Properties of Bondage Arcs

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Abstract: In this paper, we define the non-bondage number \( b_n(G) \) for any fuzzy graph and define fuzzy strong line graph. A characterization is obtained for fuzzy strong line graphs \( L_s(G) \) such that \( L_s(G) \) is tree. A necessary condition for a fuzzy double strong line graph of cycle is a fuzzy trees and the exact value of \( b_n(G) \) for any graph \( G \) is found. Moreover we define neighbourhood extension also analysis it extendable by using bondage arcs and relationships between \( b_n(G) \) and \( b \).

Keywords: (G) Minimum dominating set, maximum non-bondage number \( b_n(G) \), minimum bondage number \( b(G); L_s(G) \) is strong line graph, \( L_s(G) \) is double strong line graph, \( G \) is neighbour extendable

1 Introduction

Fuzzy graph theory was introduced by A. Rosenfeld [8] in 1975. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of neural networks, expert systems, information theory, cluster analysis, medical diagnosis, control theory, etc. Sunil Mathew, Sunitha M.S [10] has obtained the fuzzy graph-theoretic concepts like f-bonds, paths, cycles, trees and connectedness and established some of their properties. V.R. Kulli and B. Janakiram [7] have established the non-bondage number of a graph. First we give the definitions of basic concepts of fuzzy graphs and define the non-bondage and it is properties. All graphs consider here are finite, undirected, distinct labeling with no loop or multi arc and p nodes and q (fuzzy) arcs. Any undefined term in this paper may be found in Harary[5]. Among the various applications of the theory of domination that have been
considered, the one that is perhaps most often discussed concerns a communication network. Such a network consists of existing communication links between a fixed set of sites. The problem is to select a smallest set of sites at which to place transmitters so that every site in the network that does not have a transmitter is joined by a direct communication link to one that does have a transmitter. This problem reduces to that of finding a minimum dominating set in the graph corresponding to the network. This graph has a node representing each site and an arc between two nodes iff the corresponding sites have a direct communications link joining them. To minimize the direct communication links in the network, we introduce the following section.

2 Preliminaries

A fuzzy subset of a non-empty set $V$ is a mapping $\sigma : V \rightarrow [0, 1]$. A fuzzy relation on $V$ is a fuzzy subset of $E(V \times V)$. A fuzzy graph $G = (\sigma, \mu)$ is a pair of function $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$, where $\mu(u, v) \leq \sigma(u)\sigma(v)$ for all $u, v \in V$. The underlying crisp graph of $G= (\sigma, \mu)$ is denoted by $G^* = (V, E)$, where $V = \{u \in V : \sigma(u) > 0\}$ and $E = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. The order $P = \sum_{v \in D} \sigma(v)$. The graph $G = (\sigma, \mu)$ is domenoted by $G$, if unless otherwise mentioned. Let be a fuzzy graph on. The degree of a vertex $u$ is $d_G(u) = \sum_{v \neq u} \mu(uv)$. The minimum degree of $G$ is $\delta(G) = \land\{d(G)(u), \forall u \in V\}$ and the maximum degree of $G$ is $\delta(G) = \lor d(G)(u), \forall u \in V$. The strength of connectedness between two nodes $u$ and $v$ in a fuzzy graph $G$ is defined as the maximum of the strength of all paths between $u$ and $v$ and is denoted by $CONN_G(u, v)$. A $u-v$ path $P$ is called a strongest path if its strength equals $CONN_G(u, v)$. A fuzzy sub graph $H = (\tau, \rho)$ is called a fuzzy sub graph of $G$ if $\tau(x) \leq \sigma(x)$ for all $x \in V$ and $\rho(x, y) \leq (x, y)$ for all $(x, y) \in V$. A fuzzy sub graph $H = (\tau, \rho)$ is said to be a spanning fuzzy sub graph of $G$, if $\tau(x) = \sigma(x)$ for all $x$. A fuzzy $G$ is said to be connected if there exists a strongest path $A$ path $P$ of length $n$ is a sequence of distinct nodes $u_0, u_1, u_2, u_n$ such that $(u_{i-1}, u_i) > 0$ and degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then $P$ is called a cycle and it is a fuzzy cycle if there is more than one weak arc. Let $u$ be a node in fuzzy graphs $G$ then $N(u) = \{v : (u, v)\}$ is strong arc is called neighborhood of $u$ and $N[u] = N(u) \cup u$ is called closed neighborhood of $u$. Neighborhood degree of the node is defined by the sum of the weights of the strong neighbor node of $u$ is denoted by $d_s(u) = \sum_{v \in N(u)} \sigma(v)$.

