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Radio Antipodal Number of Circulant Graphs

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Abstract

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Let $diam(G)$ denote the diameter of G and $d(u, v)$ denote the distance between the vertices u and v in G. An antipodal labeling of G with diameter d is a function f that assigns to each vertex u, a positive integer $f(u)$, such that $d(u, v) + |f(u) - f(v)| \geq d$, for all $u, v \in V$. The span of an antipodal labeling f is $max\{|f(u) - f(v)| : u, v \in V(G)$. The antipodal number for G, denoted by anG, is the minimum span of all antipodal labelings of G. Determining the antipodal number of a graph G is an NP-complete problem. In this paper we determine the antipodal number of circulant graphs .

Keywords

Labeling, radio antipodal numbering, diameter, circulant.

1. Introduction

Let G be a connected graph and let k be an integer, $k \geq 1$. A radio k- labeling f of G is an assignment of positive integers to the vertices of G such that $d(u, v) + |f(u) - f(v)| \geq k + 1$ for every two distinct vertices u and v of$G$, where $d(u, v)$is the distance between any two vertices u and v of G. The span of such a function$f$, denoted by $\text{sp}(f) = max\{|f(u) - f(v)| : u, v \in V(G)\}$. Radio k −labeling was motivated by the frequency assignment problem [3]. The maximum distance among all pairs of vetices in G is the diameter of G. The radio labeling is a radio k- labeling when $k = diam(G)$. When $k = diam(G) - 1$, a radio k-labeling is called a radio antipodal labeling. In otherwords, an antipodal labeling for a graph G is a function ,$f: V(G) \to \{0,1,2, \dots\}$ such that $d(u, v) + |f(u) - f(v)| \geq diam(G)$. The radio antipodal number for G, denoted by $an(G)$, is the minimum span of an antipodal labeling admitted by G. A radio labeling is a one-to –one function, while in an antipodal labeling, two vertices of distance $diam(G)$ apart may receive the same label.

The antipodal labeling for graphs was first studied by Chartrand et al.[8], in which, among other results, general bounds of $an(G)$ were obtained. Khennoufa and Togni [10] determined the exact value of $an(P_n)$ for paths P_n. The antipodal labeling for cycles C_n was studied in [4], in which lower bounds for $an(C_n)$ are obtained. In addition, the bound for the case $n \equiv 2(mod\ 4)$ was proved to be the exact value of $an(C_n)$, and the bound for the case $n \equiv 1(mod\ 4)$ was conjectured to be the exact value as well [7]. Justie Su-tzu Juan and Daphne Der-Fen Liu [9] confirmed the conjecture mentioned above. Moreover they determined the value of $an(C_n)$ for the case $n \equiv 3(mod\ 4)$ and also for the case $n \equiv 0(mod\ 4)$. They improve the known lower bound [4] and give an upper bound. They also conjecture that the upper bound is sharp.

In this paper we obtain an upper bound for the radio antipodal number of the Circulant graphs.

Definition An undirected circulant graph denoted by $G(n; \pm\{1, 2 \dots j\})$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, $n \geq 3$, is defined as a graph with vertex set

$V = \{0, 1 \dots n-1\}$

and edge set

$E = \{(i, j) : |j - i| \equiv s(\text{mod } n),\ s \in \{1, 2, \dots, j\}\}$.

For our convenience we take the vertex set V as $\{v_1, v_2 \ \cdots \ v_n\}$ in clockwise order.

The diameters of certain classes of circulant graphs which are going to be discussed in this are given below:

1. Diameter of $G\left(n; \left\{1, 2 \dots \lfloor \frac{n}{2} \rfloor - 1\right\}\right)$ is 2 .

2. If $n \equiv 0\ (\text{mod } 4)$, then the diameter of

$G\left(n;\left\{1,\left\lfloor \frac{n}{2}\right\rfloor\right\}\right)$ is $\left\lfloor \frac{n}{4}\right\rfloor$.

3. If $n \equiv 0 \pmod 3$, then the diameter of $G\left(n;\left\{1,\frac{n}{3}\right\}\right)$ is $\left\lfloor \frac{n}{6}\right\rfloor+1$.

