm' and $R_1, R_2, \ldots, R_m$ are fixed prior to the test. At the first failure R_1 units are removed randomly from remaining n-1 units. At the second failure R_2 units are removed randomly from remaining $n-2-R_1$ units and so on. At the m^{th} failure all remaining R_m units are removed. Here, we observe failure time of m units and remaining n-m units are removed at different stages of experiment. In conventional Type-II censoring scheme $R_1=R_2=\ldots=R_{m-1} = 0$ and $R_m = n-m$. In this paper, we consider the progressive Type-II censoring scheme.

Progressive Type-II censoring scheme for various lifetime distributions has been discussed by number of researchers. Cohen [1] studied MLE of the parameters of exponential and normal distribution for progressively Type-II censored samples. Mann ([2], [3]) considered Weibull distribution with progressive censoring. Balkrishnan ([4], [5] and [6]) discussed inference for the scaled half-logistic, Gaussian and extreme value distributions under progressive Type-II censoring scheme respectively. Ng [7] studied parameter estimation for modified Weibull distribution under progressively Type-II censored samples. Balkrishnan and Aggarwala [8] gave details about progressive Type-II censoring scheme.

Balkrishnan [9] studied various distributions and inferential methods for progressively censored data. Pradhan [10] considered point and interval estimation of a k-unit parallel system based on progressive Type-II censoring scheme with exponential distribution as the component life distribution. Chein and Balkrishnan [11] discussed consistency and asymptotic normality of MLE based on progressive Type-II censored samples. Iliopoulos and Balkrishnan [12] studied likelihood inference for Laplace distribution based on progressive Type-II censored samples. Krishna and Malik [13] discussed reliability estimation in Maxwell distribution based on progressively Type-II censord data. Recently, Potdar and Shirke [14] studied inference for the scale parameter of lifetime distribution for k-unit parallel system based on progressively censored data. Potdar and Shirke [15] discussed estimation for the distribution of a k-unit parallel system with exponential distribution as the component life distribution based on Type-II progressively censored data. Potdar and Shirke [16] studied Inference for the parameters generalized inverted family of distributions.

Dempster et al. [17] introduced EM algorithm. They presented maximum likelihood estimation for incomplete data. Mclachlan and Krishnan [18] gave more details about EM algorithm. Little and Rubin [19] discussed EM algorithm for exponential family of distributions. Pradhan and Kundu [20] used EM algorithm to estimate parameters of generalized exponential distribution under Type-II censoring scheme. Ng et al. [21] used EM algorithm to estimate parameters of lognormal and Weibull distributions under Type-II censoring scheme. In this article, we used EM algorithm for estimation of the parameters of a k-unit parallel system based on progressive Type-II censoring scheme when component lifetime follows Rayleigh distribution with scale parameter λ.

Parameter estimation is based on the life times of the system. We assumed that n items put on test and failure times of $\sum_{i=1}^{m} R_i = n - m$ items are censored. Lifetimes of these censored items are unknown. We consider this data as missing and used EM algorithm to compute MLE. Louis [22] presented technique for computing observed Fisher information within EM algorithm framework. We used this technique to obtain observed Fisher information. Asymptotic normal distribution of MLE is used to construct CI for the parameter.

Maximum likelihood method is used to estimate the parameter and reliability function of the Rayleigh distribution. This estimation procedure is discussed in Section 2. By using asymptotic normality of the MLE, we derived CI and coverage probability is computed in Section 3. To investigate performance of procedure, simulation study has been made in Section 4. Results of simulation study are discussed in Section 5. In Section 6 proposed estimation methods are illustrated for real data set. Conclusions are presented in Section 7.

2. Estimation

Consider Rayleigh distribution with scale parameter λ. The PDF $g(y)$ and CDF $G(y)$ are respectively given by,

$$g(y) = \frac{2y}{\lambda^2} e^{-\left(\frac{y}{\lambda}\right)^2} \quad y \geq 0, \lambda > 0$$

$$G(y) = 1 - e^{-\left(\frac{y}{\lambda}\right)^2} \quad y \geq 0, \lambda > 0$$

Consider k unit parallel system with independent and identically distributed components. Let $Y_1, Y_2, \ldots, Y_k$ be the lifetimes, where Y_i is the lifetime of the i^{th} component with $Y_i \sim g(y)$. Life time of the system $X = \max(Y_1, Y_2, \ldots, Y_k)$. The CDF of X is

$$F(x) = \left(1 - e^{-\left(\frac{x}{\lambda}\right)^2}\right)^k \qquad x \geq 0, \ \lambda > 0. \quad (1)$$

and the PDF of X is

$$f(x) = \frac{2kx}{\lambda^2}\, e^{-\left(\frac{x}{\lambda}\right)^2}\left(1-e^{-\left(\frac{x}{\lambda}\right)^2}\right)^{k-1} \quad x \geq 0,\ \lambda > 0. \qquad (2)$$