3 Fuzzy dominating set

**Definition 3.1.** Let $G$ be a fuzzy graph and $u$ be a node in $G$ then there exist a node $v$ such that $(u, v)$ is a strong arc then $u$ dominates $v$.

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Definition 3.2. Let G be a fuzzy graph. A subset D of V is said to be a fuzzy dominating set if for every node \( v \in V \setminus D \), there exists \( u \in D \) such that \( u \) dominates \( v \).

Definition 3.3. The domination number of G is the minimum cardinality taken over all dominating sets in G and is denoted by \( \gamma(G) \), where \( \gamma(G) = \sum_{v \in D} \sigma(v) \). A dominating set with cardinality \( \gamma(G) \) is called \( \gamma \)-set of G.

4 Fuzzy non bondage number

Definition 4.1. The bondage number \( b(G) \) of a fuzzy graph \( G(V, E, \sigma, \mu) \) is minimum number of fuzzy arcs among all sets of arcs \( X = (x, y) \) subset of \( E \) such that \( \text{CONN}_{G-(x,y)}(u,v) < \text{CONN}_{G}(u,v) \) for all \( u \in V - \gamma(G) \) and \( v \in \gamma(G) \). Here \( \gamma(G) \) represent minimum dominate set.

Definition 4.2. The non-bondage number \( b_n(G) \) of a fuzzy graph \( G(V, E, \sigma, \mu) \) is maximum number of fuzzy arcs among all sets of arcs \( X = (x, y) \) subset of \( E \) such that \( \text{CONN}_{G-(x,y)}(u,v) = \text{CONN}_{G}(u,v) \) for all \( u \in V - \gamma(G) \) and \( v \in \gamma(G) \). Here \( \gamma(G) \) represent minimum dominate set.

Theorem 4.3. For any fuzzy graph G,

\[ b_n(G) = q - p + \gamma(G) \quad (1) \]

where \( q \) is total number of fuzzy arcs and \( p \) is total number of node.

Theorem 4.4. For any graph G,

\[ b_n(G) \leq q - \Delta_n \quad (2) \]

where \( \Delta_n \) is number total number of strong arcs in \( \Delta \) of \( G \).

Theorem 4.5. For any fuzzy graph, \((x, y)\) is a non bondage iff \((x, y)\) is a weakest arc of any cycle.

Remark 1. Let G: \((\sigma, \mu)\) be a fuzzy graph such that \( G^* : (\sigma^*, \mu^*) \) is a cycle and Let \( t = \min \{ \mu(u,v) : \mu(u,v) > 0 \} \). Then all arcs \((u,v)\) of G such that \( \mu(u,v) > t \) are fuzzy bondage of G. \( \mu(u,v) = t \) is only non-bondage of G.
Theorem 4.6. Let $G: (\sigma, \mu)$ be fuzzy graph such that $G^*: (\sigma^*, \mu^*)$ is a cycle. Then a node is a not fuzzy cut node of $G$ iff it is incident with a non-bondage arc.

Theorem 4.7. If $G$ is a fuzzy forest, the arcs of $F$ are not fuzzy non-bondage of $G$.

Theorem 4.8. A complete fuzzy graph with $n$ notes has $n-1$ non bondage.

Theorem 4.9. A complete fuzzy graph has no fuzzy cut nodes.

5 Exact values of $b_n(G)$ for some standard graphs

Proposition 5.1. If $P_p$ is a path with $p \geq 4$ nodes, then $b_n(P_p) = \lceil p/3 \rceil - 1$.

