

# Reconstruction of Spectrum from Nonuniformly Sampled Signals using FFT

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**Abstract** — In this paper, reconstruction of a spectrum from recurrent nonuniformly sampled signal using FFT for known nonuniform sampling ratios is described. Recurrent nonuniform sampling occurs in a very high-speed waveform digitizing system with interleaved A/D converters. We developed a reconstruction algorithm using FFT by deriving the relationship between the DTFT of a uniformly sampled signal and the DTFT of a nonuniformly sampled signal. A spectrum was reconstructed using the proposed algorithm and the result was compared to one existing algorithm that uses an alternative DTFT. The proposed method performed equally at lower computational complexity.

**Index Terms** — Recurrent Nonuniform Sampling, Spectrum Reconstruction, FFT.

## I. INTRODUCTION

In this paper, reconstruction of a spectrum from recurrent nonuniform sampling using FFT for known nonuniform sampling ratios is described. Recurrent nonuniform sampling occurs in a very high-speed waveform digitizing system with interleaved A/D converters [1]–[4].

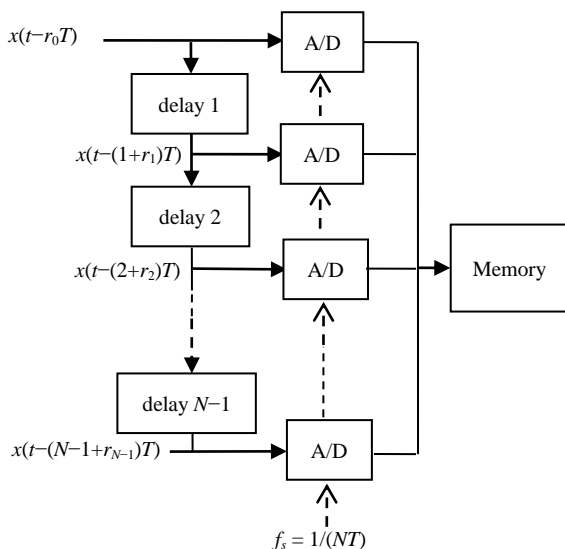


Fig. 1. Very high-speed waveform ADC system.

As shown in Fig. 1, to increase the sampling frequency  $N$  A/D converters are used. Sampling frequency of each A/D converter is  $1/NT$  [Hz] and the resulting sampling frequency of the high-speed waveform digitizer is  $1/T$  [Hz]. Ideally

each delay is exactly  $T$  seconds. However, due to the imperfection of the delays, the actual delay at the  $n$ -th converter is given by  $(n+r_n)T$  where  $r_n$  are termed the nonuniform sampling ratios and should be zero for uniform sampling. Recurrent nonuniform sampling of a continuous-time signal is shown in Fig. 2 when 3 A/D converters are used.

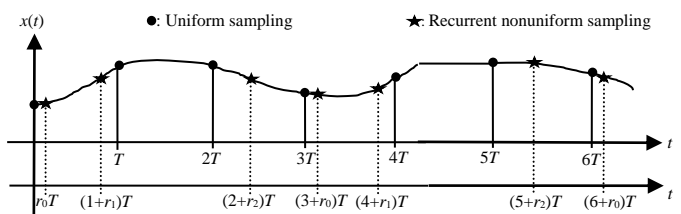


Fig. 2. Recurrent nonuniform sampling.  $T$  is the sampling interval for uniform sampling,  $r_n$  are the nonuniform sampling ratios. There is a periodic nonuniform sampling pattern. In this case the period  $N$  is 3.

A continuous-time signal  $x(t)$  is nonuniformly sampled at

$$(kN + n)T + r_n T = (kN + n + r_n)T \quad (1)$$

where  $k$  in general goes from  $-\infty$  to  $\infty$ ,  $n$  ranges from 0 to  $N-1$ , and  $T$  is the average sampling interval.