4. If $n \equiv 0 \pmod{10}$, then the diameter of $G\left(n;\left\{1,\frac{n}{5}\right\}\right)$ is $\frac{n}{10}+2$.

Theorem: The radio antipodal number of the circulant graph $G\left(n;\left\{1,2\ldots\left\lfloor \frac{n}{2}\right\rfloor-1\right\}\right)$ is given by

$$an\left(G\left(n;\left\{1,2\ldots\left\lfloor \frac{n}{2}\right\rfloor-1\right\}\right)\right)=\left\lceil \frac{n}{2}\right\rceil.$$

Proof: Let $V\left(G\left(n;\left\{1,2\ldots\left\lfloor \frac{n}{2}\right\rfloor-1\right\}\right)\right)=\{v_1,v_2 \;\cdots\; v_n\}$.

Define a mapping $f:\{v_1,v_2\ldots v_n\}\to N$ as follows:

$f(v_i)=i,\ i=1,2\ldots\left\lceil \frac{n}{2}\right\rceil,$

$f\left(v_{\left\lceil \frac{n}{2}\right\rceil+i}\right)=i,\ i=1,2\ldots\left\lfloor \frac{n}{2}\right\rfloor.$ See figure 1

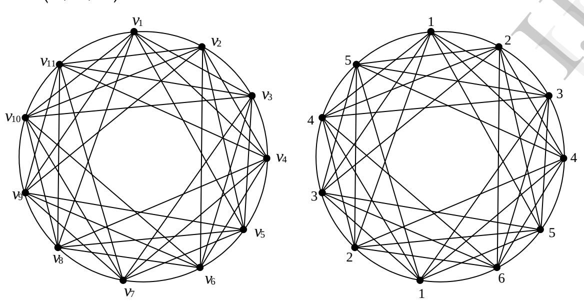

Figure 1 A circulant graph of G(11,{1,2,3,4})

First we claim that f is a radio antipodal labeling.

Case 1: If $u=v_k$ and $w=v_m,\ 1\le k\ne m\le\left\lceil \frac{n}{2}\right\rceil$, then $d(u,w)\ge 1$, $f(u)=k$ and $f(w)=m$. Hence $d(u,w)+|f(u)-f(w)|\ge 1+(k-m)\ge 2$, since $k\ne m$.

Case 2: If $u=v_{\left\lceil \frac{n}{2}\right\rceil+k}$ and $w=v_{\left\lceil \frac{n}{2}\right\rceil+m},\ 1\le k\ne m\le\left\lfloor \frac{n}{2}\right\rfloor$, then $f(u)=k$ and $f(w)=m$ and $d(u,w)\ge 1$. Therefore $d(u,w)+|f(u)-f(w)|\ge 1+(k-m)\ge 2$, since $k\ne m$.

Case 3: If $u=v_k$ and $w=v_{\left\lceil \frac{n}{2}\right\rceil+m},\ 1\le k\le\left\lceil \frac{n}{2}\right\rceil,\ 1\le m\le\left\lfloor \frac{n}{2}\right\rfloor$, then either $d(u,w)\ge 1$ and $|f(u)-f(w)|\ge 1$ or $d(u,w)=2$ and $|f(u)-f(w)|\ge 0$. In both cases, we have $d(u,w)+|f(u)-f(w)|\ge 2$. Thus $d(u,w)+|f(u)-f(w)|\ge 2$ for all $u,w\in G\left(n;\left\{1,2\ldots\left\lfloor \frac{n}{2}\right\rfloor-1\right\}\right)$.

Therefore f is a radio antipodal labeling and $an(G)\le n.$

$an\left(G\left(n;\left\{1,2\ldots\left\lfloor \frac{n}{2}\right\rfloor-1\right\}\right)\right)\ge n=\ an(f).$

Hence the radio antipodal number of $G\left(n;\left\{1,2\ldots\left\lfloor \frac{n}{2}\right\rfloor-1\right\}\right)$ is $\left\lceil \frac{n}{2}\right\rceil$.

Theorem: The radio antipodal number of $G\left(n;\left\{1,\frac{n}{2}\right\}\right),\quad n\equiv 0 \pmod 4,\ n>16,$ satisfies

$$an\left(G\left(n;\left\{1,\frac{n}{2}\right\}\right)\right)\le\left(\frac{n}{2}-1\right)\left(\frac{n}{4}-2\right).$$

Proof: We partition the vertex set $V=\{v_1,v_2\ldots v_n\}$ into four disjoint sets $V_1,\ V_2,\ V_3$ and V_4. Let $V_1=\left\{v_1,v_2\ldots v_{\frac{n}{4}}\right\}$,

$V_2=\left\{v_{\frac{n}{4}+1},v_{\frac{n}{4}+2}\ldots v_{\frac{n}{2}}\right\}$,

$V_3=\left\{v_{\frac{n}{2}+1},v_{\frac{n}{2}+2}\ldots v_{3\frac{n}{2}}\right\}$ and

$V_4=\left\{v_{3\frac{n}{2}+1},v_{3\frac{n}{2}+2}\ldots v_n\right\}$. See figure 2.

