Rayleigh-Bénard Chandrasekhar Convection in An Electrically Conducting Fluid using Maxwell-Cattaneo Law with Temperature Modulation of the Boundaries

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***Abstract* - The effect of imposed time-periodic boundary temperature (ITBT, also called temperature modulation) and magnetic field at the onset of convection is investigated by making a linear analysis. The classical Fourier heat law is replaced by the non-classical Maxwell-Cattaneo law. The classical approach predicts an infinite speed for the propagation of heat. The adopted non-classical theory involves wave type of heat transport and does not suffer from the physically unacceptable drawback of infinite heat propagation speed. The Venezian approach is adopted in arriving at the critical Rayleigh number, correction Rayleigh number and wave number for small amplitude of ITBT. Three cases of oscillating temperature field are examined: (a) symmetric, so that the wall temperatures are modulated in phase, (b) asymmetric, corresponding to out-of phase modulation and (c) only the lower wall is modulated. The temperature modulation is shown to give rise to sub-critical motion. The shift in the critical Rayleigh number is calculated as a function of frequency and it is found that it is possible to advance or delay the onset of convection by time modulation of the wall temperatures. It is shown that the system is most stable when the boundary temperatures are modulated out-of-phase. It is also found that the results are noteworthy at short times and the critical eigenvalues are less than the classical ones.**

Keywords: Time periodic boundary temperature, Rayleigh-Bénard Chandrasekhar convection, Magnetic field, Maxwell-Cattaneo law.

I. INTRODUCTION

The Classical Fourier law of heat conduction expresses that the heat flux within a medium is proportional to the local temperature gradient in the system. A well known consequence of this law is that heat perturbations propagate with an infinite velocity. This drawback of the classical law motivated Maxwell [1], Cattaneo [2], Lebon and Cloot [3], Dauby et al. [4], Straughan [5], Siddheshwar [6], Pranesh [7], Pranesh and Kiran [8, 9, 10] and Pranesh and Smita[11] to adopt a non-classical Maxwell-Cattaneo heat flux law in studying Rayleigh-Bénard/Marangoni convection to get rid of this unphysical results. This Maxwell-Cattaneo heat flux law equation contains an extra inertial term with respect to the Fourier law.

$$\tau \frac{d\vec{Q}}{dt} + \vec{Q} = -\chi \nabla T$$

where, $\vec{Q}$ is the heat flux, τ is a relaxation time and χ is the heat conductivity. This heat conductivity equation together with conservation of energy equation introduces the hyperbolic equation, which describes heat propagation with finite speed. Puri and Jordan [12, 13] and Puri and Kythe [14, 15] have studied other fluid mechanics problems by employing the Maxwell-Cattaneo heat flux law.

Thomson [16] and Chandrasekhar [17] studied theoretically and Nakagawa [18, 19] and Jirlow [20] studied experimentally, the thermal stability in a horizontal layer of fluid with magnetic field and found that vertical magnetic field delays the onset of thermal convection. The application of magnetoconvection is primarily found in astrophysics and in particular by the observation of sunspots. Motivated by the above applications several authors investigated the suppression of convection by strong magnetic fields.

One of the effective mechanisms of controlling the convection is by maintaining a non-uniform temperature gradient across the boundaries. Such a temperature gradient may be generated by (i) an appropriate heating or cooling at the boundaries, (ii) injection/suction of fluid at the boundaries, (iii) an appropriate distribution of heat source, and (iv) radiative heat transfer. These methods are mainly concerned with only space-dependent temperature gradients. However in practice, the non-uniform temperature gradients find its origin in transient heating or cooling at the boundaries. Hence, basic temperature profile depends explicitly on position and time. This problem, called the thermal modulation problem, involves solution of the energy equation under suitable time-dependent boundary conditions. This temperature profile which is a function of both space and time can be used as an effective mechanism to control the convective flow by proper adjustment of its parameters, namely, amplitude and frequency of modulation. It can be used to control the convection in material processing applications to achieve higher efficiency and to advance convection in achieving major enhancement of heat, mass and momentum transfer.

There are many studies available in the literature concerning how a time-periodic boundary temperature affects the onset of Rayleigh-Bénard convection. Since the problem of Taylor stability and Bénard stability are very similar, Venezian [21] investigated the thermal analogue of Donnelly's experiment [22] for small amplitude temperature modulations. Many researchers, under different conditions, have investigated thermal instability in a horizontal layer with temperature modulation. Some of them are Gershuni and Zhukhovitskii [23], Rosenblat and Tanaka [24], Siddheshwar and Pranesh [25, 26], Mahabaleswar [27], Bhatia and Bhadauria [28, 29], Malashetty and Mahantesh Swamy [30], Bhadauria [31], Siddheshwar and Annamma [32], Bhadauria and Atul [33], Pranesh and Sangeetha [34, 35], Pranesh [36] and Pranesh and Riya[37].

The objective of the present problem is to investigate the combined effect of magnetic field and temperature modulation on the onset of Raleigh-Bénard magnetoconvection using Maxwell-Cattaneo law.

II. MATHEMATICAL FORMULATION

Consider an infinite horizontal layer of a Boussinesquian, electrically conducting fluid, of depth'd'. Cartesian co-ordinate system is taken with origin in the lower boundary and z-axis vertically upwards. Let ΔT be the temperature difference between the lower and upper boundaries. (See Fig. 1.)

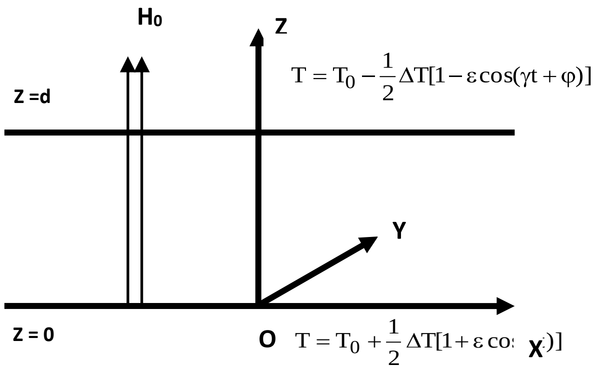

Figure 1. Schematic diagram of the Rayleigh – Bénard situation.