Proposition 5.2. If $C_p$ is a cycle with $p \geq 3$ nodes, then $b_n(C_p) = \lceil p/3 \rceil$.

Proposition 5.3. If $K_p$ is a complete graph $p \geq 3$ nodes, then $b_n(K_p) = \frac{(p-1)(p-2)}{2}$.

Proposition 5.4. If $K_{m,n}$ is a complete bipartite graph, then $b_n(K_{m,n}) = mn - m - n + 2$.

Proposition 5.5. If $W_p$ is a wheel with $p \geq 4$ nodes, then $b_n(W_p) = p - 1$.

Proposition 5.6. For any tree $T$, $b_n(T) = \gamma(T) - 1$.

6 Relationships between $b_n(G)$ and $b$

Theorem 6.1. Let $T \neq P_4$ be a tree with at least two cut nodes. Then

$$b_n(G) \geq b(T) \tag{3}$$

Theorem 6.2. For any fuzzy graph

$$b(G) \leq b_n(G) + 1 \tag{4}$$

Theorem 6.3. If $G$ be a cycle graph then

$$b_n(G) + b_n(\bar{G}) \leq \frac{p(p-3)}{2} \tag{5}$$
Proof. by theorem 6 $b_n(G) = q - p + \gamma(G)$

\[
b_n(\bar{G}) = \bar{q} - p + \gamma(\bar{G})
\]

\[
b_n(\bar{G}) + b_n(G) = q - p + \gamma(G) + \bar{q} - p + \gamma(\bar{G})
\]

\[
= q + \bar{q} - 2p + (\bar{G}) + \gamma(G)
\]

\[
= \left(\frac{p(p-1)}{2}\right) - 2p + \gamma(G) + \gamma(G)
\]

\[
\leq \left(\frac{p(p-1)}{2}\right) - 2p + p
\]

\[
\leq \left(\frac{p(p-3)}{2}\right).
\]

Theorem 6.4. For any graph $G$,

\[
b_n(\bar{G}) + b_n(G) \leq \frac{(p - 1)(p - 2)}{2}.
\] (6)

Proof. by Theorem 7 $b_n \leq q - \Delta_n$, then

\[
b_n(\bar{G}) + b_n(G) \leq \bar{q} + q - (\Delta_n + \delta_n)
\]

\[
= \left(\frac{p(p-1)}{2}\right) - (\Delta_n + \delta_n)
\]

\[
\leq \left(\frac{p(p-1)}{2}\right) - (p - 1)
\]

\[
\leq \left(\frac{(p-2)(p-1)}{2}\right).
\]

Theorem 6.5. If $G$ be a cycle graph then

\[
b(\bar{G}) + b(G) \leq \frac{(p(p - 3))}{2} + 2.
\] (7)

Proof. by Theorem 9

\[
b(G) \leq b_n(G) + 1
\]

\[
b(\bar{G}) + b(G) \leq b_n(\bar{G}) + b_n(G) + 2
\]

by Theorem 10

\[
b(\bar{G}) + b(G) \leq \frac{(p(p-3))}{2} + 2
\]

Theorem 6.6. Any graph $G$,

\[
b(\bar{G}) + b(G) \leq \frac{(p - 1)(p - 2)}{2} + 2.
\] (8)
Theorem 6.7. If \( G \) be a tree then

\[
b_n(\bar{G}) + b_n(G) \geq \gamma(\bar{G}) + \gamma(G) - 2,
\]

if \( p \geq 4 \)

Proof. \( b_n(G) \geq \gamma(G) - 1 \), then
\[
b_n(\bar{G}) + b_n(G) \geq \gamma(G) + \gamma(G) - 2.
\]

\( \square \)

7 Block

Definition 7.1. A connected fuzzy graph is called block if all nodes are satisfies the condition \( \text{CONN}_{G-v}(u, v) = \text{CONN}_G(u, v) \) for every \( u, v \) in \( G \).