Suppose 'n' systems of k-unit parallel systems are under test and we observe lifetimes of 'm' systems under progressive Type-II censoring. Let (R_1, R_2,.., R_m) is a progressive censoring scheme. The likelihood function for the observed data is

$$L(\lambda) = C\prod_{i=1}^{m} f(x_{(i)})\left[1-F(x_{(i)})\right]^{R_i}, \qquad (3)$$

$$\text{where } C = n\prod_{j=1}^{m-1}\left(n-j-\sum_{i=1}^{j} R_i\right).$$

$$L(\lambda) = C\prod_{i=1}^{m}\frac{2kx_{(i)}}{\lambda^2}\, e^{-\left(\frac{x_{(i)}}{\lambda}\right)^2} \times\left(1-e^{-\left(\frac{x_{(i)}}{\lambda}\right)^2}\right)^{k-1} \times\left[1-\left(1-e^{-\left(\frac{x_{(i)}}{\lambda}\right)^2}\right)^{k}\right]^{R_i}. \qquad (4)$$

Suppose $x_{(1)}, x_{(2)},\ldots, x_{(m)}$ is the observed data and $z_1, z_2, \ldots, z_m$ is the censored data. We note that z_i is a vector with R_i elements. That is at the i^{th} failure, we remove R_i (i=1,2,…….,m) systems. Observations on the removed systems are not available. The censored data Z=($z_1, z_2, \ldots, z_m$) can be considered as missing data. X=($x_{(1)}, x_{(2)}, \ldots, x_{(m)}$) is observed data. W=(X, Z) is the complete data set. Then complete log likelihood function is

$$L_c = n\log(2) + n\log(k) - 2n\log(\lambda) + \sum_{i=1}^{m}\log(x_i) - \frac{1}{\lambda^2}\sum_{i=1}^{m}x_i^2 + (k-1)\sum_{i=1}^{m}\log\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right] + \sum_{i=1}^{m}\sum_{j=1}^{R_i}\log(z_{ij}) - \frac{1}{\lambda^2}\sum_{i=1}^{m}\sum_{j=1}^{R_i}z_{ij}^2 + (k-1)\sum_{i=1}^{m}\sum_{j=1}^{R_i}\log\left[1-e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\right]. \qquad (5)$$

In order to obtain MLE of λ, we use EM algorithm due to Dempster et al. [17]. For the E step in EM algorithm we take Expectation of Z_{ij}. The derivative of L_c with respect to λ is taken for the M step, where

$$\frac{dL_c}{d\lambda} = -\frac{2n}{\lambda} + \frac{2}{\lambda^3}\sum_{i=1}^{m}x_i^2 - \frac{2(k-1)}{\lambda^3}\sum_{i=1}^{m}\frac{x_i^2 e^{-\left(\frac{x_i}{\lambda}\right)^2}}{1-e^{-\left(\frac{x_i}{\lambda}\right)^2}} + \frac{2}{\lambda^3}\sum_{i=1}^{m}R_i\, a(x_i,k,\lambda^0) - \frac{2(k-1)}{\lambda^3}\sum_{i=1}^{m}R_i\, b(x_i,k,\lambda^0). \qquad (6)$$

where $a(x_i,k,\lambda^0) = E(z_{ij}^2 \mid z_{ij} > x_i)$

$$= \int_{x_i}^{\infty} z^2\, \frac{\frac{2kz}{\lambda^2}\, e^{-\left(\frac{z}{\lambda}\right)^2}\left(1-e^{-\left(\frac{z}{\lambda}\right)^2}\right)^{k-1}}{1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right)^{k}}\, dz.$$

$$\text{and } b(x_i,k,\lambda^0) = E\left(\left.\frac{z_{ij}^2 e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}{1-e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}\right| z_{ij} > x_i\right)$$

$$= \int_{x_i}^{\infty}\frac{z^2 e^{-\left(\frac{z}{\lambda}\right)^2}}{1-e^{-\left(\frac{z}{\lambda}\right)^2}}\,\frac{\frac{2kz}{\lambda^2}\, e^{-\left(\frac{z}{\lambda}\right)^2}\left(1-e^{-\left(\frac{z}{\lambda}\right)^2}\right)^{k-1}}{1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right)^{k}}\, dz.$$

We have to solve equation $\frac{dL_c}{d\lambda} = 0$ to obtain λ^1 as the solution. But this equation does not have solution in the closed form. Therefore we used Newton-Raphson method and compute λ^1. By using λ^1, we computed $a(x_i,k,\lambda^1)$ and $b(x_i,k,\lambda^1)$. This ends the M step. We continue this procedure until convergence took place.