The basic governing equations are:

Continuity equation:

$$\nabla.\vec{q} = 0, \tag{1}$$

Conservation of linear momentum:

$$\left.\begin{aligned} \rho_o\left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q}\right] &= -\nabla P - \rho g\hat{k} \\ &+ \mu\nabla^2\vec{q} + \mu_m\left(\vec{H}.\nabla\right)\vec{H} \end{aligned}\right\}, \tag{2}$$

Conservation of energy:

$$\frac{\partial T}{\partial t} + \left(\vec{q}\nabla\right)T = -\nabla.\vec{Q}, \tag{3}$$

Maxwell – Cattaneo heat flux law:

$$\tau\left[\dot{\vec{Q}} + \vec{\omega}_1 \times \vec{Q}\right] = -\vec{Q} - \kappa\nabla T, \tag{4}$$

Magnetic induction equation:

$$\frac{\partial \vec{H}}{\partial t} + (\vec{q}.\nabla)\vec{H} = (\vec{H}.\nabla)\vec{q} + \gamma_m\nabla^2\vec{H}, \tag{5}$$

Equation of state:

$$\rho = \rho_o[1 - \alpha(T - T_o)]. \tag{6}$$

where, $\vec{q}$ is the velocity, $\vec{H}$ is the magnetic field, T is the temperature, P is the hydromagnetic pressure, ρ is the density, ρ_o is the density of the fluid at reference temperature $T = T_o$, $\gamma_m = \frac{1}{\mu_m\sigma}$, μ_m is magnetic permeability, g is the acceleration due to gravity, κ is the thermal conductivity, α is the coefficient of thermal expansion, μ is the dynamic viscosity, $\vec{\omega}_1 = \frac{1}{2}\nabla \times \vec{q}$, $\vec{Q}$ is the heat flux vector and τ is the constant relation time.

The lower wall $z = 0$ and upper wall $z = d$ are subjected to the temperatures

$$T = T_0 + \frac{1}{2}\Delta T[1 + \varepsilon\cos(\gamma t)], \tag{7}$$

and

$$T = T_0 - \frac{1}{2}\Delta T[1 - \varepsilon\cos(\gamma t + \varphi)]. \tag{8}$$

respectively, where ε is the small amplitude, γ is the frequency and φ is the phase angle.

III. BASIC STATE

The basic state of the fluid being quiescent is described by:

$$\left.\begin{aligned} \vec{q}_b = 0,\ P = P_b(z,t),\ \rho = \rho_b(z,t), \\ \vec{Q} = \left(0, 0, Q_b(z)\right),\ T = T_b(z,t), \\ \vec{H}_b = H_0\hat{k} \end{aligned}\right\}, \tag{9}$$

The temperature T_b, pressure P_b, $\vec{Q}_b$ heat flux and density ρ_b satisfy

$$\frac{\partial T_b}{\partial t} = \kappa\frac{\partial^2 T_b}{\partial z^2}, \tag{10}$$

$$-\frac{\partial P_b}{\partial z} = \rho_b g, \tag{11}$$

$$\vec{Q}_b = -\kappa\frac{\partial T_b}{\partial z}, \tag{12}$$

$$\rho_b = \rho_0[1 - \alpha(T_b - T_0)]. \tag{13}$$

The solution of "(10)" that satisfies the thermal boundary conditions of "(7)" and "(8)" is

$$T_b = T_s(z) + \varepsilon T_1(z,t), \tag{14}$$

The steady temperature field T_s and an oscillating field εT_1 are given by
where,

$$T_s = T_0 + \frac{\Delta T}{2d}(d - 2z), \tag{15}$$

$$T_1 = \mathrm{Re}\left\{\left[a(\lambda)e^{\frac{\lambda z}{d}} + a(-\lambda)e^{-\frac{\lambda z}{d}}\right]e^{-i\gamma t}\right\}. \tag{16}$$

In the "(16)"

$$\lambda = (1-i)\left(\frac{\gamma d^2}{2\kappa}\right)^{\frac{1}{2}}, \tag{17}$$

$$a(\lambda) = \frac{\Delta T}{2}\left(\frac{e^{-i\varphi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right). \tag{18}$$

and Re stands for the real part.

We now superpose infinitesimal perturbation on this basic state and study the stability of the system.

IV. LINEAR STABILITY ANALYSIS

Let the basic state be disturbed by an infinitesimal thermal perturbation. We now have

$$\left.\begin{array}{l} \vec{q}' = \vec{q}_b + \vec{q}',\ P = P_b + P', \\ \rho = \rho_b + \rho',\ \vec{Q} = Q_b + \vec{Q}', \\ T = T_b + T',\ \vec{H}_b = H_0\hat{k} + \vec{H}' \end{array}\right\}, \tag{19}$$

The primes indicate that the quantities are infinitesimal perturbations and subscript 'b' indicates basic state value.

Substituting "(19)" in to "(1)" to "(6)" and using the basic state solutions "(10)" to "(13)". We get the linearised equations governing the infinitesimal perturbations in the form

$$\nabla.\vec{q}' = 0, \tag{20}$$

$$\left.\begin{array}{l} \rho_0\left[\dfrac{\partial \vec{q}'}{\partial t}\right] = -\nabla P' + \mu\nabla^2\vec{q}' - \rho' g\hat{k} \\ \qquad + \mu_m H_0 \dfrac{\partial \vec{H}'}{\partial z} \end{array}\right\}, \tag{21}$$

$$\frac{\partial T'}{\partial t} + W'\frac{\partial T_b}{\partial z} = -\nabla.\vec{Q}', \tag{22}$$

$$\left.\begin{array}{l} \left[1 + \tau\dfrac{\partial}{\partial t}\right]\vec{Q}' = \dfrac{1}{2}\kappa\tau\left(\dfrac{\partial T_b}{\partial z}\right)\left[\dfrac{\partial \vec{q}'}{\partial z} - \nabla W'\right] \\ \qquad - \kappa\nabla T' \end{array}\right\}, \tag{23}$$

$$\frac{\partial \vec{H}'}{\partial t} = H_0\frac{\partial W'}{\partial z}\hat{k} + \gamma_m\nabla^2\vec{H}', \tag{24}$$

