Quasi Nonexpansive Sequences
In Dislocated Quasi - Metric Spaces

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Abstract:
We introduce the notion of a quasi - nonexpansive sequence with respect to a non - empty subset of a dislocated quasi - nonexpansive metric space and extend the results of M.A.Ahmed and F.M.Zeyada [1] to such sequences.

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Dislocated quasi - nonexpansive w.r.to F , quasi – metric spaces , asymptotically regular , decreasing sequences and metric spaces.

1. INTRODUCTION

M.A.Ahmed and F.M.Zeyada [1] established the convergence of a sequence \( \{x_n\} \), in a dislocated - quasi metric space \((X,d)\) if a map \( T : X \rightarrow X \) is quasi – nonexpansive with respect to \( \{x_n\} \). We observe that the role played by the map \( T \) in proving the convergence, is meagre. Consequently, we introduce the notion of a quasi – nonexpansive sequence with respect to a non empty subset of a dislocated quasi metric space and establish the convergence of such sequences under certain conditions.

These results extend the results of [1].

We begin with various definitions

Definition 1.1:
Let \( X \) be a non - empty set and let \( d : X \times X \rightarrow [0,\infty) \) be a function called a distance function satisfying one or more of (1.1.1) – (1.1.5).

(1.1.1): \( d(x,x) = 0 \) \( x \in X \).
(1.1.2): \( d(x,y) = d(y,x) = 0 \Rightarrow x = y \) \( x \neq y \in X \).
(1.1.3): \( d(x,y) = d(y,x) \forall x, y \in X \).
(1.1.4): \( d(x,y) \leq d(x,z) + d(z,y) \forall x, y, z \in X \).
(1.1.5): \( d(x,y) \leq \max\{d(x,z), d(z,y)\} \forall x, y, z \in X \).
(i) If \( d \) satisfies (1.1.2) and (1.1.4) then \( d \) is called a dislocated quasi metric (or) \( dq \)-metric and \( (X,d) \) is called a \( dq \)-metric space.

(ii) If \( d \) satisfies (1.1.2), (1.1.3) and (1.1.4) then \( d \) is called a dislocated metric and \( (X,d) \) is called a dislocated metric space.

(iii) If \( d \) satisfies (1.1.1), (1.1.2) and (1.1.4) then \( d \) is called a quasi metric (or) \( q \)-metric and \( (X,d) \) is called a quasi metric space (or) \( q \)-metric space.

(iv) If \( d \) satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.4) then \( d \) is called a metric and \( (X,d) \) is called a metric space.

(v) If \( d \) satisfies (1.1.1), (1.1.2), (1.1.3) and (1.1.5) then \( d \) is called an ultra metric and \( (X,d) \) is called an ultra metric space.

We observe that every ultra metric is a metric.

Let \( D \) be a subset of a quasi metric space \( (X,d) \) and \( T : D \to X \) be any mapping. Assume that \( F(T) \) is the set of all fixed points of \( T \). For a given \( x_0 \in D \), the sequence of iterates \( \{x_n\} \) is defined by

\[
(1) \quad x_n = T(x_{n-1}) = T^n(x_0), \quad \text{where } n \in N \text{ and } N \text{ is the set of all positive integers}.
\]

**Definition 1.2:** (F.M.Zeyada, G.H.Hassan and M.A.Ahmed [11])

A sequence \( \{x_n\} \) in a dislocated quasi metric space \( (X,d) \) is called Cauchy, if to each \( \varepsilon > 0 \), there exists \( n_0 \in N \), such that for all \( m,n \geq n_0 \), \( d(x_m,x_n) < \varepsilon \).

**Definition 1.3:**

A sequence \( \{x_n\} \) in a dislocated quasi metric space \( (X,d) \) is said to be dislocated quasi-convergent (or) \( dq \)-convergent to \( x \), if

\[
\lim_{n \to \infty} d(x_n,x) = \lim_{n \to \infty} d(x,x_n) = 0.
\]

In this case \( x \) is called a dislocated quasi-limit (or) \( dq \)-limit of \( \{x_n\} \) and we write \( x_n \to x \). It can be shown that \( dq \)-limit of a sequence \( \{x_n\} \), if exists is unique.

**Note:** In a dislocated quasi metric space, when we talk of \( dq \)-convergence or \( dq \)-limit, we conveniently drop the prefix “ \( dq \)” in the absence of any ambiguity.

**Definition 1.4:**

A dislocated quasi metric space \( (X,d) \) is complete, if every Cauchy sequence in it is \( dq \)-convergent.