In Newton Raphson-method, we need to choose initial value of λ. We used least square estimate. Ng [7] discussed estimation of model parameters of modified Weibull distribution based on progressively Type-II censored data where the empirical distribution function is computed as (see Meeker and Escober [23])

$$\hat{F}(x_{(i)}) = 1 - \prod_{j=1}^{i}(1-\hat{p}_j).$$

with

$$\hat{p}_j = \frac{1}{n-\sum_{k=2}^{j}R_{k-1} - j + 1}, \qquad \text{for } j = 1,2,\ldots,m.$$

The estimate of the parameters can be obtained by least square fit of simple linear regression.

$y_i = \beta x_{(i)}$ with $\beta = -1/\lambda$.

$$y_i = \ln\left[1 - \frac{\hat{F}^{\frac{1}{k}}(x_{(i-1)}) + \hat{F}^{\frac{1}{k}}(x_{(i)})}{2}\right], \qquad \text{for } i=1,2,\ldots,m.$$

$$\hat{F}(x_{(0)}) = 0.$$

The least square estimates of λ is given by

$$\hat{\lambda} = -\frac{\sum_{i=1}^{m} x_{(i)}^2}{\sum_{i=1}^{m} x_{(i)} y_i}. \quad (7)$$

We used $\hat{\lambda}$ as an initial value of λ in Newton-Raphson method. Reliability function at time t is

$$R(t) = P(X > t) = 1 - F(t).$$

$$R(t) = 1 - \left(1 - e^{-\left(\frac{t}{\lambda}\right)^2}\right)^k \qquad t \geq 0, \lambda > 0.$$

The Maximum likelihood estimate of R(t) is

$$\widehat{R(t)} = 1 - \left(1 - e^{-\left(\frac{t}{\hat{\lambda}}\right)^2}\right)^k \qquad t \geq 0, \hat{\lambda} > 0.$$

We compute observed Fisher information using the idea of missing information principle of Louis [22].

The observed information
= complete information – missing information

$$I_x(\lambda) = I_w(\lambda) - I_{w|x}(\lambda), \quad (8)$$

where,

Complete information = $I_w(\lambda) = -E\left[\frac{d^2L}{d\lambda^2}\right]$,

where, L is the log likelihood function of all n observations.

Now,

$$L = n\log(2) + n\log(k) - 2n\log(\lambda) + \sum_{i=1}^{n}\log(x_i) - \frac{1}{\lambda^2}\sum_{i=1}^{n} x_i^2 + (k-1)\sum_{i=1}^{n}\log\left[1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right].$$

and

$$\frac{dL}{d\lambda} = -\frac{2n}{\lambda} + \frac{2}{\lambda^3}\sum_{i=1}^{n} x_i^2 - \frac{2(k-1)}{\lambda^3}\sum_{i=1}^{n}\frac{x_i^2 e^{-\left(\frac{x_i}{\lambda}\right)^2}}{1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}}.$$

$$\frac{d^2L}{d\lambda^2} = \frac{2n}{\lambda^2} - \frac{6}{\lambda^4}\sum_{i=1}^{n} x_i^2 - \frac{4(k-1)}{\lambda^6}\sum_{i=1}^{n}\frac{x_i^4 e^{-\left(\frac{x_i}{\lambda}\right)^2}}{\left(1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right)^2} + \frac{6(k-1)}{\lambda^4}\sum_{i=1}^{n}\frac{x_i^2 e^{-\left(\frac{x_i}{\lambda}\right)^2}}{1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}}.$$

Complete information is given by,

$$I_w(\lambda) = -\frac{2n}{\lambda^2} + \frac{6}{\lambda^4}\sum_{i=1}^{n} E[x_i^2] + \frac{4(k-1)}{\lambda^6}\sum_{i=1}^{n} E\left[\frac{x_i^4 e^{-\left(\frac{x_i}{\lambda}\right)^2}}{\left(1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right)^2}\right] - \frac{6(k-1)}{\lambda^4}\sum_{i=1}^{n} E\left[\frac{x_i^2 e^{-\left(\frac{x_i}{\lambda}\right)^2}}{1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}}\right]. \quad (9)$$

Missing information is given by

$$I_{W|X}(\lambda) = \sum_{i=1}^{m} R_i\, I_{W|X}^{(i)}(\lambda) = -\sum_{i=1}^{m}\sum_{j=1}^{R_i} E_{Z|X}\left[\frac{d^2\log\left(f\left(Z_{ij}|X_i,\lambda\right)\right)}{d\lambda^2}\right]. \quad (10)$$

Consider

$$f_{z|X}\left(Z_{ij}|X_i,\lambda\right) = \frac{f(z_{ij};\lambda)}{1 - F(x_i;\lambda)} = \frac{\frac{2kz_{ij}}{\lambda^2} e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\left(1 - e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\right)^{k-1}}{1 - \left(1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right)^k}.$$