$$\rho' = -\alpha\rho_0 T'. \tag{25}$$

Operating divergence on the "(23)" and substituting in "(22)", on using "(20)", we get

$$\left.\begin{array}{l} \left[1 + \tau\dfrac{\partial}{\partial t}\right]\dfrac{\partial T'}{\partial t} = \dfrac{1}{2}\chi_1\left(\dfrac{\partial T_b}{\partial z}\right)\left(\nabla^2 W'\right) \\ \qquad + \kappa\nabla^2 T' - \left[1 + \tau\dfrac{\partial}{\partial t}\right]W'\dfrac{\partial T_b}{\partial z} \end{array}\right\}, \tag{26}$$

where, $\chi_1 = \kappa\tau$

The perturbation "(21)", "(24)" and "(26)" are non-dimensionalised using the following definitions:

$$\left.\begin{array}{l} (x^*, y^*, z^*) = \dfrac{(x,y,z)}{d},\ \vec{q}^* = \dfrac{\vec{q}'}{\left(\kappa/d\right)}, \\ W^* = \dfrac{W'}{\left(\kappa/d\right)},\ t^* = \dfrac{t}{\left(d^2/\kappa\right)}, \\ T^* = \dfrac{T'}{\Delta T},\ \vec{H}^* = \dfrac{\vec{H}'}{H_o} \end{array}\right\}, \tag{27}$$

Using "(25)" in "(21)", operating curl twice on the resulting equation and non-dimensionalising the resulting equation and also "(24)" and "(26)", using "(27)", we get:

$$\left.\begin{array}{l} \dfrac{1}{Pr}\dfrac{\partial}{\partial t}(\nabla^2 W) = R\nabla_1^2 T + \nabla^4 W \\ \qquad + Q\dfrac{Pr}{Pm}\nabla^2\left(\dfrac{\partial H_z}{\partial z}\right) \end{array}\right\}, \tag{28}$$

$$\frac{\partial H_z}{\partial z} = \frac{\partial W}{\partial z} + \frac{Pr}{Pm}\nabla^2 H_z, \tag{29}$$

$$\left.\begin{array}{l} \left[1 + 2C\dfrac{\partial}{\partial t}\right]\dfrac{\partial T}{\partial t} = \dfrac{\partial T_0}{\partial z}C\nabla^2 W + \nabla^2 T \\ \qquad - \dfrac{\partial T_0}{\partial z}\left[1 + 2C\dfrac{\partial}{\partial t}\right]W \end{array}\right\}. \tag{30}$$

where, the asterisks have been dropped for simplicity and the non-dimensional parameters $R,\ Q,\ Pr, Pm$ and C are as defined as

$$R = \frac{\rho_o\alpha g\Delta T d^3}{\kappa\mu} \qquad \text{(Rayleigh Number)},$$

$$Q = \frac{{\mu_m}^2\sigma{H_0}^2 d^2}{\mu} \qquad \text{(Chandrasekhar Number)}$$

$$Pr = \frac{\mu}{\rho_o\kappa} \qquad \text{(Prandtl Number)},$$

$$Pm = \frac{\mu}{\gamma_m} \qquad \text{(Magnetic Prandtl Number) and}$$

$$C = \frac{\tau\kappa}{2d^2} \qquad \text{(Cattaneo Number).}$$

In "(30)", $\left(\frac{\partial T_0}{\partial z}\right)$ is the non – dimensional from of $\left(\frac{\partial T_b}{\partial z}\right)$,

where,

$$\frac{\partial T_0}{\partial z} = -1 + \varepsilon f(z), \tag{31}$$

$$f(z) = \mathrm{Re}\left\{\left[A(\lambda)e^{\lambda z} + A(-\lambda)e^{-\lambda z}\right]e^{-i\gamma t}\right\} \tag{32}$$

and $$A(\lambda) = \frac{\lambda}{2}\left[\frac{e^{-i\varphi} - e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}\right]. \tag{33}$$

Equations "(28)" to "(30)" are solved subject to the following conditions:

$$W = \frac{\partial^2 W}{\partial z^2} = T = \frac{\partial H_z}{\partial z} = 0 \text{ at } z = 0,1 \tag{34}$$

Eliminating T and H_z from "(28)" to "(30)", we get a differential equation of order 8 for W in the form

$$\begin{bmatrix} [K_4]\nabla^2 K_2 . K_1 \\ -Q\frac{Pr}{Pm}\nabla^2 K_2 \frac{\partial^2}{\partial z^2} \end{bmatrix} W = R\nabla_1^2 \frac{\partial T_0}{\partial z}[K_3]W, \tag{35}$$

where, $K_1 = \left(\frac{\partial}{\partial t} - \frac{Pr}{Pm}\nabla^2\right)$,

$$K_2 = \left[\left(1 + 2C\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} - \nabla^2\right],$$

$$K_3 = K_1\left[C\nabla^2 - \left(1 + 2C\frac{\partial}{\partial t}\right)\right],$$

$$K_4 = \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right).$$

In dimensionless form, the velocity boundary conditions for solving "(35)" are obtained from "(28)" to "(30)" and "(34)" in the form:

$$W = \frac{\partial^2 W}{\partial z^2} = \frac{\partial^4 W}{\partial z^4} = \frac{\partial^6 W}{\partial z^6} = 0 \text{ at } z = 0,1 \tag{36}$$

V. METHOD OF SOLUTION

We seek the eigen-function W and eigenvalues R of the "(35)" for the basic temperature distribution "(31)" that departs from the linear profile $\frac{\partial T_0}{\partial z} = -1$ by quantities of order ε. Thus, the eigenvalues of the present problem differ from those of the ordinary Bénard convection by quantities of order ε. We seek the solution of "(35)" in the form

$$(R, W) = (R_0, W_0) + \varepsilon(R_1, W_1) + \varepsilon^2(R_2, W_2) + ... \tag{37}$$

