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Properties Of Multivalent Holomorphic Functions And Some Results For Univalent Functions Defined By A Generalized Salagean Operator

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Abstract

In this paper we derived properties like differential Subordination, Hadamard product, Quasi-Hadamard product of Holomorphic Univalent and Multivalent functions with positive, negative Taylor series expansion. The results for Univalent functions are defined by a generalized Salagean operator.

Key Words *Analytic function, Holomorphic function, Salagean operator, Subordination.*

1. Introduction

This paper contains the discussion of differential subordination and discussion of convolution and quasi-convolution. The differential subordination in the complex plane is the generalization of a differential inequality on real line. In convolution or Hadamard product of two power series the term "convolution" arises from the formula $h(r^2e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i(\theta-t)})g(re^{it})dt,\ \ r<1$. Convolution has the algebraic properties of ordinary multiplication. The geometric series $\ell(z)=\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$ acts as the identity element under convolution $f*\ell = f \ \ \forall f$. A lot of literature on differential subordination is available in nuture for example Hardy, Littlewood and Polya [1], Protter and Weinberger [2], Walter [3], G. Goluzin [4],R. Robinson [5] S. S. Miller and P. T. Mocanu [6] etc.The study of convolution has been taken into consideration by number of authors, Hayman [7], Epstein and Schoenberg [8], Loewner and Netanyahu [9], Suffridge [10] and Robertson [11] etc. We have also extended the concept of convolution to quasi-convolution and have obtained nice characterizations. For our convenience throughout this paper we are considering

$$\begin{pmatrix} a_{2k}\ge 0, n\ge 0, m\ge 0, 0\le\gamma\ \le 1, \frac{\sigma(\eta+\varsigma)}{\varepsilon(\delta+\tau)+p} = \frac{a\mu}{\lambda}, \\ \varsigma\ge 0,\tau\ge 0,\delta\ge 0, 0<\varepsilon\le\frac{1}{2}, 0\le\alpha<1, \\ \eta\ge 0, 0<\sigma\le\frac{1}{2}, -\frac{1}{2}\le B<A\le\frac{1}{2} \end{pmatrix},$$

$$D^{\varepsilon(\delta+\tau)+p} = H_p^{\lambda},\ \ \sigma(\eta+\varsigma) = 2a, \varepsilon(\delta+\tau)+p=\lambda$$

1.1 Preliminary Lemma

1.1.1 Let $f(z)$ be of the form $f(z) = z-\sum_{k=2}^{\infty} a_{2k}z^{2k}$, Then f(z) belongs to $T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\alpha)$ if and only if

$$\sum_{k=2}^{\infty}\{[(2k-1)+2(2k-\propto)B-2(1-\propto)A] \times \frac{\langle 1+(2k-1)\sigma(\eta+\varsigma)\rangle^n\langle 1-\gamma+\gamma[1+(2k-1)\sigma(\eta+\varsigma)]\rangle^m}{(B-2A)(1-\propto)}\Big\} a_{2k}\le 1.$$

Proof: Since $f(z)\in T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\alpha)$ And

$$T^*(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\alpha) = \left\{f: f\in T^*: \frac{(1-\gamma)z[D_*^n f(z)]' + \gamma z\left[D_*^{n+m}f(z)\right]'}{(1-\gamma)D_*^n f(z)+\gamma D_*^{n+m}f(z)} \in P(A,B,\alpha)\right\}$$

Then $\frac{(1-\gamma)z[D_*^n f(z)]' + \gamma z\left[D_*^{n+m}f(z)\right]'}{(1-\gamma)zD_*^n f(z)+\gamma D_*^{n+m}f(z)}$

$$= \frac{z-\sum_{k=2}^{\infty} 2kX^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}}{z-\sum_{k=2}^{\infty} X^n(1-\gamma+\gamma X^m a_{2k}z^{2k}} \prec \frac{1+[2(1-\propto)A+2\propto B]z}{1+2Bz}$$

Where $X = 1+(2k-1)\sigma(\eta+\varsigma)$

Now, by definition of subordination, there exists $w(z)$ which is Holomorphic Function in U with $w(0)=0$, $|w(z)|=1$ in U such that

$$\frac{z-\sum_{k=2}^{\infty} 2kX^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}}{z-\sum_{k=2}^{\infty} X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k}} \prec \frac{1+2[(1-\propto)A+\propto B]w(z)}{1+2Bw(z)}$$

then by simple calculations, we obtain

$$\frac{\sum_{k=2}^{\infty}(2k-1)X^n(1-\gamma+\gamma X^m a_{2k}z^{2k-1}}{(2B-2(1-\propto)A-)\propto B-\sum_{k=2}^{\infty} 2[(2k-\propto)B-(1-\propto)A]X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}}$$

Thus by noting |w (z)| < 1, we get

$w(z) =$

$$\left|\frac{\sum_{k=2}^{\infty}(2k-1)X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}}{2(B-A)(1-\propto)-\sum_{k=2}^{\infty}2[(2k-\propto)B-(1-\propto)A]X^n(1-\gamma+\gamma X^m)a_{2k}z^{2k-1}}\right| < 1$$

Letting $z \to 1^-$, we get

$$\frac{\sum_{k=2}^{\infty}(2k-1)X^n(1-\gamma+\gamma X^m)a_{2k}}{2(B-A)(1-\propto)-\sum_{k=2}^{\infty}2[(2k-\propto)B-(1-\propto)A]X^n(1-\gamma+\gamma X^m)a_{2k}} < 1$$

$\therefore \sum_{k=2}^{\infty}(2k-1)X^n(1-\gamma+\gamma X^m)a_{2k} < 2(B-A)(1-\propto) - \sum_{k=2}^{\infty}2[(2k-\propto)B - (1-\propto)A]X^n(1-\gamma+\gamma X^m)a_{2k}$ Then

$$\sum_{k=2}^{\infty}\left\{[(2k-1)+2(2k-\propto)B-2(1-\propto)A] \times \frac{[1+(2k-1)a\mu]^n(1-\gamma+\gamma[1+(2k-1)a\mu]^m)}{2(B-A)(1-\propto)}a_{2k}\right\} \le 1.$$

2. Analysis and Main Results

2.1 Applications of Differential Subordination Let $A(p,1)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{P+n} \quad (a_n \ge 0;\ p \in N) \qquad (1)$$

which are holomorphic in the open unit disc $U = \{z: |z| < 1\}$. Let $f(z), g(z) \in A(p,1)$, where $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{P+n}$ and $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}z^{P+n}$. Then the convolution

$$(f*g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^{P+n}\ z^{P+n} \qquad (2)$$

Let $A, B, \sigma, \eta, \varsigma$ and $\varepsilon, \delta, \tau$ be fixed real numbers. A function $f(z) \in A(p,1)$ belongs to the class $I_{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau)}(p; A, B)$ if it satisfies

$$\ell_{\varepsilon,\delta,\tau,p}(f) \prec \frac{1+2Az}{1+2Bz} \quad (z \in U) \qquad (3)$$

$$\ell_{\varepsilon,\delta,\tau,p}(f) = [1-\sigma(\eta+\varsigma)]\frac{H_p^{\lambda-1}f(z)}{z^p} + \sigma(\eta+\varsigma)\frac{D^{\varepsilon(\delta+\tau)+p}f(z)}{z^p}$$

where $H_p^{\lambda-1}f(z) = z^p + \sum_{n=1}^{\infty}\frac{\Gamma\lambda+n)}{\Gamma\lambda n!}a_{p+n}z^{p+n}$. (4)

Where Γ is a gamma function. From above equation it is follows that $z\left[H_p^{\lambda-1}f(z)\right]' = \lambda H_p^{\lambda}f(z) - (\lambda-p)H_p^{\lambda-1}f(z)$. (5)

This work is motivated by K. Piejko and J. Sokol [12] and J. Patel and S. Rout [13], where we have used the techniques of differential subordination to obtain several interesting properties. A holomorphic function f is said to be close-to-convex of order $\alpha (0 \le \alpha < 1)$. If there exists a convex function $h \in A(1,1)$ and $\beta \in R$ such that $Re\left[\frac{f'(z)}{e^{i\beta}h'(z)}\right] > \alpha\ for\ z \in U$.

Theorem 2.1.1 Let $f(z) \in A(p,1)$ then

$$z\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'' = \lambda\left[z^{1-p}H_p^{\lambda}f(z)\right]' - (\varepsilon[\delta+\tau]+p)\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'. \qquad (6)$$

Proof We know

$z\left[H_p^{\lambda-1}f(z)\right]' = \lambda H_p^{\lambda}f(z) - (\lambda-p)H_p^{\lambda-1}f(z)$.

$\therefore z\left[H_p^{\lambda-1}f(z)\right]' + 1 - pH_p^{\lambda-1}f(z) = \lambda H_p^{\lambda}f(z) - (\lambda-p)H_p^{\lambda-1}f(z) + (1-p)H_p^{\lambda-1}f(z)$.