Let,

$$\log f = \log\left[f_{z|X}\left(Z_{ij}|X_i,\lambda\right)\right].$$

Therefore,

$$\log f = \log(2) + \log(k) + \log\left(z_{ij}\right) - 2\log(\lambda) - \left(\frac{z_{ij}}{\lambda}\right)^2 + (k-1)\log\left(1 - e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\right) - \log\left[1 - \left(1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right)^k\right].$$

$$\frac{d\log f}{d\lambda} = -\frac{2}{\lambda} + \frac{2z_{ij}^2}{\lambda^3} - \frac{2(k-1)z_{ij}^2 e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}{\lambda^3\left(1 - e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\right)} - \frac{2kx_i^2 e^{-\left(\frac{x_i}{\lambda}\right)^2}\left[1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^{k-1}}{\lambda^3\left\{1 - \left[1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^k\right\}}.$$

and

$$\frac{d^2\log f}{d\lambda^2} = \frac{2}{\lambda^2} - \frac{6z_{ij}^2}{\lambda^4} - \frac{4(k-1)z_{ij}^4 e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}{\lambda^6\left[1 - e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\right]^2} + \frac{6(k-1)z_{ij}^2 e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}{\lambda^4\left[1 - e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\right]} - \frac{4kx_i^4 e^{-\left(\frac{x_i}{\lambda}\right)^2}\left[1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^{k-2}\left[1 - ke^{-\left(\frac{x_i}{\lambda}\right)^2}\right]}{\lambda^6\left\{1 - \left[1 - e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^k\right\}}$$

$$+\frac{4k^2x_i^4e^{-2\left(\frac{x_i}{\lambda}\right)^2}\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^{2(k-1)}}{\lambda^6\left\{1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^k\right\}^2}+\frac{6kx_i^2\lambda^2e^{-\left(\frac{x_i}{\lambda}\right)^2}\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^{k-1}}{\lambda^6\left\{1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^k\right\}}.$$

Hence Missing information is

$$I_{W|X}(\lambda)=\sum_{i=1}^{m}R_i\,I_{W|X}^{(i)}(\lambda)$$

$$=-\sum_{i=1}^{m}\sum_{j=1}^{R_i}E_{Z|X}\left[\frac{d^2\log\left(f(Z_{ij}|X_i,\lambda)\right)}{d\lambda^2}\right],$$

$$=-\frac{2(n-m)}{\lambda^2}+\frac{6}{\lambda^4}\sum_{i=1}^{m}\sum_{j=1}^{R_i}E[z_{ij}^2]$$

$$+\frac{4(k-1)}{\lambda^6}\sum_{i=1}^{m}\sum_{j=1}^{R_i}E\left[\frac{z_{ij}^4e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}{\left[1-e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}\right]^2}\right]$$

$$-\frac{6(k-1)}{\lambda^4}\sum_{i=1}^{m}\sum_{j=1}^{R_i}E\left[\frac{z_{ij}^2e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}{1-e^{-\left(\frac{z_{ij}}{\lambda}\right)^2}}\right]$$

$$+\frac{4k}{\lambda^6}\sum_{i=1}^{m}\sum_{j=1}^{R_i}\frac{x_i^4e^{-\left(\frac{x_i}{\lambda}\right)^2}\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^{k-2}\left[1-ke^{-\left(\frac{x_i}{\lambda}\right)^2}\right]}{1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^k}$$

$$-\frac{4k^2}{\lambda^6}\sum_{i=1}^{m}\sum_{j=1}^{R_i}\frac{x_i^4e^{-2\left(\frac{x_i}{\lambda}\right)^2}\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^{2(k-1)}}{\left\{1-\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^k\right\}^2}$$

$$-\frac{6k}{\lambda^4}\sum_{i=1}^{m}\sum_{j=1}^{R_i}\frac{x_i^2e^{-\left(\frac{x_i}{\lambda}\right)^2}\left[1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right]^{k-1}}{\left[1-\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^2}\right)^k\right]}. \quad (11)$$

Using expressions in equations (10) and (11), we obtained Fisher information.

3. Confidence Intervals

Using property of asymptotic normality of MLE we construct CI for λ. Let $\hat{\lambda}_n$ is the MLE of λ and $\hat{\sigma}^2(\hat{\lambda}_n)=\frac{1}{I(\hat{\lambda}_n)}$ is the estimated variance of $\hat{\lambda}_n$. Therefore, 100(1-α)% asymptotic CI for λ is given by

$$\left(\hat{\lambda}_n-\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)},\ \hat{\lambda}_n+\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)}\right), \quad (12)$$

where $\tau_{\alpha/2}$ is the upper $100(\alpha/2)^{th}$ percentile of standard normal distribution.