The expansion "(37)" is substituted into "(35)" and the coefficients of various powers of ε are equated on either side of the equation. The resulting system of equation is

$$LW_0 = 0, \tag{38}$$

$$\left.\begin{aligned} LW_1 = &\left(\frac{\partial}{\partial t} - \frac{Pr}{Pm}\nabla^2\right)\left[C\nabla^2 - \left(1 + 2C\frac{\partial}{\partial t}\right)\right] \\ &\left(f R_0 \nabla_1^2 W_0 - R_1 \nabla_1^2 W_0\right) \end{aligned}\right\}, \tag{39}$$

$$\left.\begin{aligned} LW_2 = &\left(\frac{\partial}{\partial t} - \frac{Pr}{Pm}\nabla^2\right)\left[C\nabla^2 - \left(1 + 2C\frac{\partial}{\partial t}\right)\right] \\ &\left(\begin{aligned} &f R_0 \nabla_1^2 W_1 - R_1 \nabla_1^2 W_1 \\ &+ f R_1 \nabla_1^2 W_0 - R_2 \nabla_1^2 W_0 \end{aligned}\right) \end{aligned}\right\}. \tag{40}$$

where,

$$\left.\begin{aligned} L = &M_1 \nabla^2 M_2 M_3 + R_0 \nabla_1^2 M_4 M_3 \\ &- Q\frac{Pr}{Pm}\nabla^2 M_5 \frac{\partial^2}{\partial z^2} \end{aligned}\right\}, \tag{41}$$

$$M_1 = \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right),$$

$$M_2 = \left[\left(1 + 2C\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} - \nabla^2\right],$$

$$M_3 = \left(\frac{\partial}{\partial t} - \frac{Pr}{Pm}\nabla^2\right),$$

$$M_4 = \left[C\nabla^2 - \left(1 + 2C\frac{\partial}{\partial t}\right)\right],$$

$$M_5 = \left[\left(1 + 2C\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} - \nabla^2\right].$$

VI. SOLUTION TO THE ZEROTH ORDER PROBLEM

The zeroth order problem is equivalent to the Rayleigh-Bénard problem with magnetic field using Maxwell-Cattaneo law in the absence of temperature modulation. The stability of the system in the absence of thermal modulation is investigated by introducing vertical velocity perturbation W_0 lowest mode of convection as

$$W_0 = \mathrm{Sin}(\pi z)\exp[i(l x + m y)], \tag{42}$$

where, l and m are wave numbers in xy-plane with $a^2 = l^2 + m^2$. Substituting "(42)" into "(38)" we obtain the expression for Rayleigh number in the form

$$R_0 = \frac{K_1^6 + Q\pi^2 K_1^2}{a^2\left[CK_1^2 + 1\right]}, \tag{43}$$

where, $K_1^2 = \pi^2 + a^2$ and $a^2 = l^2 + m^2$.

Setting $Q=0$ and $C=0$, the "(43)"reduces the classical Rayleigh-Bénard result.

$$R_0 = \frac{K_1^6}{a^2}. \quad (44)$$

VII. SOLUTION TO THE FIRST ORDER PROBLEM

Equation "(39)" for W_1 now takes the form

$$LW_1 = A_1\left[fR_0a^2\sin(\pi z) - R_1a^2\sin(\pi z)\right], \quad (45)$$

where, $A_1 = \left(CK_1^2+1\right)\left(\frac{Pr}{Pm}K_1^2\right)$.

If the above equation is to have a solution, the right hand side must be orthogonal to the null-space of the operator L. This implies that the time independent part of the RHS of the "(45)" must be orthogonal to $\sin(\pi z)$. Since f varies sinusoidal with time, the only steady term on the RHS of "(45)" is $A_1R_1a^2\sin(\pi z)$, so that $R_1 = 0$. It follows that all the odd coefficients i.e. $R_1 = R_3 = ... = 0$ in "(37)".

To solve "(45)", we expand the right-hand side using Fourier series expansion and obtain W_1 by inverting the operator L term by term as

$$W_1 = A_1R_0a^2\,\mathrm{Re}\left\{\sum \frac{B_n(\lambda)}{L(\gamma,n)}e^{-i\gamma t}\sin(\pi z)\right\}, \quad (46)$$

where,

$$B_n(\lambda) = A(\lambda)g_{n1}(\lambda) + A(-\lambda)g_{n1}(-\lambda),$$

$$B_n(\lambda) = \frac{2n\pi^2\lambda^2\left[e^{-\lambda} - e^{\lambda} + (-1)^n\left(e^{-\lambda-i\phi} - e^{\lambda-i\phi}\right)\right]}{\left[e^{\lambda}-e^{-\lambda}\right]\left[\lambda^2+(n+1)^2\pi^2\right]\left[\lambda^2+(n-1)^2\pi^2\right]}, \quad (47)$$

$$L(\gamma,n) = \left[X_1 + R_0a^2X_2 - Q\frac{Pr}{Pm}\left(n^2\pi^2\right)X_3\right] + i\gamma\left[X_4 + R_0a^2X_5 + Q\frac{Pr}{Pm}\left(n^2\pi^2\right)\right],$$

$$X_1 = \left[\begin{array}{l} 2C\gamma^2\frac{Pr}{Pm}K_n^6 - \gamma^2K_n^4 - \frac{Pr}{Pm}K_n^6 \\ -2C\gamma^4\frac{1}{Pm}K_n^2 + \frac{\gamma^2}{Pm}K_n^4 - \frac{\gamma^2}{Pr}K_n^4 \end{array}\right],$$

$$X_2 = \left[\left(CK_n^2+1\right)\frac{Pr}{Pm}K_n^2 - 2C\gamma^2\right],$$

$$X_3 = \left[K_n^6 - 2C\gamma^2K_n^2\right],$$

$$X_4 = \left[\begin{array}{l} \frac{1}{Pm}K_n^4 - \frac{\gamma^2}{Pr}K_n^2 - 2C\gamma^2\frac{1}{Pm}K_n^4 \\ -2C\gamma^2K_n^4 + \frac{Pr}{Pm}K_n^6 + K_n^6 \end{array}\right].$$

$$X_5 = \left[-\left(CK_n^2+1\right) - 2\frac{Pr}{Pm}CK_n^2\right]$$

and $K_n^2 = n^2\pi^2 + a^2$ (see Siddheshwar and Pranesh [26]).