$= \lambda H_p^{\lambda}f(z) + (1-\lambda)H_p^{\lambda-1}f(z)$. But owing to $z\left[H_p^{\lambda-1}f(z)\right]' + (1-p)H_p^{\lambda-1}f(z)$

$= z^p\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'$, We obtain, $z\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'$

$= \lambda\left[z^{1-p}H_p^{\lambda-1}f(z)\right] + (1-\lambda)\left[z^{1-p}H_p^{\lambda-1}f(z)\right]$

Differentiating both sides we get, $z\left[z^{1-p}H_p^{\lambda-1}f(z)\right]''$

$= \lambda\left[z^{1-p}H_p^{\lambda}f(z)\right]' - \lambda\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'$.

Thus theorem holds true.

Corollary 2.1.1 Let $f(z) \in A(p,1)$and $z^{1-p}D^{\varepsilon(\delta+\tau)+p-1}f(z)$ is convex univalent function. Then $z^{1-p}H_p^{\lambda-1}f(z)$ is close-to-convex of order $\frac{\lambda-1}{|\lambda|}$ With respect to $z^{1-p}H_p^{\lambda-1}f(z)$.

Proof Since $z\left[z^{1-p}H_p^{\lambda-1}f(z)\right]''$

$= \lambda\left[z^{1-p}H_p^{\lambda}f(z)\right]' - \lambda\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'$.

$\therefore z\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'' = \lambda\left[z^{1-p}H_p^{\lambda}f(z)\right]' - \lambda\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'$. We obtain

$$\frac{\left[z^{1-p}H_p^{\lambda}f(z)\right]'}{\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'} = \frac{z\left[z^{1-p}H_p^{\lambda-1}f(z)\right]''}{\lambda\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'} + 1$$

Since $z^{1-p}H_p^{\lambda-1}f(z)$ is a convex function,

$$Re\left\{\frac{\lambda\left[z^{1-p}H_p^{\lambda}f(z)\right]'}{|\lambda|\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'}\right\} = Re\left\{\frac{z\left[z^{1-p}H_p^{\lambda-1}f(z)\right]''}{|\lambda|\left[z^{1-p}H_p^{\lambda-1}f(z)\right]'} + \frac{\lambda}{|\lambda|}\right\} > Re\frac{\lambda-1}{|\lambda|}$$

Therefore, by definition of close-to-convex we get the required result.

Theorem 2.1.2 Let $f_1(z), f_2(z) \in A(p,1)$, $\ell_{\varepsilon,\delta,\tau,,p}[f_1(z)] \prec h_1(z)$ and $\ell_{\varepsilon,\delta,\tau,p}[f_2(z)] \prec h_2(z)$, where $h_1(z)$, $h_2(z)$ are convex univalent in U and if $\frac{\lambda}{a\mu} \ge 0, \lambda > a\mu > 0$ then $\ell_{\varepsilon,\delta,\tau,p}\left[H_p^{\lambda-1}(h_1*h_1)(z)\right] \prec \frac{\lambda}{a\mu}z^{-\frac{\lambda}{a\mu}}\int_0^z t^{\frac{\lambda}{a\mu}-1}[h_1(t)*h_2(t)]dt \prec. [h_1(z)*h_2(z)]$

Proof Since $\ell_{\varepsilon,\delta,\tau,p}[f_1(z)] \prec h_1(z)$ and $l_{\varepsilon,\delta,\tau,p}[f_2(z)] \prec h_2(z)$ Then we have $\ell_{\varepsilon,\delta,\tau,p}[f_1(z)] * \ell_{\varepsilon,\delta,\tau,p}[f_2(z)] \prec h_1(z)*h_2(z)$. And by [13], the convolution of convex univalent functions is also the convex univalent function. Now, let

$p(z) = \ell_{\varepsilon,\delta,\tau,p}\left[H_p^{\lambda-1}(f_1*f_1)(z)\right]$

$$= [1-a\mu]\frac{H_p^{\lambda-1}[H_p^{\lambda-1}(f_1*f_2)(z)]}{z^p} + a\mu\frac{H_p^{\lambda}[H_p^{\lambda-1}(f_1*f_2)(z)]}{z^p}$$

Then $p(z)$ is holomorphic function and $p(0) = 1$ in U. since we have $z\left[H_p^{\lambda-1}f(z)\right]'$

$= \lambda H_p^{\lambda}f(z) - (\lambda-p)H_p^{\lambda-1}f(z)$

$\therefore p(z) + \frac{a\mu z}{\lambda}p'(z) = \ell_{\varepsilon,\delta,\tau,p}\left[H_p^{\lambda-1}(f_1*f_1)(z)\right] + \frac{a\mu z}{\lambda}\left(l_{\varepsilon,\delta,\tau,p}\left[H_p^{\lambda-1}(f_1*f_1)(z)\right]\right)'$

$= \left(1-\frac{a\mu p}{\lambda}\right)z^{-p}\left[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\right] + \frac{a\mu}{\lambda}z^{1-p}\left[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\right]'$

$+\frac{a\mu z}{\lambda}\Big\{\Big(1-\frac{a\mu p}{\lambda}\Big)z^{-p}\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]'$
$+\frac{a\mu}{\lambda}z^{1-p}\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]''\Big\}$
$=\Big(1-\frac{a\mu p}{\lambda}\Big)z^{-p}\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]+\frac{a\mu}{\lambda}z^{1-p}$
$+\frac{a\mu}{\lambda}z\Big\{\Big(1-\frac{a\mu}{\lambda}p\Big)\Big[-pz^{-(p+1)}H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)$
$+z^{-p}\Big(H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\Big)'\Big]+\frac{a\mu}{\lambda}(1-p)$
$\Big[\times z^{-p}\Big(H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\Big)'$
$+z^{1-p}\Big(H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\Big)''\Big]\Big\}$
$=\Big[1-2\frac{a\mu}{\lambda}p+\Big(\frac{a\mu}{\lambda}\Big)^2p^2\Big]z^{-p}$
$\times\Big(H_p^{\lambda-1}f_1(z)*D^{\mu+p-1}f_2(z)\Big)$
$+\Big[\frac{a\mu}{\lambda}+\frac{a\mu}{\lambda}\Big(1-\frac{a\mu}{\lambda}p\Big)+\Big(\frac{a\mu}{\lambda}\Big)^2(1-p)\Big]$
$\times z^{1-p}\Big(H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\Big)'$
$+\Big(\frac{a\mu}{\lambda}\Big)^2z^{2-p}\Big(H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\Big)''.$

Now $\ell_{\varepsilon,\delta,\tau,p}[f_1(z)]*\ell_{\varepsilon,\delta,\tau,p}[f_2(z)]=$
$\Big[\Big(1-\frac{a\mu}{\lambda}p\Big)z^{-p}H_p^{\lambda-1}f_1(z)+\frac{a\mu}{\lambda}z^{1-p}\Big(H_p^{\lambda-1}f_1(z)\Big)'\Big]$
$*\Big[\Big(1-\frac{a\mu}{\lambda}p\Big)z^{-p}H_p^{\lambda-1}f_2(z)+\frac{a\mu}{\lambda}z^{1-p}\Big(H_p^{\lambda-1}f_2(z)\Big)'\Big]$
$=\Big(1-\frac{a\mu}{\lambda}p\Big)^2z^{-p}\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]$
$+2\Big(1-\frac{a\mu}{\lambda}p\Big)\frac{a\mu}{\lambda}z^{1-p}\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]'$
$+\Big(\frac{a\mu}{\lambda}\Big)^2z^{1-p}\Big(z\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]'\Big)'=$
$\Big(1-\frac{a\mu}{\lambda}p\Big)^2z^{-p}\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]$
$+\Big[2\Big(1-\frac{a\mu}{\lambda}p\Big)^2\frac{a\mu}{\lambda}+\Big(\frac{a\mu}{\lambda}\Big)^2\Big]z^{1-p}$
$\times\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]'+\Big(\frac{a\mu}{\lambda}\Big)^2z^{2-p}$
$\times\big[H_p^{\lambda-1}f_1(z)*H_p^{\lambda-1}f_2(z)\big]''$. Then
$p(z)+\frac{a\mu}{\lambda}zp'(z)=$
$\ell_{\varepsilon,\delta,\tau,p}[f_1(z)]*\ell_{\varepsilon,\delta,\tau,p}[f_2(z)]\prec h_1(z)*h_2(z).$
$\therefore p(z)\prec\frac{\lambda}{a\mu}z^{-\frac{\varepsilon(\delta+\tau)+p}{\sigma(\eta+\varsigma)}}$
$\int_0^z t^{\frac{\lambda}{a\mu}-1}[h_1(t)*h_2(t)]dt\prec[h_1(z)*h_2(z)].$