8 Strong line graph

Definition 8.1. Given a fuzzy graph \( G \), its strong line graph \( Ls(G) \) is a fuzzy graph, \( Ls(G) \) is a graph \( G \) such that
Each node of \( Ls(G) \) represents an arc of \( G \); and Two nodes of \( Ls(G) \) are adjacent if and only if their corresponding arcs are strong and share a common end point in \( G \).

Definition 8.2. Given a fuzzy graph \( G \), its double strong line graph \( L^*_s(G) \) is a fuzzy graph, \( L^*_s(G) \) is a graph \( G \) such that
Each node of \( L^*_s(G) \) represents a strong arc of \( G \); and Two nodes of \( L^*_s(G) \) (say \( u \) and \( v \)) are adjacent if and only if their corresponding arcs are strong and share a common end point in \( G \) say \( x.\rho(u, v) = \mu(k, x) \wedge \mu(x, w) \)

Theorem 8.3. For any cycle fuzzy graph, then \( Ls(G) \) has one isolate node iff \( G \) has a non-bondage arc.

Proof. Let \( Ls(G) \) has one isolated node, so \( G \) has one weakest arc \( (x, y) \) by theorem 4.5 then \( (x, y) \)is non-bondage. Conversely, let \( G \) has a non-bondage arc \( (x, y) \), clearly \( (x, y) \) weakest arc and node \( x \) and \( y \) does not common node for two strong arcs. So corresponding vertex of arc \( (x, y) \) in \( Ls(G) \) is isolated.

\( \square \)

Theorem 8.4. Let \( Ls(G) \) be fuzzy black graph such that \( Ls(G) \) is a tree. Then a node in \( Ls(G) \) iff \( G \) has non-bondage or a arc adjacent with a non-bondage arc.
Proof. Given $L_s(G)$ be fuzzy block then $CONN_{G-\nu}(u, w) = CONN_G(u, w)$ by definition here node of $L_s(G)$ is a arc of G, so we clearly that $CONN_{G-(x,y)}(r, v) = CONN_G(r, v)$, converse true trivially.

**Theorem 8.5.** A complete fuzzy graph of $L_s(G)$ is not a complete.

Proof. Given that G is compete fuzzy graph so there exist a cycle in G and every node of even pair does not adjacent with odd pair so corresponding nodes not adjacent in $L_s(G)$.

**Theorem 8.6.** Let G be a path graph with n nodes then $L_s^*(G)$ has n-2 arcs.

Proof. Given G be a path with n nodes then n-1 arcs so $L_s^*(G)$ has a path with n-1 nodes so it has n-2 arcs.

**Theorem 8.7.** Let G be complete graph with n nodes then $L_s^*(G)$ is 2(n-2) regular graph.

Proof. Given G be complete graph so every node has n-1 strong arc then every arc adjacent with 2n-4, so $L_s^*(G)$ has 2(n-2) regular graph.

9 Neighbourhood Extension

**Definition 9.1.** Let G be graph and $S_i \subseteq V$, each $S_i$ is collection of each node in G and let $G_E(\tau, \rho)$ be underlying crisp graph. If $G_E(\tau, \rho)$said to be Neighbourhood extension, then satisfied following condition.

- Each node of $G_E$ represents a strong neighbourhood set of G
- Two nodes of $G_E$ are adjacent iff their correspond neighbour set have at least one common node
  where $\rho(S_i, S_j) = \min\{\mu(x, v_i), \mu(v_j, x)\mid x \in S_i \cap S_j\}$

**Definition 9.2.** Let G* be Connected graph and if $G_E \cong G^*$ then G said be C- Neighbour Extendable graph also called strong Neighbourhood Extendable otherwise weak Neighbourhood Extendable

**Definition 9.3.** Let G* be tree graph, if $G_E \cong G^*$ then G said be t- Neighbour Extendable graph also called semi strong Neighbourhood Extendable

**Theorem 9.4.** Let G be cycle graph and if all arcs are bondage arc , then $G_E$ is complement of G.
Proof. We know that G is connected graph and each node adjacent two nodes since G is cycle, so strong neighbour of each node of G is two. Clearly, if \( v_i, v_j \), are adjacent nodes then \( N_s(v_i) \cap N_s(v_j) = \emptyset \) but alternative nodes have some same nodes, so make arcs between them it will be form complement of G.