The equation for W_2, then becomes

$$LW_2 = -A_1R_2a^2W_0 + A_2a^2f\,R_0W_1, \quad (48)$$

where,

$$A_2 = \left[\left(CK_n^2+1\right)\frac{Pr}{Pm}K_n^2 - 2C\gamma^2\right] + i\gamma\left[-\left(CK_n^2+1\right) - 2\frac{Pr}{Pm}CK_n^2\right] \quad (49)$$

We shall not solve "(48)", but will use this to determine. The solvability condition requires that the time-independent part of the right hand side of "(48)" must be orthogonal to $\sin(n\pi z)$, and this results in the following equation,

$$R_2 = \frac{R_0a^2}{2}\mathrm{Re}\sum\frac{|B_n(\lambda)|^2|A_2|^2}{|L_1(\gamma,n)|^2}\left[\frac{Y_1}{2}\right], \quad (50)$$

$$Y_1 = L_1(\gamma,n) + L_1^*(\gamma,n),$$

$$L_1(\gamma,n) = L(\gamma,n)A_2^*,$$

A_2^* and $L_1^*(\gamma,n)$ are the conjugates of A_2 and $L_1(\gamma,n)$ respectively.

VIII. MINIMUM RAYLEIGH NUMBER FOR CONVECTION

The value of Rayleigh number R obtained by this procedure is the eigenvalue corresponding to the eigen-function W which, though oscillating, remains bounded in time. Since, R is a function of the horizontal wave number 'a' and the amplitude of modulation ε, we have

$$R(a,\varepsilon) = R_0(a) + \varepsilon^2R_2(a) + ... \quad (51)$$

It was shown by Venezian [21] that the critical value of thermal Rayleigh number is computed up to $O(\varepsilon^2)$, by evaluating R_0 and R_2 at $a = a_0$. It is only when one wishes to evaluate R_4 that a_2 must be taken into account where $a = a_2$ minimizes R_2. Evaluate the critical value of R_2 (denoted by R_{2c}) one has to substitute $a = a_0$ in R_2, where a_0 is the value at which R_0 given by "(44)" is minimum.

We now evaluate R_{2c} for three cases:

(a) When the oscillating field is symmetric, so that the wall temperatures are modulated in phase with $\varphi = 0$. In this case, $B_n(\lambda) = b_n$ or 0, accordingly as n is even or odd.

(b) When the wall temperature field is antisymmetric, corresponding to out-of phase modulation with $\varphi = \pi$. In this case, $B_n(\lambda) = 0$ or b_n, accordingly as n is even or odd.

(c) When only the temperature of the bottom wall is modulated, the upper plate being held at constant temperature, with $\varphi = i\infty$. In this case, $B_n(\lambda) = \frac{b_n}{2}$ for integers values of n.

The b_n are given by

$$b_n = \frac{4n\pi^2\lambda^2}{\left[\lambda^2 + (n+1)^2\pi^2\right]\left[\lambda^2 + (n-1)^2\pi^2\right]}$$

The variable λ defined in "(17)", in terms of the dimensionless frequency, reduces to $\lambda = (1-i)\left(\frac{\gamma}{2}\right)^{\frac{1}{2}}$ and thus

$$|b_n|^2 = \frac{16n^2\pi^4\gamma^2}{\left[\gamma^2 + (n+1)^4\pi^4\right]\left[\gamma^2 + (n-1)^4\pi^4\right]}.$$

Hence from "(50)" and using the above expression of $B_n(\lambda)$, we can obtain the following expression for R_{2c} in the form

$$R_{2c} = \frac{R_0 a^2}{2} \operatorname{Re} \sum \frac{|B_n(\lambda)|^2 |A_2|^2}{|L_1(\gamma, n)|^2} \left[\frac{Y_1}{2}\right]. \quad (52)$$

where, $Y_1 = L_1(\gamma, n) + L_1^*(\gamma, n)$

In "(52)" the summation extends over even values of n for case (a), odd values of n for case (b) and for all values of n for case (c).The infinite series "(52)" converges rapidly in all cases. The variation of R_{2c} with γ for different values of C, Pr, Pm and Q are depicted in figures.

IX. SUBCRITICAL INSTABILITY

The critical value of Rayleigh number R_c is determine to be of order ε^2, by evaluating R_{0c} and R_{2c}, and is of the form

$$R_c = R_{0c} + \varepsilon^2 R_{2c} \quad (53)$$

where, R_{0c} and R_{2c} can be obtained from "(44)", "(52)" respectively.

If R_{2c} is positive, sub critical instability exists and R_c has a minimum at $\varepsilon = 0$. When R_{2c} is negative, sub critical instabilities are possible. In this case from "(47)" we have

$$\varepsilon^2 < \frac{R_{0c}}{R_{2c}} \quad (54)$$

Now, we can calculate the maximum range of ε, by assigning values to the physical parameters involved in the above condition. Thus, the range of the amplitude of modulation, which causes sub critical instabilities in different physical situations, can be explained.

X. RESULTS AND DISCUSSION

In this paper we make an analytical study of the effects of temperature modulation and magnetic field at the onset of convection in Newtonian fluid by replacing the classical Fourier law of heat conduction by non-classical Maxwell-Cattaneo law.

The analysis presented in this paper is based on the assumption that the amplitude of the modulating temperature is small compared with the imposed steady temperature difference. The validity of the results obtained here depends on the value of the modulating frequency γ. When $\gamma << 1$, the period of modulation is large and hence the disturbance grows to such an extent as to make finite amplitude effects important. When $\gamma \to \infty$, $R_{2c} \to 0$, thus the effect of modulation becomes small. In view of this, we choose only moderate values of γ in our present study.

The results have been presented in Fig. 2 to Fig. 13, from these figures we observe that the value of R_{2c} may be positive or negative. The sign of the correction Rayleigh number R_{2c} characterizes the stabilizing or destabilizing effect of modulation on R_{2c}. A positive R_{2c} means the modulation effect is stabilizing while a negative R_{2c} means the modulation effect is destabilizing compared to the system in which the modulation is absent.