Theorem 2.1.3 Let $f_1(z)\in\ell_{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(p;\ A_1,B_1)$ and $f_2(z)\in\ell_{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(p;\ A_2,B_2)$ that is $\ell_{\varepsilon,\delta,\tau,p}[f_1(z)]\prec\frac{1+A_1z}{1+B_1z}$ *and* $\ell_{\varepsilon,\delta,\tau,p}[f_2(z)]\prec\frac{1+A_2z}{1+B_2z}$. where $-1\le B_1<A_1\le1;\ -1\le B_2<A_2\le1$ $\varepsilon(\delta+\tau)+p>\sigma(\eta+\varsigma)>0$ And $\frac{\lambda}{a\mu}\ge0$ Then $\ell_{\varepsilon,\delta,\tau,p}\big[H_p^{\lambda-1}(f_1*f_2)(z)\big]\prec1+(A_1-B_1)(A_2-B_2)$ $\times\frac{\lambda}{a\mu}z^{-\frac{\lambda}{a\mu}}\int_0^z\frac{t^{\frac{\lambda}{a\mu}-1}}{1-B_1B_2t}dt=q(z).$ Where $q(z)=1+\frac{\lambda(A_1-B_1)(A_2-B_2)z}{\lambda+a\mu}\times$ $[1-B_1B_2tz]^{-1}2F1\Big(1,1;2+\frac{\lambda}{a\mu};\frac{B_1B_2z}{B_1B_2z-1}\Big).$ (7)

Proof Since $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are univalent convex functions, $\frac{1+A_1z}{1+B_1z}*\frac{1+A_2z}{1+B_2z}=\Big[1+(A_1-B_1)\frac{z}{1+B_1z}\Big]$ $*\Big[1+(A_2-B_2)\frac{z}{1+B_2z}\Big]$ $=1+(A_1-B_1)(A_2-B_2)\frac{z}{1+B_1B_2z}$. Thus, by Theorem 2.1.2 we have $\ell_{\varepsilon,\delta,\tau,p}\big[H_p^{\lambda-1}(f_1*f_2)(z)\big]$
$\prec1+(A_1-B_1)(A_2-B_2)\frac{\lambda}{a\mu}z^{-\frac{\lambda}{a\mu}}\int_0^z\frac{t^{\frac{\lambda}{a\mu}-1}}{1-B_1B_2t}dt$
$q(z)=\frac{\varepsilon(\delta+\tau)+p}{\sigma(\eta+\varsigma)}z^{-\frac{\lambda}{a\mu}}\int_0^z t^{\frac{\lambda}{a\mu}-1}\Big(1+\frac{(A_1-B_1)(A_2-B_2)t}{1-B_1B_2t}\Big)dt$
$=1+(A_1-B_1)(A_2-B_2)\frac{\lambda}{a\mu}z$
$\times\int_0^1 s^{\frac{\lambda}{a\mu}-1}(1-B_1B_2sz)^{-1}ds$ Hence we obtained the required result. Putting $A_1=A_2=B_1=B_2=1$ in Theorem 2.1.3, we have.

Corollary 2.1.2 Let $f_1(z),f_2(z)\in A(p,1)$. Let $\ell_{\varepsilon,\delta,\tau,p}[f_1(z)]\prec\frac{1+z}{1-z}$ and $\ell_{\varepsilon,\delta,\tau,p}[f_2(z)]\prec\frac{1+z}{1-z}$ then
$\ell_{\varepsilon,\delta,\tau,p}(f_1*f_2)(z)\prec1+4\frac{\lambda}{a\mu}z^{-\frac{\lambda}{a\mu}}\int_0^z\frac{t^{\frac{\lambda}{a\mu}-1}}{1+t}dt.$
Putting $\sigma(\eta+\varsigma)=1,\ \varepsilon(\delta+\tau)=0$ in above corollary 2.1.2 we have, next corollary.

Corollary 2.1.3 Let $f_1(z),f_2(z)\in A(p,1)$. Let $\ell_{0,0,0,p}[f_1(z)]\prec\frac{1+z}{1-z}$ and $\ell_{0,0,0,p}[f_2(z)]\prec\frac{1+z}{1-z}$ then $l_{0,p}(f_1*f_2)(z)\prec1+4pz^{-p}\int_0^z\frac{t^{p-1}}{1+t}dt.$

Consider the following integral transform
$F_c(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt=\sum_{n=p}^{\infty}\frac{c+p}{c+n}z^n*f(z)$ (8)
Where $f(z)\in A(p,1)$ and $c+p>0$. Now since $H_p^{\lambda-1}f(z)=\frac{z^p}{(1-z)^\lambda}*f(z)=z^p+$ $\sum_{n=1}^{\infty}\frac{\Gamma(\lambda+n)}{\Gamma\lambda n!}a_{n+p}z^{n+p}$, Then we have $z\big[H_p^{\lambda-1}f_c(z)\big]'$ $=(c+p)H_p^{\lambda-1}f(z)-cH_p^{\lambda-1}f_c(z).$ (9)

Theorem 2.1.4 Let μ,c be real numbers $(\mu\ge o)$ such that $c+p>0$. If $f_1(z),f_2(z)\in A(p,1)$ satisfy
$\frac{H_p^{\lambda-1}(f_1*f_2)(z)}{z^p}\prec\frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z}$ then
$\frac{H_p^{\lambda-1}[F_c(z)*G_c(z)]}{z^p}\prec q(z)\prec\frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z}$
Where $F_c(z)$ is defined as
$F_c(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt=\sum_{n=p}^{\infty}\frac{c+p}{c+n}z^n*f(z)$
$H_p^{\lambda-1}f(z)=\frac{z^p}{(1-z)^\lambda}*f(z)=z^p\sum_{n=1}^{\infty}\frac{\Gamma(\lambda+n)}{\Gamma\lambda n!}a_{n+p}z^{n+p}.$
$G_c(z)$ is defined as $G_c(z)=\sum_{n=p}^{\infty}\frac{c+p}{c+n}z^n*f_2(z)$ and
$q(z)=1+(1-B_1B_2z)^{-1}\frac{c+p}{c+n+1}(A_1-B_1)$
$\times(A_2-B_2)z\,2F1\Big(1,1;\ 2+c+p;\frac{B_1B_2z}{B_1B_2z-1}\Big).$

Proof Let $p(z) = \frac{H_p^{\lambda-1}(F_c * G_c)(z)}{z^p}$ then $p(z)$ is holomorphic in U with $p(0) = 1$. Since we know $Z[H_p^{\lambda-1} f_c(z)]' = (c+p)H_p^{\lambda-1} f(z)$ $-cH_p^{\lambda-1} f_c(z)$. Hence

$$p(z) + \frac{zp'(z)}{c+p} = \frac{H_p^{\lambda-1}[f_1 * f_2](z)}{z^p} \prec 1 + \frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z}$$

$$\therefore \frac{H_p^{\lambda-1}(F_c * G_c)(z)}{z^p} \prec q(z)$$

$$= (c+p)z^{-(c+p)} \int_0^z t^{c+p-1} \frac{(1+A_1t)}{(1+B_1t)} * \frac{(1+A_2t)}{(1+B_2t)} dt$$

$$\prec 1 + \frac{(A_1-B_1)(A_2-B_2)z}{1-B_1B_2z}$$ Finally we obtain

$$q(z) = 1 + (1 - B_1B_2z)^{-1} \frac{c+p}{c+n+1}(A_1 - B_1)$$

$$\times (A_2 - B_2)z \, 2F1\left(1,1;\, 2 + c + p\,; \frac{B_1B_2z}{B_1B_2z-1}\right)$$ If we put $\delta = \tau = 0, p = A_1 = A_2 = 1, B_1 = B_2 = -1$ in above Theorem 2.1.4, then we have

Corollary 2.1.4 Let $c + 1 > 0$ where c a real number. If $f_1(z), f_2(z) \in A(p, 1)$ and $\frac{(f_1 * f_2)(z)}{z} \prec 1 + \frac{4z}{1-z}$ then $\frac{[F_c(z) * G_c(z)]}{z} \prec q(z) \prec 1 + \frac{4z}{1-z}$. where

$$F_c(z) = \sum_{n=1}^{\infty} \frac{c}{c+n} z^n * f_1(z),$$

$$G_c(z) = \sum_{n=1}^{\infty} \frac{c}{c+n} z^n * f_2(z),$$

$$q(z) = 1 + 4(1-z)^{-1} \frac{c+1}{c+2}\, 2F1\left(1,1;\, c+3\,; \frac{z}{z-1}\right)$$

2.2 Convolution and Quasi-Convolution Properties

Let T denote the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \tag{10}$$

Let $P(A, B, \alpha)$ be the class of holomorphic functions in U that satisfies $f(z) \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}$ Where $-\frac{1}{2} \leq B < A \leq \frac{1}{2}, 0 \leq \alpha < 1$.