Recurrent nonuniform sampling is described in the literature [1]–[7]. In particular, an algorithm to reconstruct a spectrum from a nonuniformly sampled signal for known nonuniform sampling ratios is proposed in [4] where an alternative transform is used. The method to estimate nonuniform sampling ratio or sampling time offset is presented in [8]. In this paper, we derived a relationship between the discrete-time Fourier Transform (DTFT) of a uniformly sampled signal and the DTFT of a nonuniformly sampled signal. Our derivation closely followed the one presented in [4]. Using the relationship an algorithm for reconstruction of spectrum using FFT is proposed. The development of the algorithm also followed the one described in [4]. The experiment verified that the proposed method showed the identical performance at much lower computational complexity compared to the result described in [4].

The paper is organized as follows. In section 2, a relationship between the DTFT of a uniformly sampled signal and the DTFT of a nonuniformly sampled signal is derived

and the new algorithm that can take advantage of FFT is described. In section 3, experimental comparison between the proposed method and the existing one is made in terms of performance and computational complexity. Finally, a conclusion is made in section 4.

## II. RECONSTRUCTION METHOD USING FFT

The discrete-time Fourier transform (DTFT) of a uniformly sampled discrete-time signal or sequence,  $x[kN+n] = x(kNT+nT)$ , is

$$X(\theta) = X(e^{j\theta}) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} x[kN+n] e^{-j\theta(kN+n)} \quad (2)$$

where  $\theta (= \omega T)$  is termed the digital frequency (or normalized frequency) in radians. Most literature use  $X(e^{j\theta})$  rather than  $X(\theta)$  to denote the DTFT. However, we would like to use  $X(\theta)$  for convenience. The DTFT of the nonuniformly sampled sequence is

$$\tilde{X}(\theta) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} \tilde{x}[kN+n] e^{-j\theta(kN+n)} \quad (3)$$

where the nonuniformly sampled sequence is expressed as

$$\tilde{x}[kN+n] = x(kNT+nT+r_nT). \quad (4)$$

Using the inverse DTFT formula the DTFT of the nonuniformly sampled sequence becomes

$$\tilde{X}(\theta) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) e^{j\lambda(kN+n+r_n)} d\lambda \right] e^{-j\theta(kN+n)} \quad (5)$$

where  $X(\lambda)$  is the DTFT of the uniformly sampled sequence as in (2) except that  $\theta$  is replaced by  $\lambda$  [rad]. By changing the order of the summations and manipulating exponents,

$$\tilde{X}(\theta) = \sum_{n=0}^{N-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \left[ \sum_{k=-\infty}^{\infty} e^{j(\lambda-\theta)kN} \right] e^{j(\lambda-\theta)n+\lambda r_n} d\lambda \quad (6)$$

A train of unit impulse functions of  $t$  with the period  $P$  can be expressed in terms of its own Fourier series so that

$$\sum_{k=-\infty}^{\infty} \delta(t-kP) = \frac{1}{P} \sum_{k=-\infty}^{\infty} e^{jk \frac{2\pi}{P} t} \quad (7)$$

where  $\delta(t)$  is the unit impulse function, and the right hand side of the equation is the expression of the Fourier series.

By substituting  $P$  with  $2\pi/N$ , one obtains

$$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(t-k \frac{2\pi}{N}) = \sum_{k=-\infty}^{\infty} e^{jtkN} \quad (8)$$

By replacing  $t$  in equation (8) with  $(\lambda-\theta)$  and plugging it into equation (6) one obtains

$$\tilde{X}(\theta) = \sum_{n=0}^{N-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \left[ \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\lambda-\theta-k \frac{2\pi}{N}) \right] e^{j(\lambda-\theta)n} e^{j\lambda r_n} d\lambda \quad (9)$$