Fig. 2 is the plot of R_{2c} versus frequency of modulation γ for different values of Cattaneo number C, in the case of in-phase modulation. In the figure, we observe that as C increase R_{2c} become more and more negative, which represents Cattaneo number has a destabilizing influence. Increase in Cattaneo number leads to narrowing of the convection cells and thus lowering of the critical Rayleigh number. It is also observed from the figures that influence of Cattaneo number is dominant for small values because the convection cells have fixed aspect ratio. It is interesting to note that for a given value of C, R_{2c} decreases for small values of γ and increases for the moderate values of γ. Thus small values of γ destabilize the system and moderate values of γ stabilize the system. This is due to the fact that when the frequency of modulation is low, the effect of modulation on the temperature field is felt throughout the fluid layer. If the plates are modulated in phase, the temperature profile consists of the steady straight line section plus a parabolic profile which oscillates in time. As the amplitude of modulation increases, the parabolic part of the profile becomes more and more significant. It is known that a parabolic profile is subject to finite amplitude instabilities so that convection occurs at lower Rayleigh number than those predicted by the linear theory.

Fig. 3 is the plot of R_{2c} versus frequency of modulation γ for different values of Chandrasekhar number Q, in the case of in-phase modulation. In the figure, we observe that as Q increases R_{2c} becomes more and more negative, for small values of γ and becomes more and more positive for moderate values of γ. In making the conclusions from the figure we should recollect that Q influences R_{0c}.

We find that R_{2c} increases with increase in Q. When the magnetic field strength permeating the medium is considerably strong, it induces viscosity into the fluid, and the magnetic lines are distorted by convection. Then these magnetic lines hinder the growth of disturbances, leading to the delay in the onset of instability.

Fig. 4 is the plot of R_{2c} versus frequency of modulation γ for different values of Prandtl number Pr, in the case of in-phase modulation. In the figure, we observe that as Pr increases R_{2c} becomes more and more negative. We can infer from this is that the effect of increasing the viscosity of the fluid is to destabilize the system.

Fig. 5 is the plot of R_{2c} versus frequency of modulation γ for different values of magnetic Prandtl number Pm, in the case of in-phase modulation. In the figure, we observe that as Pm increase R_{2c} becomes more and more negative.

We now discuss the results pertaining to out-of-phase modulation. Comparing Fig. 2 to Fig. 5 and Fig. 6 to Fig. 9 respectively we find that R_{2c} is positive for the out-of-phase modulation where as it is negative for in-phase modulation. Thus, Q, Pr and Pm have opposing influence in in-phase and out-of-phase modulations. However, C has identical influence on R_{2c} in both in-phase and out-of-phase and these can be seen in Fig. 2 and Fig. 6. The above results are due to the fact that the temperature field has essentially a linear gradient varying in time, so that the instantaneous Rayleigh number is super critical for half cycle and subcritical during the other half cycle.

The above results on the effect of various parameters on R_{2c} for out-of-phase modulation do not qualitatively change in the case of temperature modulation of just the lower boundary. This is illustrated with the help of Fig. 10 to Fig. 13.

From the above result, we can conclude that the system is more stable when boundary temperature is modulated in out-of-phase in compare to only lower wall modulation and in-phase modulation. In-phase temperature modulation leads to sub critical motions. Sub critical motions are ruled out in the case of out-of-phase modulation and lower wall temperature modulation.

Fig. 14 is the plot of Rayleigh number R versus amplitude of modulation ε_1 for different values of γ, in the case of in-phase modulation. From the figure we observe that has amplitude of modulation ε_1 increases the Rayleigh number R also increases thus amplitude of modulation stabilizes the system. It can be clearly seen that as γ increases R increases for smaller values of γ and decreases for moderate values of γ which reconforms our earlier discussion on this.

The effect of ε_1 on R with respect to out-of-phase modulation and only lower wall modulation is illustrated in Fig. 15 and Fig. 16. We observe that as ε_1 increases R increases thus increase in ε_1 stabilizes the system. It is also observed from these figures that the frequency of modulation γ destabilizes the system.

The results of this study are useful in controlling the convection by thermal modulation with Maxwell-Cattaneo law.

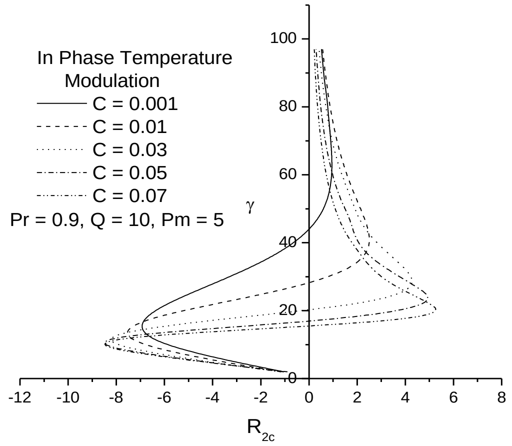

Figure 2:
Plot of R_{2c} versus frequency of modulation γ for different values of Cattaneo number C.

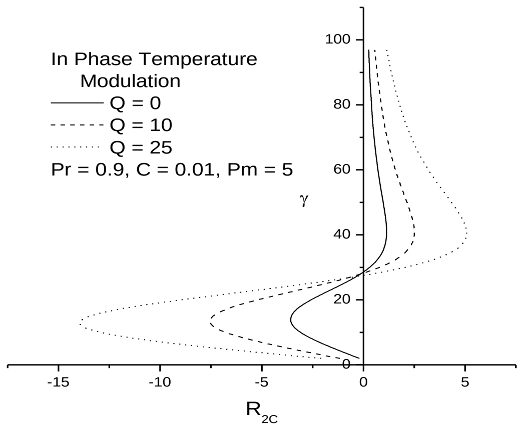

Figure 3:
Plot of R_{2c} versus frequency of modulation γ for different values of Chandrasekhar number Q.

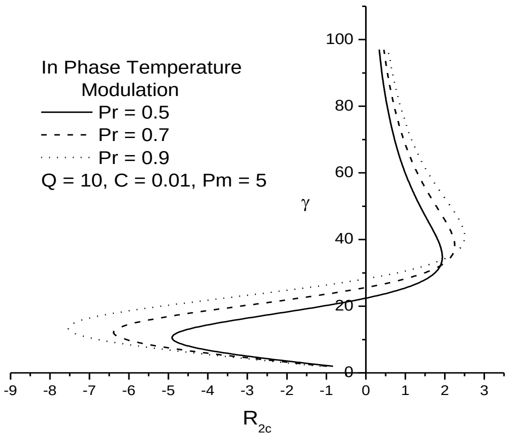

Figure 4: Plot of R_{2c} versus frequency of modulation γ for different values of Prandtl number Pr.