Meeker and Escober [23] reported that the asymptotic CI for λ based on $\ln(\hat{\lambda}_n)$ has better coverage probability. An approximate 100(1-α)% CI for ln(λ) is

$$\left(\begin{array}{c}\ln(\hat{\lambda}_n)-\tau_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}^2(\ln(\hat{\lambda}_n))},\\ \ln(\hat{\lambda}_n)+\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\ln(\hat{\lambda}_n))}\end{array}\right), \quad (13)$$

where $\hat{\sigma}^2(\ln(\hat{\lambda}_n))$ is the estimated variance of $\ln(\hat{\lambda}_n)$ which is approximately obtained by $\hat{\sigma}^2(\ln(\hat{\lambda}_n))\approx\frac{\hat{\sigma}^2(\hat{\lambda}_n)}{\hat{\lambda}_n^{\,2}}$ Hence, an approximate 100(1-α)% CI for λ is

$$\left(\hat{\lambda}_n e^{\left(-\frac{\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)}}{\hat{\lambda}_n}\right)},\ \hat{\lambda}_n e^{\left(\frac{\tau_{\alpha/2}\sqrt{\hat{\sigma}^2(\hat{\lambda}_n)}}{\hat{\lambda}_n}\right)}\right). \quad (14)$$

4. Simulation Study

A simulation study is carried out to study the performance of MLE by considering bias and MSE for various progressively Type-II censoring scheme. Approximate CIs based on MLE and log-transformed MLE are compared through their coverage probability. Balkrishnan and Sandhu [24] presented algorithm for sample generation from progressively Type-II censored scheme. Using this algorithm, we generate samples from the distribution of a k-unit parallel system with Rayleigh distribution as the component life distribution.

Algorithm –

1. Generate i.i.d. observations (W_1, W_2, ……, W_m) from U(0, 1).
2. For (R_1, R_2, …. , R_m) censoring scheme, set E_i= 1/ (i + R_m+R_{m-1}+…. + R_{m-i+1}) for i=1,2,….,m.
3. Set $V_i = W_i^{E_i}$ for i=1,2, ….. , m.
4. Set U_i= 1– V_m V_{m-1} ….. V_{m-i+1}) for i=1,2, …. , m. Then (U_1, U_2, …… , U_m) is the progressively Type-II censored sample from U(0,1).
5. For given values of the parameter λ, set

$$x_{(i)}=\left[-\lambda^2\log\left(1-(U_i)^{1/k}\right)\right]^{1/2},\ \text{for i=1, …, m.} \quad (15)$$

Then ($x_{(1)}$, $x_{(2)}$, …… , $x_{(m)}$) is the required progressively Type-II censored sample from the distribution of a k-unit parallel system with Rayleigh distribution as the component life distribution. For simulation study we consider 30 different progressively Type-II censored schemes as mentioned in Table I.

Table 1.
Progressively Type-II censored schemes used for simulation study

n	m	Scheme no.	Scheme
5	2	[1]	(3,0)
		[2]	(0,3)
		[3]	(1,2)
		[4]	(2,1)
15	5	[5]	(10, 4*0)
		[6]	(4*0, 10)
		[7]	(2,2,2,2,2)
15	10	[8]	(5,9*0)
		[9]	(9*0,5)
		[10]	(3,2, 8*0)
20	10	[11]	(10,9*0)
		[12]	(9*0,10)
25	10	[13]	(15,9*0)
		[14]	(9*0,15)
		[15]	(5,5,5,7*0)
25	15	[16]	(10,14*0)
		[17]	(14*0,10)
30	10	[18]	(20, 9*0)
		[19]	(9*0,20)
30	15	[20]	(15, 14*0)
		[21]	(14*0,15)
		[22]	(5,5,5,12*0)
30	20	[23]	(10, 19*0)
		[24]	(19*0,10)
		[25]	(0,5,5,17*0)
50	20	[26]	(30,19*0)
		[27]	(19*0,30)
50	35	[28]	(15,34*0)
		[29]	(34*0,15)
		[30]	(5,5,5,32*0)

In Table 1, scheme (a, b) stands for R_1= a and R_2=b. Similar meaning holds for schemes described through completely specified vector, while scheme (10, 4*0) means R_1=10 and rest four R_i's are zero. i.e. $R_2=R_3=R_4=R_5=0$.

Simulation was carried out for 3-unit parallel system and 5-unit parallel system (i.e. k=3, k=5) with λ=1. EM algorithm and Newton-Raphson method are used to compute MLE. For each particular progressive censoring scheme, 10,000 sets of observations were generated. The bias, the MSE and the coverage probability for the corresponding approximate CIs for λ are displayed in the Tables 2 and 3 for k=3 and k=5 respectively.Further, reliability estimate for different time period t= 1 given in Tables 4 and 5 for k=3 and k=5 respectively.