Consider T_2^* as as subclass of T that consists of functions of the form $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ (11)

Define $T(n, m, \gamma, \sigma, \eta, \varsigma, A, B, \alpha)$

$$= \left\{f : f \in T : \frac{(1-\gamma)z[D^n f(z)]' + \gamma z[D^{n+m} f(z)]'}{(1-\gamma)D^n f(z) + \gamma D^{n+m} f(z)} \in P(A, B, \alpha)\right\} \tag{12}$$

$T^*(n, m, \gamma, \sigma, \eta, \varsigma, A, B, \alpha)$

$$= \left\{f : f \in T^* : \frac{(1-\gamma)z[D_*^n f(z)]' + \gamma z[D_*^{n+m} f(z)]'}{(1-\gamma)D_*^n f(z) + \gamma D_*^{n+m} f(z)} \in P(A, B, \alpha)\right\} \tag{13}$$

$$S^*(n, \sigma, \eta, \varsigma, A, B, \alpha) = T(n, m, 0, \sigma, \eta, \varsigma, A, B, \alpha) \tag{14}$$

$$K(n, m, \sigma, \eta, \varsigma, A, B, \alpha) = T(n, m, 1, \sigma, \eta, \varsigma, A, B, \alpha) \tag{15}$$

$$S_2^*(n, \sigma, \eta, \varsigma, A, B, \alpha) = T_2^*(n, m, 0, \sigma, \eta, \varsigma, A, B, \alpha) \tag{16}$$

$$K_2^*(n, m, \sigma, \eta, \varsigma, A, B, \alpha) = T_2^*(n, m, 1, \sigma, \eta, \varsigma, A, B, \alpha) \tag{17}$$

$$D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\sigma(\eta + \varsigma)]^n a_k z^k$$

$$D_*^n f(z) = z + \sum_{k=2}^{\infty} [1 + (2k-1)\sigma(\eta + \varsigma)]^n a_{2k} z^{2k} \tag{18}$$

If $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ and $g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}$ $(a_{2k}, b_{2k} \geq 0)$ then the convolution is defined by $f(z) * g(z) = z - \sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k}$ (19)

In the present work, we propose to give some interesting generalized work of U. H. Naik and S. R. Kulkarni [15], also there are many researchers who have studied some proper -ties of univalent functions with negative coefficients of type $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ like, Silverman and Berman [16], Padmanabhan and Ganeshan [17] studied the convolution properties of univalent functions with negative coefficients, Joshi and Kulkarni [18] also studied the properties of univalent functions with missing coefficients. We shall make use of above lemma, in our study.

Theorem 2.2.1 If $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ and $g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}$ Where $a_{2k} \geq 0$, $b_{2k} \geq 0$ Such that $f(z), g(z) \in T_2^*(n, m, \gamma, \sigma, \eta, \varsigma, A, B, \alpha)$, then $q(z) = z - \sum_{k=2}^{\infty} a_{2k}\, b_{2k} z^{2k} \in T_2^*(n, m, \gamma, \sigma, \eta, \varsigma, A_1, B_1, \alpha)$ With $A_1 \leq 1 - 2j, B_1 \geq \frac{j+A_1}{1-j}$. Where

$$j = \frac{1}{[3 + 2(4-\propto)B - 2(1-\propto)A]^2 (1+3a\mu)^n}$$

$$\times \frac{12(1-\propto)(B-A)^2}{[1-\gamma+\gamma(1-3a\mu)^m - 4(B-A)^2(1-\propto)^2]}$$

Proof Since f and g belong to, $T_2^*(n, m, \gamma, \sigma, \eta, \varsigma, A, B, \alpha)$ therefore by lemma 1.1.1

$$\therefore \sum_{k=2}^{\infty} \{[(2k-1) + 2(2k-\propto)B - 2(1-\propto)A]$$
$$\times \left[\frac{(1+(2k-1)a\mu)^n(1-\gamma+\gamma[1+(2k-1)a\mu])^m}{2(B-A)(1-\propto)}\right] a_{2k}\} \leq 1 \tag{20}$$

$$\sum_{k=2}^{\infty} [(2k-1) + 2(2k-\propto)B - 2(1-\propto)A] X^n$$
$$\times (1 - \gamma + \gamma X^m)[2(B-A)(1-\propto)]^{-1} a_{2k} \leq 1 \tag{21}$$

$$\sum_{k=2}^{\infty} [(2k-1) + 2(2k-\propto)B - 2(1-\propto)A] X^n$$
$$\times (1 - \gamma + \gamma X^m)[2(B-A)(1-\propto)]^{-1} b_{2k} \leq 1 \tag{22}$$

Where $X = 1 + (2k-1)\sigma(\eta + \varsigma)$.

We contemplate to find A_1, B_1, such that $-1 \leq A_1 < B_1 \leq 1$ for $q(z) \in T_2^*(n, m, \gamma, \sigma, \eta, \varsigma, A_1, B_1, \alpha)$, that is,

$$\sum_{k=2}^{\infty} [(2k-1) + (2k-\propto)B_1 - (1-\propto)A_1] X^n$$
$$\times (1 - \gamma + \gamma X^m)[(B_1 - A_1)(1-\propto)]^{-1} a_{2k} b_{2k} \leq 1 \tag{23}$$

By using Cauchy-Schwarz inequality, we get

$$\sum_{k=2}^{\infty} V(a_{2k} b_{2k})^{\frac{1}{2}} \leq (\sum_{k=2}^{\infty} V a_{2k})^{\frac{1}{2}} (\sum_{k=2}^{\infty} V\, b_{2k})^{\frac{1}{2}} \leq 1, \tag{24}$$

$$V = [(2k-1) + 2(2k-\propto)B - 2(1-\propto)A]$$
$$\times X^n (1 - \gamma + \gamma X^m)[2(B-A)(1-\propto)]^{-1} \tag{25}$$

If $V_1(a_{2k} b_{2k}) \leq V(a_{2k} b_{2k})^{\frac{1}{2}}$, then (23) is true, where

$$V_1 = [(2k-1) + (2k-\propto)B_1 - (1-\propto)A_1] X^n$$
$$\times (1 - \gamma + \gamma X^m) \times [(B_1 - A_1)(1-\propto)]^{-1} \tag{26}$$

Or $V_1(a_{2k} b_{2k})^{\frac{1}{2}} \leq V \quad (k = 2,3,4,\cdots)$

In view of (3.24), we have $(a_{2k} b_{2k})^{\frac{1}{2}} \leq V^{-1}$ (27)

Thus, to find V_1 such that $V_1 = V^2$ (28)

$$\therefore [(2k-1) + (2k-\propto)B_1 - (1-\propto)A_1] X^n$$
$$\times (1 - \gamma + \gamma X^m) \leq V^2[(B_1 - A_1)(1-\propto)] \tag{29}$$

$$\therefore A_1 = \frac{V^2(1-\propto)B_1 - [(2k-1) + (2k-\propto)B_1] X^n (1-\gamma+\gamma X^m)}{(1-\propto)[V^2 - X^n(1-\gamma+\gamma X^m)]} \tag{30}$$

It is clear that $V^2 \geq X^n(1 - \gamma + \gamma X^m)$ for $k \geq 1$

From (30) we can get

$\frac{(B_1-A_1)}{(B_1-1)} \geq \frac{(2k-1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)[V^2-X^n(1-\gamma+\gamma X^m)]} \quad for\ k \geq 2$ (31)

The right hand side of (31) decreases as k increases, then it has maximum for $k = 2$, then (31) is true if

$\frac{(B_1-A_1)}{(B_1-1)} \geq \frac{1}{[3+2(4-\alpha)B-2(1-\alpha)A]^2(1+3a\mu)^n}$

$\times \frac{12(1-\alpha)(B-A)^2}{[1-\gamma+\gamma(1-3a\mu)^m]-4(B-A)^2(1-\alpha)^2} = j$ (32)

We can see that j < 1. Fixing A_1 in (32), we have

$B_1 \geq \frac{j+A_1}{1-j}$, (33)

$-1 \leq A_1 < B_1 \leq 1$. Then the proof is complete.

Corollary 2.2.1 If $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ and $g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}$ where $a_{2k} \geq 0$, $b_{2k} \geq 0$ such that $f(z), g(z) \in S_2^*(n,\sigma,\eta,\varsigma,A,B,\alpha)$, then $q(z) = z - \sum_{k=2}^{\infty} a_{2k}\, b_{2k} z^{2k} \in S_2^*(n,\sigma,\eta,\varsigma,A_1,B_1,\alpha)$ With, $-1 \leq A_1 < B_1 \leq 1, -\frac{1}{2} \leq B < A \leq \frac{1}{2}$

where $A_1 \leq 1 - 2j_1, B_1 \geq \frac{j_1+A_1}{1-j_1}$,

$j_1 = \frac{12(1-\alpha)(B-A)^2}{[3+2(4-\alpha)B-2(1-\alpha)A]^2(1+3a\mu)^n-4(B-A)^2(1-\alpha)^2}$

This corollary is entirely new and not found in the literature. By letting n $=\alpha=$ 0 in Corollary 2.2.1, we have the following result due to [15].