**Definition 1.5:**

Let \( (X,d) \) be a dislocated quasi metric space. Let \( \phi \neq A \subseteq X \).

Then \( d(x,A) = \inf_{a \in A} \{d(x,a), d(a,x)\} \).

**Definition 1.6:** (M.A.Ahmed and F.M.Zeyada [1], definition 2.1)

Let \( (X,d) \) be a quasi-metric space and \( \phi \neq D \subset X \). The mapping \( T : D \to X \) is said to be quasi-nonexpansive w.r.t. a sequence \( \{x_n\} \) of \( D \), if for all \( n \in N \cup \{0\} \) and for every \( p \in F(T) \),

\[
d(x_{n+1},p) \leq d(x_n,p), \quad \text{where } F(T) = \text{the fixed point set of } T.
\]
The following results are proved in (F.M.Zeyada, G.H.Hassan and M.A.Ahmed [11])

Lemma 1.7: (F.M.Zeyada, G.H.Hassan and M.A.Ahmed [11])
Let \((X, d)\) be a dislocated quasi metric space.
Then every \(dq\)-convergent sequence in \(X\) is Cauchy.
It may be noted that the converse of lemma 1.7 is not true.

Lemma 1.8: (F.M.Zeyada, G.H.Hassan and M.A.Ahmed [11])
Let \((X, d)\) be a dislocated quasi metric space. If \(\{x_n\}\) is a sequence in \(X\)
dq-converging to \(x \in X\), then every subsequence of \(\{x_n\}\) dq-converges to \(x\).

Lemma 1.9: (F.M.Zeyada, G.H.Hassan and M.A.Ahmed [11])
Dislocated quasi-limits in a dq-metric space are unique.
(M.A.Ahmed and F.M.Zeyada [1]) proved the following results.

Theorem 1.10: (M.A.Ahmed and F.M.Zeyada [1], Theorem 2.1)
Let \(\{x_n\}\) be a sequence in a subset \(D\) of a q-metric space \((X, d)\) and
\(T: D \to X\) be a map such that \(F(T) \neq \phi\). Then
(a) \(\lim_{n \to \infty} d(x_n, F(T)) = 0\) if \(\{x_n\}\) converges to a unique point in \(F(T)\);
(b) \(\{x_n\}\) converges to a unique point in \(F(T)\) if \(\lim_{n \to \infty} d(x_n, F(T)) = 0\),
\(F(T)\) is a closed set, \(T\) is quasi-nonexpansive w.r.t \(\{x_n\}\) and \(X\) is complete.

Theorem 1.11: (M.A.Ahmed and F.M.Zeyada [1], Theorem 2.2)
Let \(\{x_n\}\) be a sequence in a subset \(D\) of a complete q-metric space \((X, d)\) and
\(T: D \to X\) be a map such that \(F(T) \neq \phi\) is a closed set. Assume that
(i) \(T\) is quasi-nonexpansive w.r.t \(\{x_n\}\);
(ii) \(\lim_{n \to \infty} d(x_n, x_{n+1}) = 0\);
(iii) if the sequence \(\{y_n\}\) satisfies \(\lim_{n \to \infty} d(y_n, y_{n+1}) = 0\), then
\[\lim\inf_{n} d(y_n, F(T)) = 0\] or \(\lim\sup_{n} d(y_n, F(T)) = 0\).
Then \(\{x_n\}\) converges to a unique point in \(F(T)\).

Note: The presence of conditions (ii) and (iii) guarantees that \(\lim_{n \to \infty} d(x_n, F(T)) = 0\).
We show in Example 2.6 that condition (ii) alone may not guarantee that
\(\lim_{n \to \infty} d(x_n, F(T)) = 0\).

2. MAIN RESULTS
In this section, we introduce the notion of a quasi-nonexpansive sequence with respect
to a non-empty subset of a dislocated quasi-nonexpansive metric space and extend
the results in [1] to such spaces.