Define a mapping $f: V\left(G\left(n;\left\{1,\frac{n}{2}\right\}\right)\right)\to N$ as follows:

$f(v_{2i-1})=(i-1)\left(\frac{n}{4}-2\right)+1,\ i=1,2\ldots\left\lceil \frac{n}{8}\right\rceil,$

$f(v_{2i})=\left(\left\lceil \frac{n}{8}\right\rceil+i-1\right)\left(\frac{n}{4}-2\right),\ i=1,2\ldots\left\lfloor \frac{n}{8}\right\rfloor,$

$f\left(v_{\frac{n}{4}+2i-1}\right)=(i-1)\left(\frac{n}{4}-2\right)+1,\ i=1,2\ldots\left\lceil \frac{n}{8}\right\rceil,$

$$f\left(v_{\frac{n}{4}+2i}\right)=\left(\left\lceil \tfrac{n}{8}\right\rceil+i-1\right)\left(\tfrac{n}{4}-2\right),\ i=1,2\ldots\left\lfloor \tfrac{n}{8}\right\rfloor,$$

$$f\left(v_{n-\left(2\left\lceil \frac{n}{8}\right\rceil+2i-3\right)}\right)=\left(\tfrac{n}{4}+i-1\right)\left(\tfrac{n}{4}-2\right)+1,\ i=1,2\ldots\left\lceil \tfrac{n}{8}\right\rceil,$$

$$f\left(v_{n-\left(2\left\lceil \frac{n}{8}\right\rceil+2i-2\right)}\right)=\left(\left\lceil \tfrac{n}{8}\right\rceil+\tfrac{n}{4}+i-1\right)\left(\tfrac{n}{4}-2\right),\ i=1,2\ldots\left\lfloor \tfrac{n}{8}\right\rfloor,$$

$$f\left(v_{n-2(i-1)}\right)=\left(\tfrac{n}{4}+i-1\right)\left(\tfrac{n}{4}-2\right)+1,\ i=1,2\ldots\left\lceil \tfrac{n}{8}\right\rceil,$$

$$f\left(v_{n-(2i-1)}\right)=\left(\left\lceil \tfrac{n}{8}\right\rceil+\tfrac{n}{4}+i-1\right)\left(\tfrac{n}{4}-2\right),\ i=1,2\ldots\left\lfloor \tfrac{n}{8}\right\rfloor.$$

We claim that $d(u,w)+\left|f(u)-f(w)\right|\geq \frac{n}{4}$ for all $u,w\in V(G\left(n;\left\{1,\frac{n}{2}\right\}\right)$.

Case 1: $u,w\in V_1$

Subcase 1.1: If $u=v_{2l-1}$ and $w=v_{2m-1,}$ $1\leq l\neq m\leq\left\lceil \frac{n}{8}\right\rceil$, then $d(u,w)\geq 2,\ f(u)=(l-1)\left(\frac{n}{4}-2\right)+1$ and $f(w)=(m-1)\left(\frac{n}{4}-2\right)+1$. Therefore $d(u,w)+\left|f(u)-f(w)\right|\geq 2+\left|(l-m)\left(\frac{n}{4}-2\right)\right|\geq \frac{n}{4}$.

Subcase 1.2: If $u=v_{2l}$ and $w=v_{2m}$, $1\leq l\neq m\leq\left\lfloor \frac{n}{8}\right\rfloor$, then $d(u,w)\geq 2$, $f(u)=\left(\left\lceil \frac{n}{8}\right\rceil+l-1\right)\left(\frac{n}{4}-2\right)$ and $f(w)=\left(\left\lceil \frac{n}{8}\right\rceil+m-1\right)\left(\frac{n}{4}-2\right)$. Therefore $d(u,w)+\left|f(u)-f(w)\right|\geq 2+\left|(l-m)\left(\frac{n}{4}-2\right)\right|\geq \frac{n}{4}$.