Corollary 2.2.2 If $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ and $g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}$ where $a_{2k} \geq 0$, $b_{2k} \geq 0$ such that f(z), $g(z) \in S_2^*(0,0,0,0,A,B,0)$, then $q(z) = z - \sum_{k=2}^{\infty} a_{2k}\, b_{2k} z^{2k} \in S_2^*(0,0,0,0,A_1,B_1,0)$. Where $a_{2k} \geq 0, b_{2k} \geq 0, -\frac{1}{2} \leq B < A \leq \frac{1}{2}$. With $A_1 \leq 1 - 2j_2, B_1 \geq \frac{j_2+A_1}{1-j_2}$, hence

$j_2 = \frac{12(B-A)^2}{[3+8B-2A]^2-4(B-A)^2}$.

Theorem 2.2.2 If $f(z) \in T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\alpha)$ and $g(z) \in T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,C,D,\alpha)$ then $f(z) * g(z) \in T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,E,F,\alpha)$. Where $E \leq 1 - 2j, F \geq \frac{j+E}{1-j}$ with

$j = \frac{[6(1-\alpha)(B-A)(D-C)]}{[3+2(4-\alpha)B-2(1-\alpha)A][3+(4-\alpha)D-(1-\alpha)C](1+3a\mu)^n}$

$\times \frac{1}{\{[1-\gamma+\gamma(1+3a\mu)^m]-2(B-A)(D-c)(1-\alpha)^2\}}$

Proof By making use of Theorem 2.2.1, and Lemma 1.1.1 we obtain,

$\frac{\{2k(F+1)-[1-\alpha F+(1-\alpha)E]\}X^n(1-\gamma+\gamma X^m)}{(F-E)(1-\alpha)}$

$\leq \frac{\{2k(2B+1)-[1+2\alpha B+2(1-\alpha)A]\}X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\alpha)}$

$\times \frac{\{2k(D+1)-[1+\alpha D+(1-\alpha)C]\}X^n(1-\gamma+\gamma X^m)}{(D-C)(1-\alpha)} = d,$ (34)

Where $X = [1 + (2k-1)a\mu]$, $a\mu \geq 0$.

Then by simple calculations we have

$\frac{F-E}{F+1} \geq \frac{(2k-1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)[d-X^n(1-\gamma+\gamma X^m)]}$ (35)

The right hand side of (35) decreases as k increases and it has maximum for k = 2, then we obtain

$\frac{F-E}{F+1} \geq \frac{[6(1-\alpha)(B-A)(D-C)]}{[3+2(4-\alpha)B-2(1-\alpha)A][3+(4-\alpha)D-(1-\alpha)C](1+3a\mu)^n}$

$\times \frac{1}{\{[1-\gamma+\gamma(1+3a\mu)^m]-2(B-A)(D-C)(1-\alpha)^2\}} = j.$ (36)

It is clear that $j < 1$. Now fixing E we get $F \geq \frac{E+j}{1-j}$, $\therefore F \leq 1\ and\ E \leq 1 - 2j$.

Corollary 2.2.3: If $f(z) \in S_2^*(n,\sigma,\eta,\varsigma,A,B,\alpha)$ And $g(z) \in S_2^*(n,\sigma,\eta,\varsigma,C,D,\alpha)$ Then $f(z) * g(z) \in S_2^*(n,\sigma,\eta,\varsigma,E,F,\alpha)$,where $E \leq 1 - 2j_1, F \geq \frac{j_1+E}{1-j_1}$ with $j_1 = \frac{[6(1-\alpha)(B-A)(D-C)]}{[3+2(4-\alpha)B-2(1-\alpha)A]}$

$\times \frac{1}{\{[3+(4-\alpha)D-(1-\alpha)C](1+3\sigma(\eta+\varsigma))^n-2(B-A)(D-c)(1-\alpha)^2\}}$

This corollary is entirely new and not found in the literature. By putting *α = n = 0* in above Corollary, we have the following result due to [15].

Corollary 2.2.4: If $f(z) \in S_2^*(0,0,0,0,A,B,0)$ and $g(z) \in S_2^*(0,0,0,0,C,D,0)$ Then $f(z) * g(z) \in S_2^*(0,0,0,0,E,F,0)$, where $E \leq 1 - 2j_2, F \geq \frac{j_2+E}{1-j_2}$ with $j_2 = \frac{[6(B-A)(D-C)]}{\{[3+8B-A][3+4D-C]-2(B-A)(D-c)\}}$.

Corollary 2.2.5 If $f(z) \in K_2^*(n,m,\sigma,\eta,\varsigma,A,B,\alpha)$ And $g(z) \in K_2^*(n,m,\sigma,\eta,\varsigma,C,D,\alpha)$, then $f(z) * g(z) \in K_2^*(n,m,\sigma,\eta,\varsigma,E,F,\alpha)$. Where $E \leq 1 - 2j_3, F \geq \frac{j_3+E}{1-j_3}$ with

$j_3 = \frac{1}{[3+2(4-\alpha)B-2(1-\alpha)A][3+(4-\alpha)D-(1-\alpha)C]}$

$\times \frac{[6(1-\alpha)(B-A)(D-C)]}{\{(1+3a\mu)^{m+n}-2(B-A)(D-c)(1-\alpha)^2\}}$. This corollary is entirely new and not found in the literature. Letting $\alpha = n = 0, \sigma = \frac{1}{2}, \eta = 1, \varsigma = 1,$ in above Corollary, we have the following result due to [15].

Corollary 2.2.6 If $f(z) \in K_2^*(0,1,1/2,1,1,A,B,0)$ and $g(z) \in K_2^*(0,1,1/2,1,1,C,D,0)$. Then $f(z) * g(z) \in K_2^*(0,1,1/2,1,1,E,F,0)$. Where $E \leq 1 - 2j_4, F \geq \frac{j_4+F}{1-j_4}$ with

$j_4 = \frac{[6(B-A)(D-C)]}{\{4[3+8B-2A][3+4D-C]-2(B-A)(D-c)\}}$

Theorem 2.2.3 Let $f(z) = z - \sum_{k=2}^{\infty} a_{2k} z^{2k}$ Where, $a_{2k} \geq 0 \in T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,A_1,B_1,\alpha)$ and $g(z) = z - \sum_{k=2}^{\infty} b_{2k} z^{2k}$ with $|b_{2i}| \leq 1,\ i \geq 1$. Then $f(z) *\ g(z) \in T(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\alpha)$.

Proof By assumption, we have

$\sum_{k=2}^{\infty}\{[(2k-1) + 2(2k-\alpha)B - 2(1-\alpha)A]$

$\times\left[\frac{\langle 1-\gamma+\gamma[1+(2k-1)\lambda]\rangle^{m}}{2(B-A)(1-\propto)} a_{2k}\right] \leq 1\}$. And since $|b_{2i}| \leq 1 \ for \ i \geq 1$, then

$\sum_{k=2}^{\infty}\left[\frac{\langle 1+(2k-1)a\mu\rangle^{n}\langle 1-\gamma+\gamma[1+(2k-1)a\mu]\rangle^{m}}{2(B-A)(1-\propto)}\right] a_{2k} b_{2k}$.

$\leq \sum_{k=2}^{\infty}\{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]$

$\frac{\langle 1+(2k-1)a\mu\rangle^{n}\langle 1-\gamma+\gamma[1+(2k-1)a\mu]\rangle^{m}}{2(B-A)(1-\propto)}\} a_{2k}|b_{2k}| \leq 1$.

That is $f(z) * g(z)$

$= z-\sum_{k=2}^{\infty} a_{2k} b_{2k} z^{2k} \in T(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\alpha)$

Corollary 2.2.7 Let $f(z)=z-\sum_{k=2}^{\infty} a_{2k} z^{2k}$. Where $a_{2k} \geq 0 \in S_2^*(n,\sigma,\eta.\varsigma,A,B,\alpha)$ and $g(z)=z-\sum_{k=2}^{\infty} b_{2k} z^{2k}$ With $|b_{2i}| \leq 1,\ i \geq 1$. Then $f(z) * g(z) \in S^*(n,\sigma,\eta,\varsigma,A,B,\alpha)$.
This corollary is entirely new and not found in the literature. By putting $n=\propto=0$ in above Corollary, we have the following result due to [15].

Corollary 2.2.8 Let $f(z)=z-\sum_{k=2}^{\infty} a_{2k} z^{2k}$, Where $a_{2k} \geq 0 \in S_2^*(0,0,0,0,A,B,0)$ and $g(z)=z-\sum_{k=2}^{\infty} b_{2k} z^{2k}$ With $|b_{2i}| \leq 1,\ i \geq 1$. Then $f(z) * g(z) \in S^*(0,0,0,0,A,B,0)$

Corollary 2.2.9 Let $f(z)=z-\sum_{k=2}^{\infty} a_{2k} z^{2k}$. Where $a_{2k} \geq 0 \in K_2^*(n,m,\sigma,\eta,\varsigma,A,B,\alpha)$. And $g(z)=z-\sum_{k=2}^{\infty} b_{2k} z^{2k}$ with $|b_{2i}| \leq 1,\ i \geq 1$. then $f(z) * g(z) \in K(n,m,\sigma,\eta,\varsigma,A,B,\alpha)$.This corollary is entirely new and not found in the literature. By putting $n=\propto=0$ and $\sigma=\frac{1}{2},\eta=\varsigma=1$ in above Corollary, we have the following result due to [15].