Definition 2.1:
Let \((X, d)\) be a dislocated quasi metric space, \(\phi \neq F \subset X\) and \(\{x_n\} \subset X\)
such that \(x_n \notin F, \forall n = 1, 2, 3, \ldots\),
Then \(\{x_n\}\) is said to be quasi-nonexpansive w.r.t \(F\), if
\[d(x_{n+1}, p) \leq d(x_n, p)\]
and
\[d( p, x_{n+1}) \leq d ( p, x_n) \forall p \in F \text{ and } n = 1, 2, 3, \ldots,\]
Lemma 2.2:
Let \((X, d)\) be a dislocated quasi metric space, \(\phi \neq F \subseteq X\) and \(\{x_n\} \subset X\), \(x_n \notin F \ \forall \ n\). Suppose that \(\{x_n\}\) is quasi-nonexpansive w.r.to \(F\). Then
\[
\lim_{n \to \infty} d(x_n, F) = 0 \Rightarrow d(x_n, x_{n+1}) \to 0 \quad \text{and} \quad d(x_{n+1}, x_n) \to 0.
\]

Proof:
Let \(\epsilon > 0\). Then there exists \(p \in F\) and positive integer \(M\) such that
\[
d(x_M, p) < \frac{\epsilon}{2} \quad \text{and} \quad d(p, x_M) < \frac{\epsilon}{2} \quad \forall \ n \geq M.
\]∴ \(\{x_n\}\) is quasi-nonexpansive w.r.to \(F\),
\[
d(x_{n+1}, p) \leq d(x_n, p) \leq \cdots \leq d(x_M, p) < \frac{\epsilon}{2}
\]
and
\[
d(p, x_{n+1}) \leq d(p, x_n) \leq \cdots \leq d(p, x_M) < \frac{\epsilon}{2}
\]
Now
\[
d(x_n, x_{n+1}) \leq d(x_n, p) + d(p, x_{n+1})
\]
\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]
\[
= \epsilon \quad \forall \ n \geq M.
\]
and
\[
d(x_{n+1}, x_n) \leq d(x_{n+1}, p) + d(p, x_n)
\]
\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]
\[
= \epsilon \quad \forall \ n \geq M.
\]
∴ \(d(x_n, x_{n+1}) \to 0\) and \(d(x_{n+1}, x_n) \to 0\).

Lemma 2.3:
Suppose \(\{x_n\}\) is quasi-nonexpansive w.r.to \(F\). Then
\[
\lim_{n \to \infty} d(x_n, F) = 0 \Rightarrow \{x_n\}\ is a Cauchy sequence.
\]

Proof:
Let \(\epsilon > 0\). Then there exists \(a\) positive integer \(M\) such that
\[
d(x_n, F) < \frac{\epsilon}{2} \quad \forall \ n \geq M.
\]
Now
\[
d(x_M, F) < \frac{\epsilon}{2} \Rightarrow \exists \ p \in F \ \exists \ d(x_M, p) < \frac{\epsilon}{2} \quad \text{and} \quad d(p, x_M) < \frac{\epsilon}{2}
\]
∴ \(d(x_n, p) \leq d(x_M, p) < \frac{\epsilon}{2}\)
and
\[
d(p, x_n) \leq d(p, x_M) < \frac{\epsilon}{2} \quad \forall \ n \geq M.
\]
Now suppose \(m, n \geq M\). Then
\[
d(x_m, x_n) \leq d(x_m, p) + d(p, x_n)
\]
\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2}
\]
\[
= \epsilon
\]
and
\[ d(x_n, x_m) \leq d(x_m, p) + d(p, x_n) \]
\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} \]
\[ = \epsilon \]
\[ \therefore \{ x_n \} \text{ is a Cauchy sequence.} \]

**Lemma 2.4:**

Let \((X, d)\) be a dislocated quasi metric space and \(\{x_n\}\) be a sequence in \(X\). Assume that \(F\) be a non-empty subset of \(X\). If \(\{x_n\}\) is quasi-nonexpansive w.r.t \(F\), then \(d(x_n, F)\) is a monotonically decreasing sequence in \([0, \infty)\).

**Proof:**

Since \(\{x_n\}\) is quasi-nonexpansive w.r.t \(F\),
\[ d(x_{n+1}, p) \leq d(x_n, p) \rightarrow (2.4.1) \text{ for all } n \in N \cup \{0\} \text{ and for every } p \in F. \]
From (2.4.1), taking the infimum over \(p \in F\), we get that
\[ d(x_{n+1}, F) \leq d(x_n, F) \text{ for all } n \in N \cup \{0\}. \]
Hence \(\{d(x_n, F)\}\) is a monotonically decreasing sequence in \([0, \infty)\).

**Lemma 2.5:**

Let \((X, d)\) be a dislocated quasi metric space and \(\{x_n\}\) be a sequence in \(X\).
Suppose \(\{x_n\}\) is quasi-nonexpansive w.r.t \(F \neq \phi\) satisfying \(\lim_{n \to \infty} d(x_n, F) = 0\).
Then \(\{x_n\}\) is a Cauchy sequence.