Subcase 1.3: If $u=v_{2l-1}$ and $w=v_{2m}$, then $d(u,w)\geq 2$. Also $f(u)=(l-1)\left(\frac{n}{4}-2\right)+1$ and $f(w)=\left(\left\lceil \frac{n}{8}\right\rceil+m-1\right)\left(\frac{n}{4}-2\right)$. Therefore

$$d(u,w)+\left|f(u)-f(w)\right|\geq 2+\left|(l-1)\left(\tfrac{n}{4}-2\right)+1-\left(\left(\left\lceil \tfrac{n}{8}\right\rceil+m-1\right)\left(\tfrac{n}{4}-2\right)\right)\right|\geq 2+\left|\tfrac{n}{4}-2\right|\geq \tfrac{n}{4}.$$

Similarly we can prove for the cases if $u,w\in V_2$, or $u,w\in V_3$ or $u,w\in V_4$.

Case 2: $u\in V_1$ and $w\in V_2$

Subcase 2.1: If $u=v_{2l-1}$ and $w=v_{\frac{n}{4}+2l-1}$, $1\leq l\leq\left\lceil \frac{n}{8}\right\rceil$, then $d(u,w)=\frac{n}{4}$ and $\left|f(u)-f(w)\right|=0$. Therefore $d(u,w)+\left|f(u)-f(w)\right|\geq \frac{n}{4}+|0|\geq \frac{n}{4}$.

Subcase 2.2: If $u=v_{2l}$ and $w=v_{\frac{n}{4}+2m}$ such that $1\leq l\leq\left\lfloor \frac{n}{8}\right\rfloor$, $1\leq m\leq\left\lfloor \frac{n}{8}\right\rfloor$, $l=m+1$ then $d(u,w)\geq 2$ and $\left|f(u)-f(w)\right|\geq \frac{n}{4}-2$. Therefore $d(u,w)+\left|f(u)-f(w)\right|\geq 2+\left|\frac{n}{4}-2\right|\geq \frac{n}{4}$.

Subcase 2.3: If $u=v_l$ and $w=v_m, l\neq m$, $l\neq m+1$, then $d(u,w)\geq 1$ and $\left|f(u)-f(w)\right|\geq \frac{n}{4}-1$. Therefore $d(u,w)+\left|f(u)-f(w)\right|\geq \frac{n}{4}$.

Similarly we can prove the remaining cases as in case 2. Hence f is a radio antipodal labeling and that

$$an\left(G\left(n;\left\{1,\frac{n}{2}\right\}\right)\right)\leq\left(\frac{n}{2}-1\right)\left(\frac{n}{4}-2\right).$$

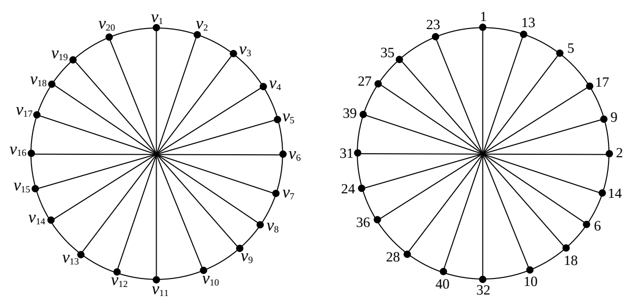

Figure 2 A circulant graph with diameter 5

Theorem: The radio antipodal number of $G\left(n;\left\{1,\frac{n}{3}\right\}\right)$, $n\equiv 0 \pmod 3$ satisfies

$$an\left(G\left(n;\left\{1,\frac{n}{3}\right\}\right)\right)\leq\begin{cases}\left(\frac{n}{6}\right)\left(\frac{n}{2}-1\right)+\left\lceil \frac{n}{12}\right\rceil, & \text{if } n \text{ is even}\\ \left(\left\lceil \frac{n}{6}\right\rceil-1\right)\left(\frac{n-1}{2}\right)+1, & \text{if } n \text{ is odd}\end{cases}.$$

Proof: We partition the vertex set $V=\{v_1,v_2\ldots v_n\}$ into two sets V_1 and V_2, where $V_1=\left\{v_1,v_2\ldots v_{\left\lceil \frac{n}{2}\right\rceil}\right\}$ and

$V_2 = \left\{ v_{\lceil \frac{n}{2} \rceil +1}, v_{\lceil \frac{n}{2} \rceil +2} \dots v_n \right\}$.