Table 2.
Bias, MSE and Coverage Probability for k=3 and λ=1

Scheme No.	Bias	MSE	Confidence level (MLE)		Confidence level (log MLE)	
			90%	95%	90%	95%
[1]	-0.0195	0.0406	0.8653	0.9075	0.8806	0.9323
[2]	-0.0196	0.0351	0.8683	0.9118	0.8865	0.9376
[3]	-0.0210	0.0359	0.8675	0.9080	0.8831	0.9335
[4]	-0.0222	0.0377	0.8651	0.9092	0.8832	0.9350
[5]	-0.0082	0.0174	0.8861	0.9327	0.8919	0.9432
[6]	-0.0079	0.0133	0.8829	0.9324	0.8893	0.9414
[7]	-0.0083	0.0143	0.8847	0.9297	0.8919	0.9414
[8]	-0.00274	0.0096	0.8951	0.9425	0.8961	0.9482
[9]	-0.0023	0.0082	0.8945	0.9421	0.8944	0.9497
[10]	-0.0036	0.0096	0.8929	0.9416	0.8966	0.9457
[11]	-0.0039	0.0093	0.8927	0.9422	0.8977	0.9513
[12]	-0.0027	0.0076	0.8908	0.9407	0.8949	0.9462
[13]	-0.0051	0.0093	0.8897	0.9408	0.8959	0.9455
[14]	-0.0035	0.0067	0.8990	0.9460	0.9052	0.9505
[15]	-0.0037	0.0087	0.8945	0.9412	0.8990	0.9464
[16]	-0.0024	0.0063	0.9017	0.9456	0.9025	0.9515
[17]	-0.00157	0.0053	0.8943	0.9430	0.8963	0.9471
[18]	-0.0038	0.0091	0.8927	0.9431	0.8990	0.9492
[19]	-0.0045	0.0066	0.8919	0.9437	0.8977	0.9453
[20]	-0.00314	0.0064	0.8952	0.9433	0.8963	0.9468
[21]	-0.0039	0.0048	0.9022	0.9496	0.9039	0.9534
[22]	-0.0031	0.0061	0.8952	0.9425	0.8981	0.9471
[23]	-0.0018	0.0049	0.8939	0.9423	0.8951	0.9457
[24]	-0.00096	0.0041	0.8988	0.9463	0.9008	0.9490
[25]	-0.0016	0.0047	0.8985	0.9451	0.9026	0.9482
[26]	-0.0013	0.0048	0.8960	0.9451	0.8966	0.9489
[27]	-0.0016	0.0033	0.8981	0.9449	0.8999	0.9498
[28]	-0.00001	0.0028	0.9002	0.9498	0.9021	0.9509
[29]	-0.00034	0.0024	0.8971	0.9475	0.8961	0.9485
[30]	-0.0004	0.0027	0.9022	0.9525	0.9019	0.9537

From Table II and III we observe the following for the MLE of the scale parameter.

- The bias and MSE of the MLE decrease with increase in sample size n as well as with increase in the effective sample size m.
- The bias and MSE of the MLE decrease as k (no. of units in parallel system) increases.

Table 3.
Bias, MSE and Coverage Probability for k=5 and λ=1

Scheme No.	Bias	MSE	Confidence level (MLE)		Confidence level (log MLE)	
			90%	95%	90%	95%
[1]	-0.0160	0.0258	0.8733	0.9171	0.8831	0.9362
[2]	-0.0091	0.0217	0.8807	0.9290	0.8909	0.9407
[3]	-0.0122	0.0224	0.8788	0.9248	0.8879	0.9403
[4]	-0.0113	0.0237	0.8792	0.9256	0.8917	0.9367
[5]	-0.0046	0.0115	0.8879	0.9397	0.8941	0.9449
[6]	-0.0042	0.0078	0.8925	0.9418	0.8969	0.9482
[7]	-0.0044	0.0087	0.8918	0.9412	0.8932	0.9458
[8]	0.0009	0.0065	0.8985	0.9472	0.9002	0.9492
[9]	-0.0014	0.0053	0.8997	0.9460	0.8992	0.9492
[10]	0.00014	0.0064	0.8958	0.9442	0.8957	0.9475
[11]	-0.0019	0.0063	0.8962	0.9458	0.8978	0.9479
[12]	-0.0023	0.0046	0.8970	0.9450	0.8979	0.9477
[13]	-0.0017	0.0061	0.8971	0.9473	0.8991	0.9495
[14]	-0.0026	0.0042	0.8977	0.9456	0.8994	0.9504
[15]	-0.0017	0.0055	0.9009	0.9463	0.9006	0.9502
[16]	-0.0005	0.0043	0.8993	0.9465	0.8998	0.9491
[17]	-0.0007	0.0034	0.8958	0.9481	0.8978	0.9481
[18]	-0.0016	0.0061	0.8963	0.9438	0.8965	0.9490
[19]	-0.0016	0.0039	0.8944	0.9452	0.8969	0.9454
[20]	-0.0016	0.0042	0.9013	0.9490	0.9029	0.9507
[21]	-0.0013	0.0031	0.8962	0.9461	0.8965	0.9484
[22]	-0.0018	0.0040	0.8970	0.9440	0.8997	0.9478
[23]	-0.0001	0.0033	0.8968	0.9464	0.8978	0.9484
[24]	0.0006	0.0027	0.9026	0.9502	0.9045	0.9512
[25]	-0.00002	0.0031	0.9005	0.9524	0.9020	0.9529
[26]	-0.0014	0.0032	0.8973	0.9435	0.8958	0.9446
[27]	-0.0011	0.0021	0.8971	0.9448	0.8968	0.9460
[28]	-0.0008	0.0019	0.8980	0.9499	0.8989	0.9500
[29]	0.0004	0.0015	0.9047	0.9488	0.9032	0.9497
[30]	-0.0003	0.0019	0.9004	0.9463	0.9009	0.9489