**Proof:**

Since \(\{x_n\}\) is a quasi-nonexpansive w.r.t \(F \neq \phi\), to each \(\epsilon > 0\), there exists \(p \in F\) and positive integer \(M\) such that
\[ d(x_m, p) < \frac{\epsilon}{2} \text{ and } d(p, x_n) < \frac{\epsilon}{2} \forall m, n \geq M. \]
Suppose \(m, n \geq M\). Then
\[ d(x_m, x_n) \leq d(x_m, p) + d(p, x_n) \]
\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} \]
\[ = \epsilon \]

and
\[ d(x_n, x_m) \leq d(x_m, p) + d(p, x_n) \]
\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} \]
\[ = \epsilon \]
\[ \therefore \{x_n\} \text{ is a Cauchy sequence.} \]

The following example shows that converse of Lemma 2.5 is not true.

**Example 2.6:**

\(X = \{(-1,0), (1,0)\}\) and the segment \([0,1) , (0,2)]\) of the \(Y\)-axis \(d\) is the usual Euclidean distance in \(R^2\).
\(F = \{(-1,0), (1,0)\}\), \(x_n = (0, 1+ \frac{1}{n})\), \(n = 1, 2, 3 \ldots\)
Then \(\{x_n\}\) is dislocated quasi-nonexpansive w.r.t \(F\)
\(d(x_n, F) \geq d(x_{n+1}, F)\).
\[ d(x_n, x_{n+1}) \to 0 \quad \text{and} \quad x_n \to (0,1) \notin F. \]

Now we state and prove our first main result, which is an extension of Theorem 1.10 to quasi–nonexpansive sequences.

**Theorem 2.7:**

Let \{x_n\} be a sequence in a subset \(D\) of a dislocated quasi–metric space \((X, d)\) and \(\phi \neq F \subset D \quad (x_n \notin F \forall n). \) Then

(a) \( \lim_{n \to \infty} d(x_n, F) = 0, \) if \{x_n\} converges to a point in \(F\)

(b) \{x_n\} converges to a unique point in \(F, \) if \( \lim_{n \to \infty} d(x_n, F) = 0, \)

\(F\) is a closed set, \(\{x_n\}\) is quasi–nonexpansive w.r.t \(F\) and \(X\) is complete.

**Proof of (a):**

Since \{x_n\} converges to a point in \(F,\) there exists a point \(p \in F\) such that

\[ \lim_{n \to \infty} d(x_n, p) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(p, x_n) = 0 \]

\[ \therefore \text{Given } \epsilon > 0, \text{ there exists a positive integer } M \text{ such that} \]

\[ d(x_n, p) < \frac{\epsilon}{2} \quad \text{and} \quad d(p, x_n) < \frac{\epsilon}{2} \quad \text{for every } n \geq M. \]

\[ \therefore d(p, p) \leq d(p, x_n) + d(x_n, p) \]

\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n \geq M. \]

\[ \therefore d(p, p) < \epsilon \quad \text{for every } \epsilon > 0, \]

\[ \therefore d(p, p) = 0. \]

Now

\[ d(x_n, F) \leq d(x_n, p) < \frac{\epsilon}{2} \quad \forall \quad n \geq M. \]

\[ \therefore \lim_{n \to \infty} d(x_n, F) = 0 \]

\[ \therefore \text{(a) holds.} \]

**Proof of (b):**

Let \((X, d)\) be a complete dislocated quasi–metric space and \{x_n\} be a sequence in \(X\) and \(\phi \neq F \subset X. \) Assume that \{x_n\} is quasi–nonexpansive w.r.t \(F,\) \(F\) is closed and \(\lim_{n \to \infty} d(x_n, F) = 0.\) Then \{x_n\} is a Cauchy sequence by lemma 2.5, hence there exists \(p\) such that \{x_n\} converges to \(p.\)

Let \(\epsilon > 0.\) There exists a positive integer \(M\) such that

\[ d(x_n, F) < \frac{\epsilon}{2} \quad \text{for every } n \geq M \quad \text{and} \]

\[ d(x_n, p) < \frac{\epsilon}{2} \quad \text{and} \quad d(p, x_n) < \frac{\epsilon}{2} \quad \forall \quad n \geq M. \]

\[ \therefore \text{There exists } q_M \in F \text{ such that} \]

\[ d(x_M, q_M) < \frac{\epsilon}{2} \quad \text{and} \quad d(q_M, x_M) < \frac{\epsilon}{2} \]

\[ \therefore d(p, q_M) \leq d(p, x_M) + d(x_M, q_M) \]