Corollary 2.2.10
Let $f(z) \in K_2^*\left(0,1,\frac{1}{2}2,1,1,A,B,0\right)$.
$f(z)=z-\sum_{k=2}^{\infty} a_{2k} z^{2k}$ Where $a_{2k} \geq 0$, and $g(z)=z-\sum_{k=2}^{\infty} b_{2k} z^{2k}$ Where $|a_{2i}| \leq 1,\ i \geq 1$.
Then $f(z) * g(z) \in K(0,1,1/2,1,1,A,B,0)$.

Theorem 2.2.4 Let $f,g \in T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\propto)$, then $q(z)=$
$z-\sum_{k=2}^{\infty}\left(a_{2k}{}^2+b_{2k}{}^2\right) \in T_2^*(n,m,\gamma,\sigma,\eta,\varsigma,A,B,\propto)$.
where $A_1 \leq 1-2j \ and \ B_1 \geq \frac{A_1+j}{1-j}$ with

$j=\frac{1}{[3+2(4-\propto)B-2(1-\propto)A]^2(1+3a\mu)^n}$
$\times\frac{[24(1-\propto)(B-A)^2]}{\{[1-\gamma+\gamma(1+3a\mu)^m]-8(B-A)^2(1-\propto)^2\}}$.

Proof By assumption, we have

$\sum_{k=2}^{\infty}\left[\frac{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\propto)}\right] a_{2k} \leq 1$

$\sum_{k=2}^{\infty}\left[\frac{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\propto)}\right] b_{2k} \leq 1$

Where $X=1+(2k-1)a\mu$, thus

$\sum_{k=2}^{\infty}\left[\frac{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\propto)} a_{2k}\right]^2$
≤ 1.

$\left(\sum_{k=2}^{\infty}\left[\frac{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\propto)} a_{2k}\right]\right)^2$
≤ 1

$\therefore \sum_{k=2}^{\infty}\left[\frac{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\propto)} b_{2k}\right]^2$
≤ 1 (37)

Then we may write

$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\propto)}\right]^2$
$\times(a_{2k}{}^2+b_{2k}{}^2) \leq 1$ (38)

Therefore, the inequality (38) holds if

$\frac{[(2k-1)+(2k-\propto)B_1-(1-\propto)A_1]X^n(1-\gamma+\gamma X^m)}{(B_1-A_1)(1-\propto)}$

$\leq \frac{1}{2}\left[\frac{[(2k-1)+2(2k-\propto)B-2(1-\propto)A]X^n(1-\gamma+\gamma X^m)}{2(B-A)(1-\propto)}\right]^2=\frac{V^2}{2}$.

And by simplification, the last inequality gives

$\frac{(B_1-A_1)}{(B_1+1)} \geq \frac{2(2k-1)X^n(1-\gamma+\gamma X^m)}{(1-\alpha)[V^2-2X^n(1-\gamma+\gamma m)]}$ (39)

The right hand side of (39) decreases as k increases and if we put k = 2, we obtain

$\frac{(B_1-A_1)}{(B_1+1)} \geq \frac{[24(1-\propto)(B-A)^2]}{[3+2(4-\propto)B-2(1-\propto)A]^2(1+3a\mu)^n}$
$\times\frac{1}{\{[1-\gamma+\gamma(1+3a\mu)^m]-8(B-A)^2(1-\propto)^2\}}=j$ (40)

Now by fixing A_1 in (40), we have $B_1 \geq \frac{A_1+j}{1-j}$ and $B_1 \leq 1$ give $A_1 \leq 1-2j$. with j given in (40).

Corollary 2.2.11 Let $f,g \in S_2^*(n,\sigma,\eta,\varsigma,A,B,\propto)$, then $q(z)=$
$z-\sum_{k=2}^{\infty}(a_{2k}{}^2+b_{2k}{}^2)z^{2k} \in S_2^*(n,\sigma,\eta,\varsigma,A_1,B_1,\propto)$
Where $A_1 \leq 1-2j_1$ and $B_1 \geq \frac{A_1+j_1}{1-j_1}$ With

$j_1=\frac{[24(1-\propto)(B-A)^2]}{\{[3+2(4-\propto)B-2(1-\propto)A]^2(1+3a\mu)^n-8(B-A)^2(1-\propto)^2\}}$.

By putting $n=\propto=0$ in Corollary 2.2.11, we have the following result due to [15].

Corollary 2.2.12 Let $f,g \in S_2^*(0,0,0,0,A,B,0)$, then $q(z)=$
$z-\sum_{k=2}^{\infty}\left(a_{2k}{}^2+b_{2k}{}^2\right)z^{2k} \in S_2^*(0,0,0,0,A_1,B_1,0)$.
Where $A_1 \leq 1-2j_2$ and $B_1 \geq \frac{A_1+j_2}{1-j_2}$ with

$j_2=\frac{24[(B-A)^2]}{\{[3+8B-2A]^2-8(B-A)^2\}}$. Let the class T (n, p) of functions of the form $f(z)=z^p-\sum_{k=n+p}^{\infty} a_k z^k$ where $(n,p \in N, a_k \geq 0)$ (41)

$f(z)$ will be Holomorphic and multivalent in the unit disk $u=\{z:|z|<1\}$. Consider the generalized Ruscheweyh derivative $\mathcal{J}_p^{\sigma,\eta,\varsigma,\lambda,\varepsilon,\delta,\tau} f(z)$ defined as

$\mathcal{J}_p^{\sigma,\eta,\varsigma,\lambda,\varepsilon,\delta,\tau} f(z)$
$= z^p-\sum_{k=n+p}^{\infty} \Omega_p^{\sigma,\eta,\varsigma,\lambda,\varepsilon,\delta,\tau}(k)\, a_k z^k$ (42)

$\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau} f(z)=$
$\frac{\Gamma(k+1+\lambda-2p)\Gamma(v+2+\lambda-p-a\mu)\Gamma(k+v-p+2)}{\Gamma(k-p+1)\Gamma(k+v-2p+2+\lambda-a\mu)\Gamma(v+2)\Gamma(1+\lambda-p)}$ (43)

$v,\varepsilon,\delta,\tau \in R, \sigma=\varepsilon, \eta=\delta, \varsigma=\tau$ and $p=v=1$

We have Ruscheweyh derivative to univalent function. Now we define a class $AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ consisting of functions $f(z)$ o f the form (41) satisfying the condition

$$Re\left[\frac{\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau)}f(z)}{(1-\gamma)z^2\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau)}f(z)\right]''+\gamma z\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau)}f(z)\right]'+\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau)}f(z)}\right]$$
$$>\alpha \qquad (44)$$

$z\in u$, $0\le\gamma<\frac{1}{2}$, $0\le\alpha<\frac{1}{(1-\gamma)p(p-1)+\gamma p+1}$ and $\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)$ as defined in (42). Also Let

$$f_i(z)=z^p-\sum_{k=n+p}^{\infty}a_{k,i}z^k \qquad (i=1,2)$$

be in the class $T(n,p)$ consisting of the family of functions that are holomorphic in u. Then the quasi-convolution $(f_1*f_2*\cdots*f_n)$ of the functions $f_1,f_2,\cdots,f_n$ is defined by $(f_1*f_2*\cdots*f_n)(z)$

$$=z^p-\sum_{k=n+p}^{\infty}\prod_{i=1}^{n}a_{k,i}z^k, \qquad (45)$$

Where $\prod_{i=1}^{n}a_{k,i}=a_{k,1}a_{k,2}\cdots a_{k,n}$, $(n\in N)$.

Theorem 2.2.5 Let $f(z)\in T(n,p)$. Then $f(z)\in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ if and only if

$$\sum_{k=n+p}^{\infty}\{1-\alpha[(1-\gamma)k(k-1)+\gamma k+1]\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)a_k$$
$$<1-\propto[(1-\gamma)p(p-1)+\gamma p+1] \qquad (46)$$

$0\le\alpha<\frac{1}{(1-\gamma)p(p-1)+\gamma p+1}$ and $\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)$ as defined in (43). The result holds true.