Let us assume that a particular frequency  $\theta_0$  is inside the interval  $[0, 2\pi/N)$ . Using the sifting property equation (9) becomes when  $N$  is even

$$\begin{aligned} \tilde{X}(\theta_0) &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-\frac{N}{2}}^{\frac{N-1}{2}} X\left(\theta_0+k \frac{2\pi}{N}\right) e^{jk \frac{2\pi}{N} n} e^{j\left(\theta_0+k \frac{2\pi}{N}\right) r_n} \\ &= \sum_{k=-\frac{N}{2}}^{\frac{N-1}{2}} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\theta_0+k \frac{2\pi}{N}\right) r_n} e^{j \frac{2\pi}{N} kn} \right] X\left(\theta_0+k \frac{2\pi}{N}\right) \end{aligned} \quad (10)$$

When  $N$  is odd, equation (9) becomes

$$\tilde{X}(\theta_0) = \sum_{k=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\theta_0+k \frac{2\pi}{N}\right) r_n} e^{j \frac{2\pi}{N} kn} \right] X\left(\theta_0+k \frac{2\pi}{N}\right) \quad (11)$$

We would like to continue with the case when  $N$  is even. Let us consider the frequency  $\theta_0 + 2\pi/N$ . With this frequency equation (9) becomes by the same way as in (10)

$$\begin{aligned} \tilde{X}\left(\theta_0+\frac{2\pi}{N}\right) &= \sum_{k=-\left(\frac{N}{2}+1\right)}^{\frac{N}{2}-2} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\theta_0+\frac{2\pi}{N}+k \frac{2\pi}{N}\right) r_n} e^{j \frac{2\pi}{N} kn} \right] X\left(\theta_0+\frac{2\pi}{N}+k \frac{2\pi}{N}\right) \\ &= \sum_{k=-\left(\frac{N}{2}+1\right)}^{\frac{N}{2}-2} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\theta_0+(k+1) \frac{2\pi}{N}\right) r_n} e^{j \frac{2\pi}{N} kn} \right] X\left(\theta_0+(k+1) \frac{2\pi}{N}\right) \\ &= \sum_{k=-\frac{N}{2}}^{\frac{N-1}{2}} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\theta_0+k \frac{2\pi}{N}\right) r_n} e^{j \frac{2\pi}{N} (k-1)n} \right] X\left(\theta_0+k \frac{2\pi}{N}\right) \end{aligned} \quad (12)$$

In general, for  $m = 0$  to  $N-1$ ,

$$\tilde{X}\left(\theta_0+m \frac{2\pi}{N}\right) = \sum_{k=-\frac{N}{2}}^{\frac{N-1}{2}} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\theta_0+k \frac{2\pi}{N}\right) r_n} e^{j \frac{2\pi}{N} (k-m)n} \right] X\left(\theta_0+k \frac{2\pi}{N}\right) \quad (13)$$

Equation (13) can be rewritten as

$$\tilde{X}\left(\theta_0+m \frac{2\pi}{N}\right) = \sum_{k=-\frac{N}{2}}^{\frac{N-1}{2}} \left[ \sum_{n=0}^{N-1} \frac{1}{N} e^{j\left(\theta_0+k \frac{2\pi}{N}\right) r_n} e^{-j \frac{2\pi}{N} (m-k)n} \right] X\left(\theta_0+k \frac{2\pi}{N}\right) \quad (14)$$

Let us define the following.

$$B(k, m) = \sum_{n=0}^{N-1} \frac{1}{N} e^{j\left(\theta_0+k\frac{2\pi}{N}\right)r_n} e^{-j\frac{2\pi}{N}nm} \quad (15)$$