We provide the labeling both when n is even and odd. See figure 3 and 4

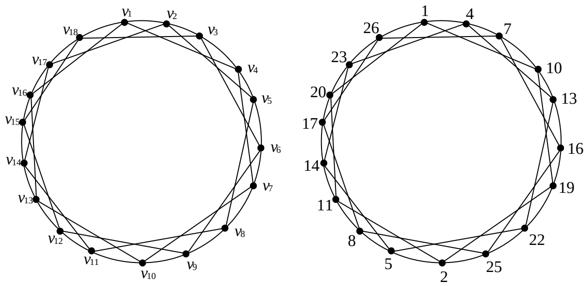

Figure 3 A circulant graph of G(18,{1,2,3,4,5,6}) with diameter 4

Case 1: n is even

Define a mapping $f : V\left(G\left(n;\left\{1,\frac{n}{3}\right\}\right)\right) \to N$ as follows:

$$f(v_i) = \left(\tfrac{n}{6}\right)(i-1)+1,\ i=1,2\dots\tfrac{n}{2},$$

$$f\left(v_{\frac{n}{2}+i}\right) = \left(\tfrac{n}{6}\right)(i-1)+\left\lceil \tfrac{n}{12} \right\rceil+1,\ i=1,2\dots\tfrac{n}{2}.$$

Case 2: n is odd

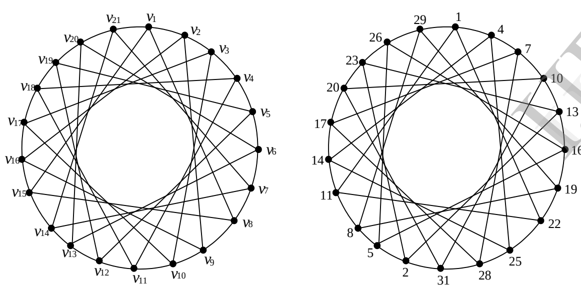

Figure 4 A circulant graph of G(21,{1,2…7}) with diameter 4

Define a mapping $f : V\left(G\left(n;\left\{1,\frac{n}{3}\right\}\right)\right) \to N$ as follows:

$$f(v_i) = \left(\left\lceil \tfrac{n}{6} \right\rceil -1\right)(i-1)+1,\ i=1,2\dots\tfrac{n+1}{2},$$

$$f\left(v_{\frac{n+1}{2}+i}\right) = \left(\left\lceil \tfrac{n}{6} \right\rceil -1\right)(i-1)+\left\lceil \tfrac{n}{12} \right\rceil,\ i=1,2\dots\tfrac{n-1}{2}.$$

proof is similar to the above class.

Theorem: The radio antipodal number of $G\left(n;\left\{1,\frac{n}{5}\right\}\right),\ n \equiv 0 \pmod{10}$ satisfies

$$an\left(G\left(n;\left\{1,\frac{n}{5}\right\}\right)\right) \le \frac{n}{20}(n+8).$$

Proof: We partition the vertex set $V = \{v_1, v_2 \dots v_n\}$ into two sets V_1 and V_2, where $V_1 = \left\{ v_1, v_2 \dots v_{\lceil \frac{n}{2} \rceil} \right\}$ and $V_2 = \left\{ v_{\lceil \frac{n}{2} \rceil +1}, v_{\lceil \frac{n}{2} \rceil +2} \dots v_n \right\}$. See figure 5

Define a mapping $f : V\left(G\left(n;\left\{1,\left\lfloor \frac{n}{5} \right\rfloor\right\}\right)\right) \to N$ as follows:

$$f(v_i) = \left(\tfrac{n}{10}+1\right)(i-1)+1,\ i=1,2\dots\tfrac{n}{2},$$

$$f\left(v_{\frac{n}{2}+i}\right) = \left(\tfrac{n}{10}+1\right)(i-1)+1,\ i=1,2\dots\tfrac{n}{2}.$$

proof is similar.

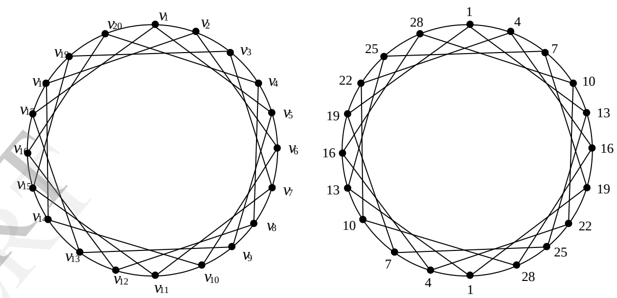

Figure 5 A circulant graph of G(20,{1,2,3,4})

4. Conclusion

The study of radio antipodal number of graphs has gained momentum in recent years. Very few graphs have been proved to have radio antipodal labeling that attains the radio antipodal number. In this paper we have determined the bounds of the radio antipodal number of the lobster and extended mesh. Further study is taken up for various other classes of graphs.

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