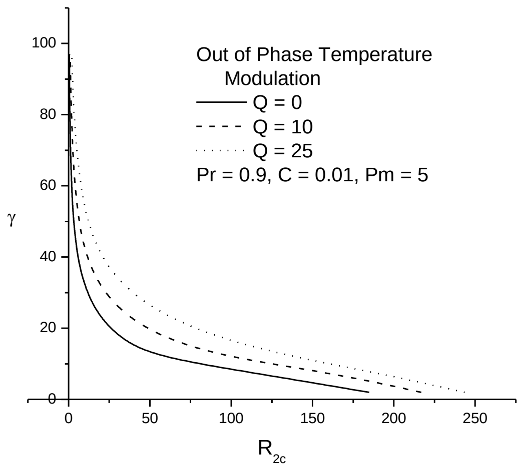

Figure 7: Plot of R_{2c} versus frequency of modulation γ for different values of Chandrasekhar number Q.

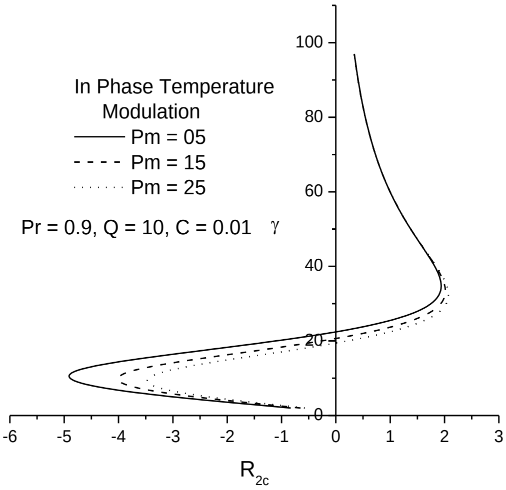

Figure 5: Plot of R_{2c} versus frequency of modulation γ for different values of Magnetic Prandtl number Pm.

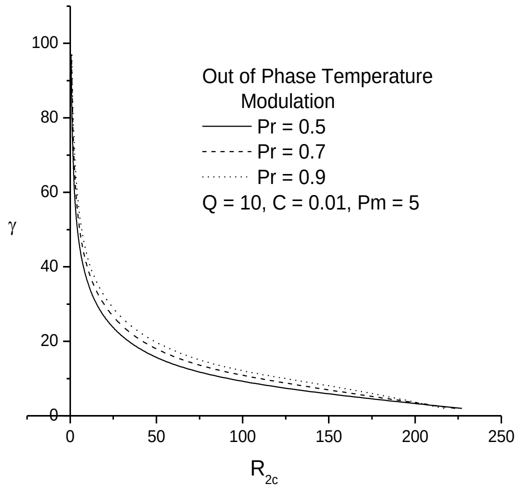

Figure 8: Plot of R_{2c} versus frequency of modulation γ different values of Prandtl number Pr.

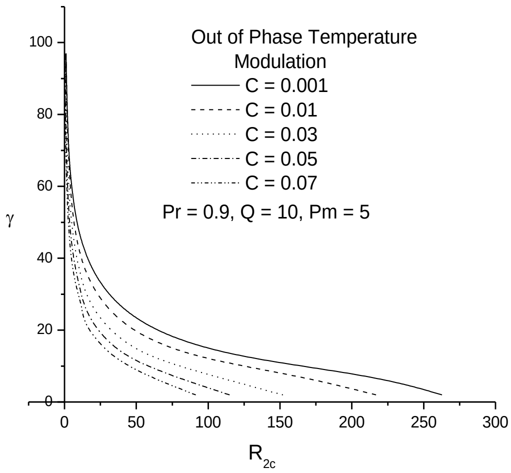

Figure 6: Plot of R_{2c} versus frequency of modulation γ for different values of Cattaneo number C.

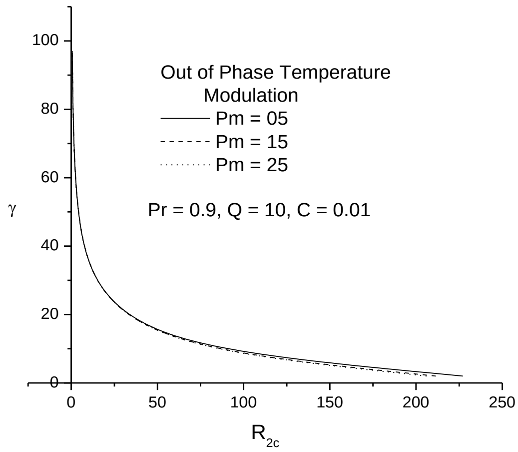

Figure 9: Plot of R_{2c} versus frequency of modulation γ for different values of Magnetic Prandtl number Pm.

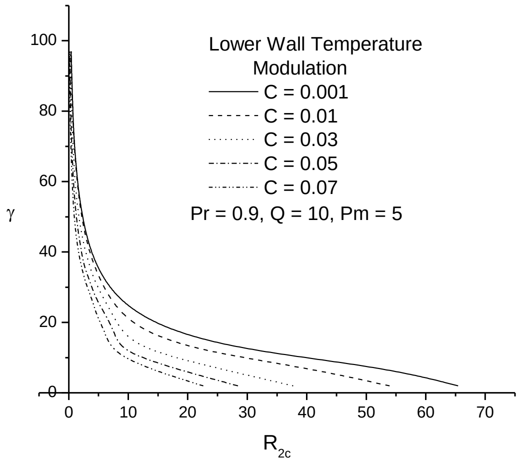

Figure 10: Plot of R_{2c} versus frequency of modulation γ for different values of Cattaneo number C.