In other words,  $B(k, m)$  for  $m = 0, 1, \hat{e}, N-1$  is the DFT of the sequence given by

$$\left\{ \frac{1}{N} e^{j\left(\theta_0+k\frac{2\pi}{N}\right)r_0}, \frac{1}{N} e^{j\left(\theta_0+k\frac{2\pi}{N}\right)r_1}, \dots, \frac{1}{N} e^{j\left(\theta_0+k\frac{2\pi}{N}\right)r_{N-1}} \right\} \quad (16)$$

where  $k = -N/2, -(N/2-1), \hat{e}, 0, 1, \hat{e}, N/2-2, N/2-1$ .  
Now equation (14) becomes

$$\tilde{X}\left(\theta_0+m\frac{2\pi}{N}\right) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} B(k, (m-k) \bmod N) X\left(\theta_0+k\frac{2\pi}{N}\right) \quad (17)$$

Equation (17) can be rewritten as

$$\tilde{X}\left(\theta_0+m\frac{2\pi}{N}\right) = \sum_{n=0}^{N-1} A(n, (m-n) \bmod N) X\left(\theta_0+n\frac{2\pi}{N}\right) \quad (18)$$

where

$$A(n, m) = \begin{cases} B(n, m) & \text{for } 0 \leq n \leq \frac{N}{2}-1 \\ B(n-N, m) & \text{for } \frac{N}{2} \leq n \leq N-1 \end{cases} \quad (19)$$

In matrix form, equation (18) becomes

$$\tilde{\mathbf{X}} = \mathbf{A}\mathbf{X} \quad (20)$$

where

$$\mathbf{X} = \begin{bmatrix} X(\theta_0) \\ X\left(\theta_0+\frac{2\pi}{N}\right) \\ \vdots \\ X\left(\theta_0+(N-2)\frac{2\pi}{N}\right) \\ X\left(\theta_0+(N-1)\frac{2\pi}{N}\right) \end{bmatrix}, \tilde{\mathbf{X}} = \begin{bmatrix} \tilde{X}(\theta_0) \\ \tilde{X}\left(\theta_0+\frac{2\pi}{N}\right) \\ \vdots \\ \tilde{X}\left(\theta_0+(N-2)\frac{2\pi}{N}\right) \\ \tilde{X}\left(\theta_0+(N-1)\frac{2\pi}{N}\right) \end{bmatrix}, \text{ and}$$

$$\mathbf{A} = \begin{bmatrix} A(0,0) & A(1,N-1) & \cdots & A(N-2,2) & A(N-1,1) \\ A(0,1) & A(1,0) & \cdots & A(N-2,3) & A(N-1,2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A(0,N-2) & A(1,N-3) & \cdots & A(N-2,0) & A(N-1,N-1) \\ A(0,N-1) & A(1,N-2) & \cdots & A(N-2,1) & A(N-1,0) \end{bmatrix}$$

The reconstruction of  $\mathbf{X}, \hat{\mathbf{X}}$ , can be obtained by

$$\hat{\mathbf{X}} = \mathbf{A}^{-1}\tilde{\mathbf{X}}. \quad (21)$$

In practice, instead of finding the inverse matrix and multiplying it to the vector, equation (20) is solved using Gaussian elimination for  $\mathbf{X}$ .

An algorithm to reconstruct the spectrum from the nonuniformly sampled signal is as follows.

- 1) Compute the DFT,  $\tilde{X}(k)$ , of the nonuniformly sampled signal using FFT by padding appropriate number of zeros as necessary. Now the number of the DFT coefficients is  $LN$ .
- 2) Do the following for  $l = 0, 1, 2, \dots, L-1$ .
  - i) Let  $\theta_0 = l\frac{2\pi}{LN}$  and compute  $B(k, m)$  using DFT as in equation (15).
  - ii) Form matrix  $\mathbf{A}(l)$  using equations (19) and (20).
  - iii) Solve the following for reconstruction of the DFT,  $\hat{\mathbf{X}}$ , using Gaussian elimination.

$$\mathbf{A}(l) \begin{bmatrix} \hat{X}(l) \\ \hat{X}(L+l) \\ \hat{X}(2L+l) \\ \vdots \\ \hat{X}((N-1)L+l) \end{bmatrix} = \begin{bmatrix} \tilde{X}(l) \\ \tilde{X}(L+l) \\ \tilde{X}(2L+l) \\ \vdots \\ \tilde{X}((N-1)L+l) \end{bmatrix} \quad (21)$$

- 3) The resulting DFT sequence,

$\{\hat{X}(0), \hat{X}(1), \dots, \hat{X}(N-1), \hat{X}(N), \dots, \hat{X}(LN-1)\}$  is the reconstruction of the spectrum.