Proof Let $f(z)\in T(n,p)$. and suppose that $f(z)\in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$, then

$$Re\left\{\frac{\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)}{(1-\gamma)z^2\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right]''+\gamma z\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right]'+\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)}\right\}$$
$$>\alpha \qquad (z\in u).$$

$$\left|\begin{matrix}(1-\alpha)\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)-(1-\gamma)z^2\propto\\ \times\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right]''-\gamma z\propto\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right]'\end{matrix}\right|>0.$$

By using the definition of $\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)$, we obtain

$$|z|^p\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}-$$
$$\sum_{k=n+p}^{\infty}\left\{\begin{matrix}1-\propto[(1-\gamma)k(k-1)+\gamma k+1]\\ \times\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)a_k|z|^k\end{matrix}\right\}>0$$

Letting $z\to 1^-$ on real values yields

$$\sum_{k=n+p}^{\infty}\left\{\begin{matrix}1-\propto[(1-\gamma)k(k-1)+\gamma k+1]\\ \times\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)a_k\end{matrix}\right\}$$
$$<1-\propto[(1-\gamma)p(p-1)+\gamma p+1]$$

Where

$$\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)=\frac{\Gamma(k-2p+1+\lambda)\Gamma(v+2+\lambda-p-a\mu)\Gamma(k+v-p+2)}{\Gamma(k-p+1)\Gamma(k+v-2p+2+\lambda-a\mu)\Gamma(v+2)\Gamma(1+a\mu)}.$$

Conversely, suppose (46) holds true, then

$$w=\frac{\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)}{(1-\gamma)z^2\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right]''+\gamma z\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right]'+\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)}$$

Thus by simple calculations we get the required result. For sharpness the function $f(z)$ is given by following

$$f(z)=\frac{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]}{\{1-\propto[(1-\gamma)(p+p)(n+p-1)+\gamma(p+n)+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}z^{n+p}.$$

Corollary 2.2.13 Let $f(z)\in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$, then

$$a_k\le\frac{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]}{\{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)},$$

$k\ge n+p$. (47)

Also, consider the class $ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ consisting of all functions $f(z)\in T(n,p)$ such that $zf'^{(z)}\in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$.

Theorem 2.2.6 The function $f\in ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ if and only if

$$\sum_{k=n+p}^{\infty}k\{1-\propto(\sigma(\eta+\varsigma)+1)-\propto(k-1)$$
$$\times(2-a\mu+(k-2)[(1-a\mu)])$$
$$\times\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)a_kz^k$$
$$\le p[1-\alpha(1-a\mu)p(p-1)+a\mu p+1].$$

Where $0\le\alpha<\frac{1}{\propto\{(1-a\mu)p(p-1)+a\mu p+1\}}$, $0\le a\mu<\frac{1}{2}$, and $\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)$ as defined in (43)

Corollary 2.2.14 Let $f(z)\in ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$, then

$$a_k\le\frac{1}{\{(n+p)(1-\propto[(1-a\mu)(n+p)(n+p-1)+a\mu(n-p)+1])\}}$$
$$\times\frac{\{p(1-\propto[(1-\gamma)p(p-1)+\gamma p+1])\}}{\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}$$
$$\le\frac{p[z-(p^2+2)]}{(n+p)\{2-[(p+n)^2+2]\}B_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}.$$

Theorem 2.2.7: Let $f\in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ then

$$|z|^p-\frac{\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}z^{n+p}$$
$$\le\left|\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right|\le|z|^p+$$
$$\frac{\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-a\mu[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}|z|^{n+p} \qquad (48)$$

$$\therefore p|z|^{p-1}-\frac{(p+n)\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}|z|^{n+p-1}$$
$$\le\left|\left[\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right]'\right|\le p|z|^{p-1}+$$
$$\frac{(p+n)\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}|z|^{n+p-1} \qquad (49)$$

Where $0\le\gamma<\frac{1}{2}$, $0\le\alpha<\frac{1}{(1-\gamma)p(p-1)+\gamma p+1}$, $\sigma(\eta+\varsigma)>-p$, $v,\varepsilon,\delta,\tau\in R$, $z\in u$.

$$\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)=\frac{\Gamma(n+1+\lambda-p)\Gamma(v+2+\lambda-p-a\mu)\Gamma(n+v+2)}{\Gamma(n+1)\Gamma(n+v+2+\lambda-p-a\mu)\Gamma(v+2)\Gamma(1+\lambda-p)}.$$

Proof For $f\in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$, we have

$$\sum_{k=n+p}^{\infty}a_k\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)$$
$$\le\frac{\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}.$$

Therefore

$$\left|\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right|=\left|z^p-\sum_{k=n+p}^{\infty}a_k\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)z^k\right|$$
$$\le|z|^p+|z|^{p+n}\sum_{k=n+p}^{\infty}a_k\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)$$
$$\le|z|^p-\frac{\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}|z|^{n+p}$$

And $\left|\mathcal{J}_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}f(z)\right| \geq |z|^p - \frac{\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}|z|^{n+p}$.
Similarly, we can prove the relation (49).

Theorem 2.2.8 Let $f_i(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ then $(f_1 * f_2 *\cdots* f_\ell)(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ and $0 \leq \xi < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1} - \frac{n}{T_1(n+p,l)}$. Where $f_i(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,i}z^k \quad (i = 1,2,\cdots,\ell \in N)$.
$T_1(n+p,\ell) = \prod_{i=1}^{\ell}\frac{1-\alpha_i[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}}$ $\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p) - 1$ For $0 \leq \propto_i < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1}$.
Thus result holds true.

Proof By induction on ℓ For $\ell = 1$, the result is true. For $\ell = 2$, we have

$$\sum_{k=n+p}^{\infty}\frac{\{1-\alpha_1[(1-\gamma)k(k-1)+\gamma k+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)}{\{1-\alpha_1[(1-\gamma)p(p-1)+\gamma p+1]\}}a_{k,1} \leq 1.$$

$$\& \sum_{k=n+p}^{\infty}\frac{\{1-\alpha_2[(1-\gamma)k(k-1)+\gamma k+1]\}\Omega_p^{\sigma(\eta+\varsigma),\varepsilon(\delta+\tau)}(k)}{\{1-\alpha_2[(1-\gamma)p(p-1)+\gamma p+1]\}}a_{k,2} \leq 1$$

By Cauchy-Schwarz inequality we have

$$\sum_{k=n+p}^{\infty}\left(\prod_{i=1}^{2}\frac{\{1-\alpha_i[(1-\gamma)k(k-1)+\gamma k+1]\}}{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}}a_{k,i}\right)^2 \times \Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k) \leq 1.$$

We have only to find the largest ξ such that

$$\sum_{k=n+p}^{\infty}\frac{\{1-\xi[(1-\gamma)k(k-1)+\gamma k+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)}{\{1-\xi[(1-\gamma)p(p-1)+\gamma p+1]\}}a_{k,1}a_{k,2} \leq 1$$

Such that

$$\frac{\{1-\xi[(1-\gamma)k(k-1)+\gamma k+1]\}}{\{1-\xi[(1-\gamma)p(p-1)+\gamma p+1]\}}\sqrt{a_{k,1}a_{k,2}} \leq \left(\prod_{i=1}^{2}\frac{\{1-\alpha_i[(1-\gamma)k(k-1)+\gamma k+1]\}}{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}}\right)^{\frac{1}{2}}.$$

Consequently, we have to find ξ such that $\frac{\{1-\xi[(1-\gamma)k(k-1)+\gamma k+1]\}}{\{1-\xi[(1-\gamma)p(p-1)+\gamma p+1]\}} \leq \Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)\prod_{i=1}^{2}\frac{\{1-\alpha_i[(1-\gamma)k(k-1)+\gamma k+1]\}}{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}}$.

Thus $(f_1 * f_2)(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$, for $0 < \xi \leq \frac{1}{(1-\gamma)p(p-1)+\gamma p+1} - \frac{n}{T_1(k)}$ Where

$$T_1(k) = \Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)\prod_{i=1}^{2}\frac{1-\alpha_i[(1-\gamma)k(k-1)+\gamma k+1]}{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}} - 1.$$

So for $k \geq n+p$ we get $0 < \xi \leq \frac{1}{(1-\gamma)p(p-1)+\gamma p+1} - \frac{n}{T_1(n+p)}$. Where

$$T_1(n+p) = \Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p) \times \prod_{i=1}^{2}\frac{1-\alpha_i[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]}{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}} - 1.$$

Now suppose the result is true for any $\in$ N . Then we must show that $(f_1 * f_2 *\cdots* f_\ell)(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$, where $0 < \xi \leq \frac{1}{(1-\gamma)p(p-1)+\gamma p+1} - \frac{n}{M_1(n+p,\ell+1)}$ and

$$M_1(n+p,\ell+1) = \frac{\{1-\xi[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\}}{\{1-\xi[(1-\gamma)p(p-1)+\gamma p+1]\}} \times \frac{\{1-\alpha_{l+1}[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)\}}{\{1-\alpha_{l+1}[(1-\gamma)p(p-1)+\gamma p+1]\}} - 1,$$

then by mathematical induction, we obtain the result which is true for any positive integer. We want to prove that $(f_1 * f_2 *\cdots* f_\ell)(z) = z^p - A_{n+p}z^{n+p}$. Where

$$A_{n+p} = \prod_{i=1}^{\ell}\frac{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}}{1-\alpha_i[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)} - 1$$

$0 \leq \propto_i < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1}$. This is the required condition and this completes the proof of theorem. For sharpness take the function

$$f_i(z) = z^p - \frac{\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}}{\{1-\alpha_i[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}z^{n+p}$$

Similarly we can prove the result for $ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\xi,p,n)$ in next Theorem.