\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

\[ d(p, q_M) < \epsilon \]

and similarly we have

\[ d(x_{n+1}, F) \leq d(x_{n+1}, P) \leq d(x_n, p) \forall \quad p \in F \]

\[ d(x_{n+1}, F) \leq d(x_{n+1}, P) \leq d(x_n, p) \forall \quad p \in F \]
\[ d(q_M, p) \leq d(q_M, x_M) + d(x_M, p) \]
\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
\[
\therefore \quad d(q_M, p) < \varepsilon
\]
\[
\therefore \quad p \text{ is a limit point of } F
\]
\[
\therefore \quad p \in F, \text{ since } F \text{ is closed}
\]
Since limits are unique (by lemma 1.9), \( \{x_n\} \) converges to a unique point \( p \in F \).

Hence (b) holds.

The following theorem which is an analogue of Theorem 1.11 establishes the convergence of the sequence.

**Theorem 2.8**:  
Let \((X, d)\) be a complete dislocated - quasi metric space. Assume that \( \{x_n\} \) is a sequence in \( X \) and \( \phi \neq F \subseteq X \). Further assume that there is a mapping \( \phi : [0, \infty) \to [0,1) \) such that \( \phi \) is monotonically increasing and
\[ d(x_{n+1}, F) \leq \phi( d(x_n, F)) d(x_n, F) \quad \text{for } n = 1, 2, 3 \ldots \to (2.8.1) \]
Then \( \{x_n\} \) is Cauchy and \( \{x_n\} \) converges to a point \( q \). If further \( F \) is closed then \( q \in F \).

**Proof**:  
By hypothesis
\[ d(x_{n+1}, F) \leq \phi( d(x_n, F)) d(x_n, F) \leq d(x_n, F), \]
so that \( \{d(x_n, F)\} \) is decreasing and hence \( \{\phi(d(x_n, F))\} \) is decreasing since \( \phi \) is increasing.
\[
\therefore \quad d(x_{n+1}, F) \leq \phi(d(x_n, F)) d(x_n, F) \leq \phi(d(x_{n-1}, F)) \phi(d(x_{n-1}, F)) \phi(d(x_{n-1}, F)) \ldots \phi(d(x_1, F)) d(x_1, F)
\]
\[
\leq \phi(d(x_1, F)) \phi(d(x_1, F)) \ldots \phi(d(x_1, F)) d(x_1, F)
\]
\[
= (\phi(d(x_1, F)))^n d(x_1, F) \to 0 \text{ as } n \to \infty
\]
( \( \therefore \quad \phi(d(x_n, F) < 1) \)
Thus \( d(x_n, F) \to 0 \) as \( n \to \infty \).

\[
\therefore \quad \text{By lemma 2.5, } \{x_n\} \text{ is Cauchy sequence and hence converges to a point } q
\]
since \( X \) is complete. If \( F \) is closed by (Theorem (2.7) (b)) follows that \( q \in F \).

**The following Example shows that**

Theorem 2.8 may not hold good if (2.8.1) is replaced by
\[ d(x_{n+1}, F) \leq d(x_n, F) \quad \text{for } n = 1, 2, 3, \ldots \to (2.8.2) \]
even if we assume that (even in a metric space)
\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \quad \to (2.8.3) \]

**Example 2.9**:  
Let \( X \) be the subset of \( R \times R \) consisting of the points \((-1,0), (1,0)\) and the segments of the \( Y \) axis joining the two points \((0,1)\) and \((0,2)\).
Hence \( X = \{(-1,0), (1,0) \text{ and } \{(0,y)/1 \leq y \leq 2\}\} \)
Let \( d \) be the Euclidean metric in \( R^2 \).
Take \( F = \{(-1,0), (1,0)\} \) and \( x_n = \{(0,1 + \frac{1}{n})/n = 1, 2, \ldots \} \)
Then \( \{x_n\} \) is quasi - nonexpansive w.r.to to \( F \),
\[ d(x_{n+1}, F) \leq d(x_n, F) \quad \text{for } n = 1, 2, 3, \ldots \]
\[ d(x_n, x_{n+1}) \to 0, \ F \text{ is closed but } \{d(x_n, F)\} \text{ does not converges to } 0. \]

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