### III. EXPERIMENTAL RESULT

In [4], an alternative discrete-time Fourier transform,  $\tilde{X}_d(\theta)$ , is defined as follows and the procedure similar to one described in the previous section is used for reconstruction of the spectrum.

$$\tilde{X}_d(\theta) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} x(kNT+nT+r_nT) e^{-j\theta(kN+n+r_n)} \quad (22)$$

In [4] instead of constructing a new  $N$  by  $N$  matrix equation for each  $l$  as in (21), only one  $N$  by  $N$  inverse matrix is used in all  $l$ . However, FFT cannot be used to compute the alternative DFT and that results in heavy computational complexity.

The following continuous-time signal as in [4] was used for our experiment.

$$x(t) = \sin(2\pi t) \quad (23)$$

The average sampling interval  $T = 0.11$  [sec], the number of A/D converters  $N = 8$ , and the number of samples  $LN = 512$  ( $L = 64$ ) in this experiment. The  $r_n$  are chosen as follows.

$$r_n = \{0.1, -0.26, 0.12, -0.14, 0.15, 0.22, -0.11, 0.13\} \quad (24)$$

## IV. CONCLUSION

In this paper, we derived a relationship between the DTFT of a uniformly sampled signal and the DTFT of a nonuniformly sampled signal. Using the relationship an algorithm for reconstruction of a spectrum from the nonuniformly sampled signal using FFT is developed. The experiment verified that the proposed method showed identical performance at much lower computational complexity compared to an existing method.

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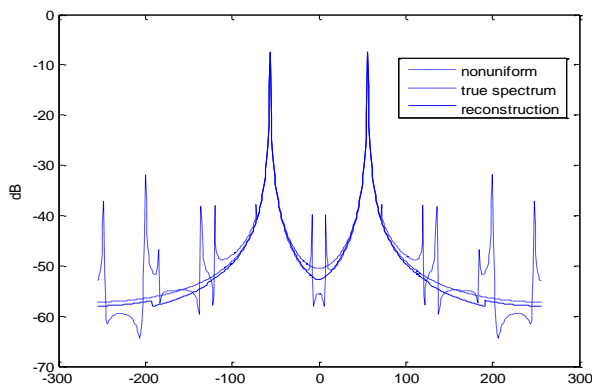


Fig. 3. Plots of reconstructed, nonuniform, and true spectra.

The plots of reconstructed, nonuniform, and true spectra are shown in Fig. 3 which is identical to the result obtained using the algorithm described in [4]. TABLE 1 shows the number of complex multiplications required for reconstruction.

TABLE 1. Required number of complex multiplications

	Proposed method	Method in [4]
To compute $\tilde{X}(k)$ or $\tilde{X}_d(k)$	$\frac{LN}{2} \log_2(LN)$ for FFT	$(LN)^2$ for alternative DFT
To build $N \times N$ matrix	$N^3 L$ or $N \left( \frac{N}{2} \log_2 N \right) L$ if FFT is used	$2N$ (for $N \times N$ matrix) + $N^3$ (for inverse matrix)
To reconstruct spectrum	$\left( \frac{N^3}{3} + \frac{N^2}{2} \right) L$	$N^2 L$

When  $L = 64$ ,  $N = 8$  and the number of samples  $LN = 512$ , the proposed method needs 21,440 complex multiplications. If we use FFT to build  $N \times N$  matrices while the method of [4] needs 266,768 multiplications. The complexity ratio is about 12.4. The ratio will increase as the number of samples increases (or  $L$  increases).