**Corollary 9.5.** Let G be cycle graph with n nodes then G is strong neighbour extendable if n is odd, otherwise weak neighbour extendable

**Proof.** Case 1: If n is odd, by using theorem 9.4 there exist two paths in \( G_E \) ie one is \( v_1, v_3, ... v_n, v_2 \) say \( P_1 \) another path say \( P_2 \) is \( v_2, v_4, ... v_{n-1}, v_1 \) so \( P_1 \) and \( P_2 \) has same nodes say \( v_1 \) and \( v_2 \) clearly \( G_E \) is connected graph and G is strong neighbour extendable

Case 2: If n is even, by using theorem 9.4 there exist two paths in \( G_E \) ie one is \( v_1, v_3, ... v_{2n-1}, v_1 \) say \( P_1 \) another path say \( P_2 \) is \( v_2, v_4, ... v_{2n}, v_2 \) so \( P_1 \) and \( P_2 \) does not have same nodes. Clearly \( G_E \) is disconnected graph and G weak neighbour extendable

**Theorem 9.6.** Let G be complete graph with n nodes, then G is strong neighbour extendable

**Proof.** We know that G complete graph then every node has n-1 strong arcs so strong neighbour set of every node has n-1 nodes \( N_s(v_i) = \{v_1, v_2, ... v_{i-1}, v_{i+1}, ... v_n\} \forall v_i \in G \) then \( N_s(v_i) \cap N_s(v_j) = \{v_1, v_2, ... v_{i-1}, v_{i+1}, ... v_{j-1}, v_{j+1}, ... v_n\} \forall v_i, v_j \in G \), this implies that \( G_E \) is complete graph, so G is strong neighbour extendable

**Theorem 9.7.** Let \( P_n \) be path with n nodes then G is not neighbour extendable.

**Proof.** Let \( P_n \) be path with n nodes then there exist unique path, so every arcs are strong arc and two nodes have one strong neighbour other nodes have two strong neighbours, but their neighbour sets are distinct, so \( G_E \) is null graph therefore G is not neighbour extendable.

**Theorem 9.8.** Let G be complete bipartite graph then G is not strong neighbour extendable

**Proof.** Given G is complete bipartite graph so node set is partition of two set say \( V_1 \) and \( V_2 \) then each node of \( V_1 \) is strong neighbour of every node in \( V_2 \) there two distinct path form in \( G_E \). One path connect every node in \( V_1 \) another path connect every node in \( V_2 \) so \( G_E \) is disconnect graph therefore G is not strong neighbour extendable.

**Theorem 9.9.** Let G be wheel graph with p nodes then G is Strong neighbour Extendable

**Proof.** Given that G is wheel graph then the domination number of G is one (say \( v_i \)) clearly \( v_i \) incident with p-1 bondage so strong neighbourhood set of every node of G has node \( v_i \) therefore \( G_E \) is connected graph implies that G is strong neighbour extendable.

**Definition 9.10.** (Deficiency Number) Let G be fuzzy graph but G is not strong neighbour extendable then the deficiency number is required number of arcs to make G is strong neighbour extendable.
Theorem 9.11. Let G be graph with 2n nodes which G is not neighbour extendable then deficiency number of G is one.

Proof. Given G is cycle with even nodes by using corollary 9.5 there exist two distinct cycle so add one strong arc between any node from one cycle and another cycle then $G_E$ is connected graph therefore neighbour extendable. □

Theorem 9.12. Let G be complete bipartite graph which G is not neighbour extendable then deficiency number of G is two .

Proof. Given G is complete bipartite graph by using theorem 9.8 there exist two open path so need two arcs for make $G_E$ is cycle therefore G is strong neighbour extendable. □

10 Conclusion

$L^*(G)$ is not tree, if all arc of G are strong. Above non bondage value ($\neq 0$) is not true for all graphs because $K_{1,n}$ or star graph and $P_3$ non-bondage value is 0 and also bondage number is equal to 1 for such above graphs

References