Table 4.
Bias, MSE and Coverage Probability of reliability estimate for k=3, t=1 and λ=1, R(t)=0.7474

Scheme No.	Bias	MSE	Confidence level (MLE)	
			90%	95%
[1]	-0.0648	0.0427	0.8258	0.8716
[2]	-0.0627	0.0374	0.8424	0.8867
[3]	-0.0620	0.0378	0.8390	0.8881
[4]	-0.0662	0.0395	0.8414	0.8880
[5]	-0.0302	0.0164	0.8702	0.9193
[6]	-0.0265	0.0125	0.8774	0.9299
[7]	-0.0267	0.0132	0.8780	0.9246
[8]	-0.0166	0.0085	0.8856	0.9342
[9]	-0.0143	0.0071	0.8878	0.9356
[10]	-0.0157	0.0083	0.8811	0.9310
[11]	-0.0174	0.0082	0.8867	0.9360
[12]	-0.0146	0.0066	0.8820	0.9335
[13]	-0.0168	0.0082	0.8819	0.9323
[14]	-0.0141	0.0060	0.8853	0.9356
[15]	-0.0170	0.0078	0.8872	0.9336
[16]	-0.0107	0.0054	0.8892	0.9361
[17]	-0.0095	0.0043	0.8937	0.9421
[18]	-0.0155	0.0077	0.8918	0.9422
[19]	-0.0116	0.0055	0.8873	0.9384
[20]	-0.0113	0.0055	0.8869	0.9348
[21]	-0.0084	0.0040	0.8922	0.9403
[22]	-0.0104	0.0051	0.8916	0.9407
[23]	-0.0086	0.0041	0.8868	0.9412
[24]	-0.0064	0.0034	0.8884	0.9386
[25]	-0.0084	0.0039	0.8901	0.9415
[26]	-0.0073	0.0039	0.8890	0.9408
[27]	-0.0058	0.0028	0.8929	0.9448
[28]	-0.0049	0.0022	0.8984	0.9472
[29]	-0.0040	0.0019	0.8964	0.9465
[30]	-0.0044	0.0022	0.8986	0.9453

Table 5. Bias, MSE and Coverage Probability of reliability estimate for k=5, t=1 and λ=1, R(t)=0.8991

Scheme No.	Bias	MSE	Confidence level (MLE)	
			90%	95%
[1]	-0.0512	0.0203	0.8512	0.8869
[2]	-0.0455	0.0169	0.8506	0.8911
[3]	-0.0447	0.0168	0.8550	0.8920
[4]	-0.0471	0.0185	0.8511	0.8920
[5]	-0.0226	0.0062	0.8791	0.9179
[6]	-0.0164	0.0040	0.8852	0.9279
[7]	-0.0169	0.0045	0.8826	0.9256
[8]	-0.0113	0.0029	0.8840	0.9260
[9]	-0.0095	0.0023	0.8917	0.9341
[10]	-0.0124	0.0031	0.8800	0.9239
[11]	-0.0118	0.0028	0.8878	0.9287
[12]	-0.0082	0.0019	0.8895	0.9318
[13]	-0.0107	0.0028	0.8782	0.9237
[14]	-0.0086	0.0019	0.8870	0.9329
[15]	-0.0107	0.0025	0.8840	0.9281
[16]	-0.0078	0.0017	0.8933	0.9346
[17]	-0.0063	0.0013	0.8925	0.9366
[18]	-0.0112	0.0027	0.8936	0.9341
[19]	-0.0073	0.0016	0.8940	0.9370
[20]	-0.0072	0.0017	0.8894	0.9333
[21]	-0.0058	0.0013	0.8939	0.9380
[22]	-0.0080	0.0017	0.8900	0.9365
[23]	-0.0061	0.0013	0.8936	0.9399
[24]	-0.0047	0.0011	0.8875	0.9383
[25]	-0.0060	0.0013	0.8939	0.9404
[26]	-0.0056	0.0012	0.8939	0.9369
[27]	-0.0043	0.0008	0.8976	0.9457
[28]	-0.0031	0.0007	0.8970	0.9424
[29]	-0.0028	0.0006	0.8914	0.9415
[30]	-0.0039	0.0007	0.8961	0.9444