Theorem 2.2.9 If $f_i(z) \in ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha_i,p,n)$ for each $(i = 1,2,\cdots\cdots,\ell)$ Then $(f_1 * f_2 *\cdots* f_\ell)(z) \in ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$
$\therefore 0 < \beta \leq \frac{1}{(1-\alpha\mu)p(p-1)+\alpha\mu p+1} - \frac{n}{T_2(n+p,\ell)}$

$$T_2(n+p,\ell) = \prod_{i=1}^{\ell}\frac{(n+p)\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}{p\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}} \times \{1-\alpha_i[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\} - 1.$$

The result is sharp for the functions: $f_i(z)$ for $(i = 1,2,\cdots\cdots,\ell)$ given by

$$f_i(z) = z^p - \frac{p\{1-\alpha_i[(1-\gamma)p(p-1)+\gamma p+1]\}}{(n+p)\{1-\alpha_i[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}z^{n+p}$$

Put $\alpha_i = \propto \forall\ (i = 1,2,\cdots\cdots,\ell)$ in Them. 2.2.8, we get

Corollary 2.2.15 If $f_i(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\propto,p,n)\forall(i = 1,2,\cdots\cdots,\ell \in N)$.
$\therefore (f_1 * f_2 *\cdots* f_\ell)(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\beta,p,n)$.
Where

$$\beta = \frac{1}{(1-\gamma)p(p-1)+\gamma p+1} - \frac{n}{\left(\frac{\{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}{\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}\right)^{\ell} - 1}$$

$0 \leq \propto < \frac{1}{(1-\gamma)p(p-1)+\gamma p+1}$ Then result is sharp for the function $f_i(z)$ for all $(i = 1,2,\cdots\cdots,\ell \in N)$ given by

$$f_i(z) = z^p - \frac{\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}{\{1-\propto[(1-\gamma)(n+p)(n+p)-1)+\gamma(n+p+1)]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}z^{n+p}.$$

Put $\alpha_i = \propto$ for $(i = 1,2,\cdots\cdots,\ell)$ in Theorem 2.2.9, then next corollary.

Corollary 2.2.16: If $f_i(z) \in ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\propto,p,n)$ for (i = 1,2,$\cdots\cdots$, $\ell \in$ N). Then $(f_1 * f_2 *\cdots* f_\ell)(z) \in ABS(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\beta,p,n)$

where $\beta = \frac{1}{(1-\gamma)p(p-1)+\gamma p+1} - \frac{n}{\left(\frac{(n+p)(\{1-\propto[(1-\gamma)(n+p)(n+p-1)+\gamma(n+p)+1]\})\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(n+p)}{p\,\{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]\}}\right)^{\ell}}$.

Theorem 2.2.10: Let $f_1(z), f_2(z), f_3(z), \cdots, f_\ell(z)$ defined by $f_i(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,i} z^k \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\propto,p,n)$. Where $(i = 1,2,\cdots,\ell \in N)$. Then arithmetic mean of f_i $(i = 1,2,\cdots,\ell \in N)$ defined by $h(z) = \frac{1}{\ell}\sum_{i=1}^{\infty} f_i(z)$, is also in $AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\propto,p,n)$ $(i = 1,2,\cdots,\ell \in N)$.

Proof: By definition of $h(z)$ we have $h(z) = \frac{1}{\ell}\sum_{i=1}^{\infty}\left(z^p - \sum_{k=n+p}^{\infty} a_{k,i} z^k\right) = z^p - \sum_{k=n+p}^{\infty}\left(\frac{1}{\ell}\sum_{i=1}^{\infty} a_{k,i}\right) z^k$. Using Theorem 2.2.5, $\sum_{k=n+p}^{\infty}\{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]\} \times \Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)\left(\frac{1}{\ell}\sum_{i=1}^{\infty} a_{k,i}\right) = \frac{1}{\ell}\sum_{i=1}^{\infty}\left(\sum_{k=n+p}^{\infty}\{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]\} \times \Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)\right) a_{k,i} \le \frac{1}{\ell}\sum_{i=1}^{\infty} 1-\propto[(1-\gamma)p(p-1)+\gamma p+1]$,
Then we obtain $h(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$.

Theorem 2.2.11: Let $f(z)$ and $g(z)$ be in the class $AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$, then $h(z)) = tf(z) + (1-t)g(z)$, $0 \le t \le 1$, also belongs to $AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$.

Proof: By definition of h(z) we have $h(z) = z^p - \sum_{k=n+p}^{\infty}[ta_k + (1-t)b_k]z^k$, where $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ And $g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$, $(a_k, b_k \ge 0)$.Using theorem 2.2.5

$$\sum_{k=n+p}^{\infty}\frac{\{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)}{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]} \times [ta_k + (1-t)b_k] = t\sum_{k=n+p}^{\infty}\frac{\{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)}{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]}a_k + (1-t)\sum_{k=n+p}^{\infty}\frac{\{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]\}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)}{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]}b_k \le 1$$

then $h(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$. consider the generalized Jung-Kim-Srivastava integral operator $F_\zeta^q(z) = z^p - \sum_{k=n+p}^{\infty}\frac{\Gamma(q+\zeta+p)\Gamma(\zeta+k)}{\Gamma(\zeta+p)\Gamma(q+\zeta+k)}a_k z^k$. Where $q \ge 0, \zeta > -1$. [19], then we have the next theorem.

Theorem 2.2.12: Let $f \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$ be defined by (41) and $q \ge 0, \zeta > -1$ then $F_\zeta^q(z)$ defined above also belongs to $AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$.
.

Proof: By Theorem 3.3.5, we have

$$\sum_{k=n+p}^{\infty}\frac{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]}{1-\propto[(1-\gamma)p(p-1)+\gamma p-1]}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k) \times \left[\frac{\Gamma(q+\zeta+p)\Gamma(\zeta+k)}{\Gamma(\zeta+p)\Gamma(q+\zeta+k)}\right]a_k \le \sum_{k=n+p}^{\infty}\frac{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]}{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)a_k \le 1.$$

Since $\frac{\Gamma(q+\zeta+p)\Gamma(\zeta+k)}{\Gamma(\zeta+p)\Gamma(q+\zeta+k)} \le 1$ for $k \ge n+p$, then $F_\zeta^q(z) \in AB(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,p,n)$

Theorem 2.2.13: Let $F_\zeta^q(z)$ be defined, having Taylor series expansion of the form, $F_\zeta^q = z^p - \sum_{k=n+p}^{\infty}\frac{\Gamma(q+\zeta+p)\Gamma(\zeta+k)}{\Gamma(\zeta+p)\Gamma(q+\zeta+k)}a_k z^k$ *Then* $F_\zeta^q(z)$ is starlike of order $\beta(0 \le \beta < p)$. in $|z| \le R_1(\sigma,\eta,\varsigma,\varepsilon,\delta,\tau,\gamma,\alpha,\beta,p,n) = \inf_{k \ge n+p}\left[\frac{(p-\beta)\Gamma(q+\zeta+k)\Gamma(\zeta+p)}{(k-\beta)\Gamma(\zeta+k)\Gamma(q+\zeta+p)}\frac{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]}{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]}\right]^{\frac{1}{k-p}} \times \left[\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)\right]^{\frac{1}{k-p}}$.

Proof: We must show that $\left|\frac{z F_\zeta^{'q}(z)}{F_\zeta^q(z)} - p\right| < p - \beta$. Or $\left|\frac{z F_\zeta^{'q}(z)}{F_\zeta^q(z)} - p\right| \le \sum_{k=n+p}^{\infty}\frac{\Gamma(q+\zeta+p)\Gamma(\zeta+k)}{\Gamma(\zeta+p)\Gamma(q+\zeta+k)}(k-p)a_k|z|^{k-p} < (p-\beta)\left(1 - \sum_{k=n+p}^{\infty}\frac{\Gamma(q+\zeta+p)\Gamma(\zeta+k)}{\Gamma(\zeta+p)\Gamma(q+\zeta+k)}a_k|z|^{k-p}\right)$

then $|z|^{k-p} \le \frac{(p-\beta)\Gamma(q+\zeta+k)\Gamma(\zeta+p)}{(k-\beta)\Gamma(\zeta+k)\Gamma(q+\zeta+p)} \times \frac{1-\propto[(1-\gamma)k(k-1)+\gamma k+1]}{1-\propto[(1-\gamma)p(p-1)+\gamma p+1]}\Omega_p^{\sigma,\eta,\varsigma,\varepsilon,\delta,\tau}(k)$. Hence proof.

3. References

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