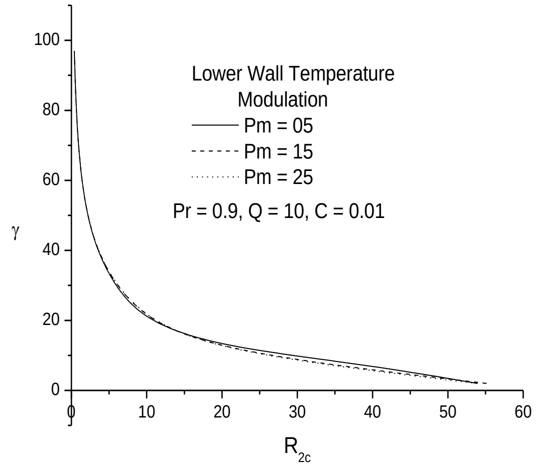

Figure 13: Plot of R_{2c} versus frequency of modulation γ for different values of Magnetic Prandtl number Pm.

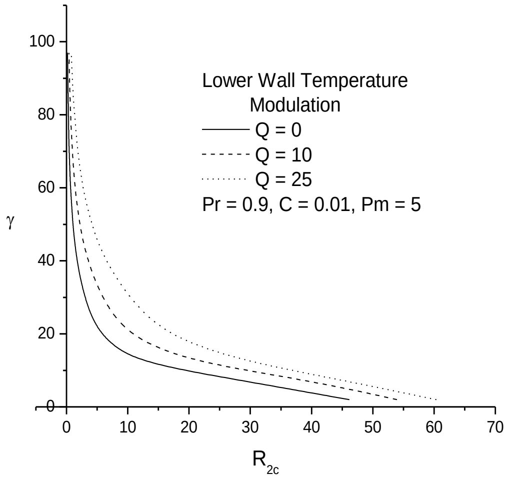

Figure 11: Plot of R_{2c} versus frequency of modulation γ for different values of Chandrasekhar number Q.

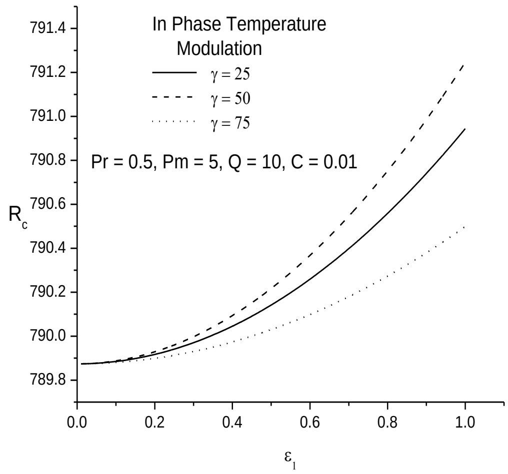

Figure 14: Plot of critical Rayleigh number $R_c = R_{0c} + \varepsilon_1^2 R_{2c}$ versus amplitude of modulation ε_1 for in phase temperature modulation for different values of frequency of modulation γ .

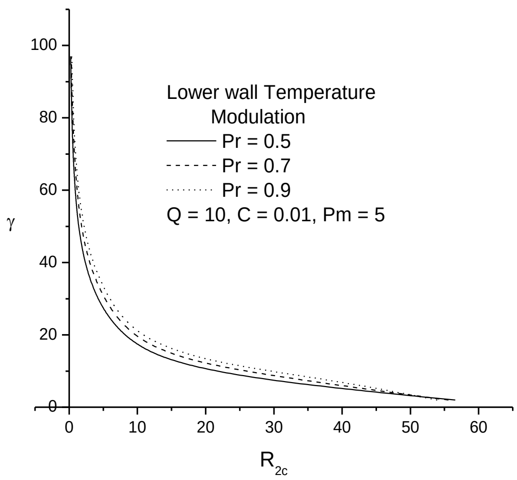

Figure 12: Plot of R_{2c} versus frequency of modulation γ different values of Prandtl number Pr.

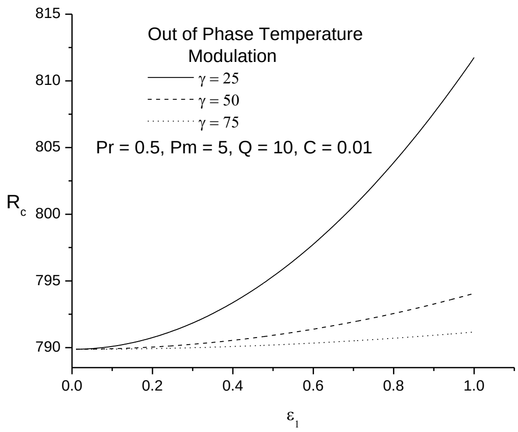

Figure 15: Plot of R_c versus ε_1 for out of phase temperature modulation for different values of γ .

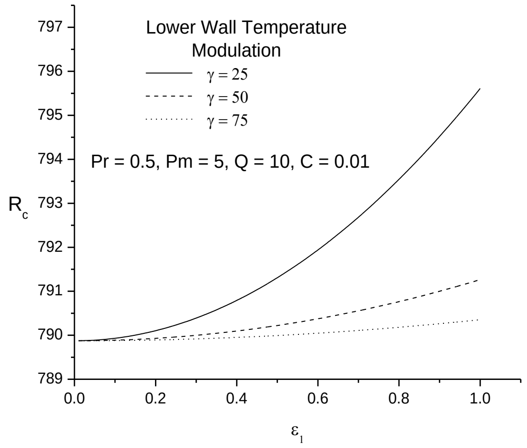

Figure 16: Plot of R_c versus ε_1 for lower wall temperature modulation for different values of γ .

XI. CONCLUSIONS

Following conclusions are drawn from the problem:

1. The system is more stable when boundary temperature is modulated in out-of-phase.
2. In-phase temperature modulation leads to sub critical motions.
3. The results of the study throw light on an external means of controlling Rayleigh-Bénard convection either advancing or delaying convection by thermal modulation.
4. It is observed that for large frequencies, the effect of modulation disappears.
5. The non – classical Maxwell – Cattaneo heat flux law involves a wave type heat transport and does not suffer from the physically unacceptable drawback of infinite heat propagation speed. The classical Fourier flux law overpredicts the critical Rayleigh number compared to that predicted by the non-classical law. Overstability is the preferred mode of convection.

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