6. Results and discussion

The bias and MSE of the MLE decrease with increase in sample size ‘n’ as well as with increase in effective sample size ‘m’. The bias and MSE of the MLE decrease as k (no. of units in parallel system) increases. The MSE of the MLE is relatively small for conventional Type-II censoring scheme as compared with progressive Type-II censoring scheme. There is negligible difference between bias in case of conventional Type-II censoring scheme and progressive Type-II censoring scheme. Coverage probability in case of progressive Type-II censoring scheme are better than confidence levels in case of conventional Type-II censoring scheme. Same trend is observed in confidence levels for the log-transformed MLE. Coverage probability for k=5 is better than coverage probability for k=3 in case of small sample size. Same trend is observed in coverage probability for the log-transformed MLE. There is negligible difference in coverage probability for different k for large sample size in both MLE and log-transformed MLE case. Coverage probability using log-transformed MLE are close to nominal levels as compared to the coverage probability of MLE for small size, while for large sample size both are similar.

The bias and MSE of the MLE of Reliability decrease with increase in sample size ‘n’ as well as with increase in the effective sample size ‘m’. The bias and MSE of the MLE of Reliability decrease as k (no. of units in parallel system) increases. There is negligible difference between the bias and MSE of the MLE of Reliability in case of conventional Type-II censoring scheme and progressive Type-II censoring scheme. For small sample size, coverage probability in case of conventional Type-II censoring are better than coverage probability in case of progressive Type-II censoring, whereas for large sample size coverage probability in case of progressive Type-II censoring are better.

6. Real Life Data

Consider the following data which represent the number of revolutions to failure (in hundreds of millions) for each of 23 ball bearings given by Lieblein and Zelen [25] (also given by Lawless [26]).
0.1788, 0.2892, 0.33, 0.4152, 0.4212, 0.4560, 0.4840, 0.5184, 0.5196, 0.5412, 0.5556, 0.6780, 0.6864, 0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 105.12, 105.84, 127.92, 128.04, 173.40

According to Raqab and Madi [27], scale parameter Rayleigh distribution satisfactory fit to this data. We consider this data as outcome for life time of two unit parallel systems and three unit parallel systems. Reliability estimate for k=2, 3 and different time periods (t=0.5, 1, 1.5, 2) is given in Tables 6.

Table 6. Reliability Estimate for k=2, 3

k	n	m	Scheme	MLE	R(t)^			
					t=0.5	t=1	t=1.5	t=2
2	11	7	(1,1,1,1,3*0)	0.6141	0.7651	0.1361	0.0051	0
			(4,6*0)	0.7332	0.8617	0.2871	0.0302	0.0012
			(6*0,4)	0.7155	0.8507	0.2635	0.0245	0.0008
			(3*0,1,1,1,1)	0.7606	0.8769	0.3236	0.0405	0.0020
		11	(11*0)	0.7380	0.8645	0.2935	0.0319	0.0013
3	7	4	(1,1,1,0)	0.7064	0.9388	0.3523	0.0327	0.0010
			(3,3*0)	0.8906	0.9802	0.6321	0.1657	0.0192
			(3*0,3)	0.7451	0.9523	0.418	0.0512	0.0022
		7	(7*0)	0.7894	0.9639	0.4898	0.0789	0.0049

From Table 6 we observe the following.

Reliability estimate increase with increase in effective sample size 'm'. Reliability estimate increase as k (no. of units in parallel system) increases. Reliability estimate in case of progressive Type-II censoring scheme are better than reliability estimate in case of conventional Type-II censoring scheme.

7. Conclusion

The study reveals that for small sample size 'n' and the smallest effective sample size 'm', EM algorithm method works well. Overall both conventional Type-II censoring scheme and progressive Type-II censoring scheme give better results. According to MSE, conventional Type-II censoring method are superior to the progressive Type-II censoring method, while CIs perform well in case of progressive Type-II censoring methods. In this study both conventional and progressive censoring methods give better performance. In many situations units are removed or lost from the experiment before the completion. For example individuals from clinical trial may drop from the experiment. In such situations we have no alternative but to use progressive censoring method and analyze the data accordingly.

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K. G. Potdar is an Assistant Professor of Statistics at the Ajara Mahavidyalaya, Ajara. He received the B.Sc. in 1990, M.Sc. in 1992 and M.Phil. in 2008 in Statistics from Shivaji University, Kolhapur, Maharashtra. His area of interest includes reliability theory and inference.

D. T. Shirke is Professor of Statistics at the Shivaji University, Kolhapur. He received the B.Sc. in 1985, M.Sc. in 1987 and Ph.D. in Statistics in 1993 from Shivaji University, Kolhapur, Maharashtra. His areas of interest include reliability theory, inference, industrial statistics, survival analysis and data mining.