# Proper Colourings in Magic and Antimagic Graphs

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**Abstract - In this paper, the new concept of proper colourings in vertex magic and anti-magic graphs has been introduced. A bijection mapping that assigns natural number to verties or edges of a graph is called a labeling and graph labeling that have weights associated with each edge and or vertex has been taken for consideration. If all the vertex weights have the same value then the labeling is called magic. If the weight is different for every vertex then we call the labeling as antimagic. In this paper,mainly new inequalities on chromatic number related with magic and anti-magic graphs are being established.**

**Keywords : vertex magic labeling, regular graphs, proper colourings,anti-magic graphs.**

## 1. INTRODUCTION

In this paper, we consider only finite, simple and undirected graphs. The graph G has vertex set V = V(G) and edge set E = E(G) and we take $\in$ = |E| and v = |V|. Magic labeling was introduced by Sedlacek in 1963[5]. The notion of an antimagic graph was introduced by Hartsfield and Ringel in 1989[3]. Vertex magic graphs are graphs labeled with numbers in which every vertex and its incident edges adds upto the same number. This number is called magic number, vertex magic labeling are coloured with proper colourings. Let r be the degree of the vertex of a regular graph.

*Main Results*

*1. Colourings in r-regular magic graphs for r ≥ 2*

The following are some of the basic definitions which are to be referred and has established some results.

**Definition 1.1**

A Bijection f : V(G) ∪ E(G) → {1, 2, 3, . . . , v+∈} is called a vertex magic labeling if there is a constant K, such that vertex weight of every vertex in G is equal to the constant

$$K = f(x) + \sum_{y \in N(x)} f(xy)$$

**Definition 1.2**

The chromatic number of a graph G is the minimum number of colours that is required to colour G. It is denoted by $\psi(G)$.

**Definition 1.3**

If every vertex in a graph G has the same degree r, that is $\delta = \Delta = r$, then G is called a regular graph of degree r, or an r-regular graph.

**Theorem 1.1**

If G is a vertex magic r-regular graphs then the chromatic number satisfies the double inequality $\frac{K-1}{v+\in} \le \psi(G) \le \frac{1}{2}nr$

**Proof**

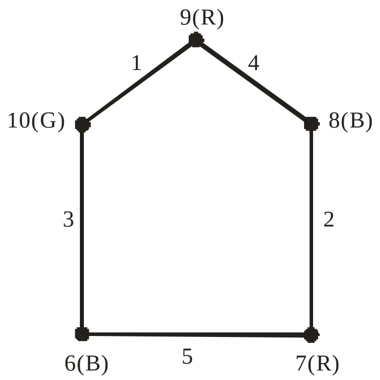
**Case: i**

Let $C_n$ be a cycle graph.
Let 'n' be an odd integer.
Let V(G) be the set of all vertices in the cycle graph and E(G) be the set of an edges in the cycle graph.
The cycle graph consists of 'n' vertices and 'n' edges.
Define f : V ∪ E → {1, 2, 3, . . . , 2n} as follows :
For every vertex $v_i$, the vertex magic labeling is defined for $1 \le i \le n$ respectively as follows:

$$f(v_i) = (2n + 1) - i \text{ for } 1 \le i \le n$$

For every edge $e_i$, the magic labeling is defined as follows for $1 \le i \le n$

$$f(e_i) = \begin{cases} \frac{i+1}{2} & \text{if i is odd} \\ \frac{n+1+i}{2} & \text{if i is even} \end{cases}$$



Magic number, K = 14

Let $C_n$ be a graph with vertex magic labeling, then the vertices $v_1$, $v_2$, $v_3$, $v_4$, $v_5$ are coloured with different colours by proper colouring and the number of colours used for colouring this graph is 3.

Therefore, $\psi(G) = 3$

Therefore, $\frac{K-1}{v+\in} \leq \psi(G)$ . . . (1)

The general condition of r-regular graph is denoted as $\frac{1}{2}nr$

Therefore, $\psi(G) \leq \frac{1}{2}nr$ . . . (2)

from equation (1) & (2)

It is easily verified that $\frac{K-1}{v+\in} \leq \psi(G) \leq \frac{1}{2}nr.$

Thus, an odd cycle satisfies the condition for colourings in vertex magic labeling for 2-regular graphs.

**Case: ii**

Let $C_n$ be an even cycle

Let 'n' be an even integer

Let $V(C_n)$ be the set of all vertices and $E(C_n)$ be the set of all edges in G.

Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$

Let $E(C_n) = \{v_n v_1\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\}$

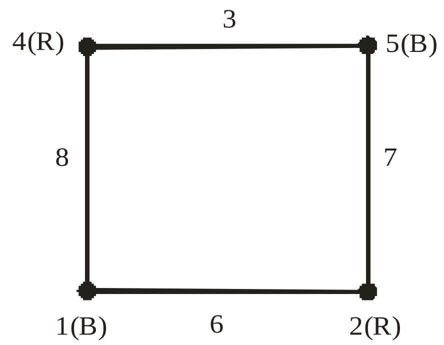
Define $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 2n\}$ as follows :

For every vertex $v_i$, the vertex magic labeling is defined as follows.

$$f(v_i) = \begin{cases} i+3 & \text{if } 1 \leq i \leq 2 \\ \frac{i+1}{2} & \text{if } i = 3 \\ i-3 & \text{if } i = 4 \end{cases}$$

For every edge $e_i$, the edge labelings is defined as follows :

$$f(e_i) = \begin{cases} i+2 & \text{if } i = 1 \\ 2n-i+1 & \text{if } 2 \leq i \leq 3 \\ 2i & \text{if } i = 4 \end{cases}$$



Magic Number, K = 15

Let $C_n$ be a cycle graph with vertex magic labeling, then the vertices $v_1$, $v_2$, $v_3$, $v_4$ are coloured with different colours by proper colouring. The colours used for colouring this graph is 2.

Therefore $\psi(G) = 2$

$\therefore \frac{K-1}{v+\in} \leq \psi(G)$ . . . (3)

The general condition of r-regular graph is denoted as $\frac{1}{2}nr$

Therefore, $\psi(G) \leq \frac{1}{2}nr$ . . . (4)

from equation (3) & (4)

It is easily verified that $\frac{K-1}{v+\in} \leq \psi(G) \leq \frac{1}{2}nr.$

Thus, an even cycle satisfies the condition for colourings in vertex magic labeling for 2-regular graphs.

**Case: iii**

Let the graph G be generalized Petersen graph

Let 'n' be an even integer.

Let $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$

Let $E(C_n) = \{e_1, e_2, e_3, \ldots, e_n\}$

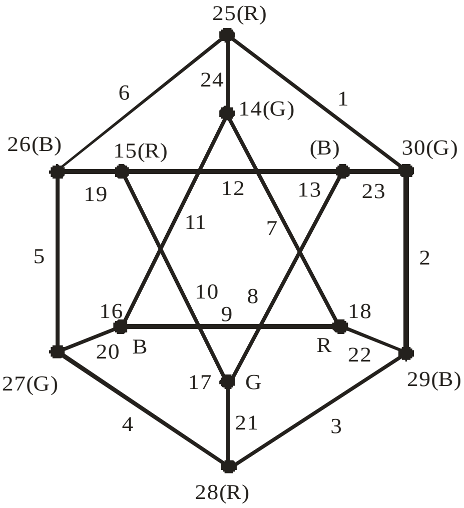
Define $f : V \cup E \rightarrow \{1, 2, 3, \ldots, 2n+6\}$ as follows :

For every vertex $v_i$ is defined respectively as follows :

$$f(v_i) = \begin{cases} 2n+i & ; \quad i = 1 \\ 2n+8-i & ; \quad 2 \leq i \leq 6 \\ 2n-i-3 & ; \quad 7 \leq i \leq 8 \\ 2n-i+3 & ; \quad 9 \leq i \leq 12 \end{cases}$$

For every edge $e_i$, is defined respectively as follows :

$$f(e_i) = \begin{cases} i & \text{for } 1 \leq i \leq 12 \\ 3n-i+1 & \text{for } 13 \leq i \leq 18 \end{cases}$$




Magic number, K = 56

The generalized Petersen Graph is labeled with vertex magic, then the vertices are coloured with different colours by proper colouring and the number of colours used for colouring this graph is 3.

Therefore, $\psi(G) = 3$

Therefore, $\dfrac{K - 1}{v + \in} \leq \psi(G)$ . . . (5)

The general condition of r-regular graph is denoted as $\dfrac{1}{2}nr$.

Therefore, $\psi(G) \leq \dfrac{1}{2}nr$ . . . (6)

from equation (5) & (6)

$$\frac{K - 1}{v + \in} \leq \psi(G) \leq \frac{1}{2}nr.$$

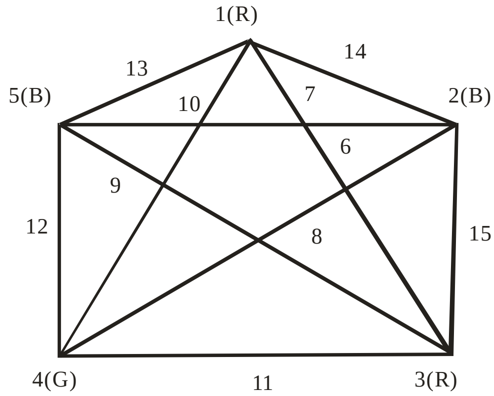
Thus, a generalized Petersen Graph satisfies the condition for colourings in vertex magic labeling for 3-regular graphs.

**Case: iv**

Let G be a complete graph
Let ‘n’ be an odd integer.
Let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertices.
Let $E = \{e_1, e_2, e_3, \dots, e_n\}$ be the edges.

Define $f : V \cup E \to \{1, 2, 3, \dots, 3n\}$ as follows :

$f(v_i) = i$ for $1 \leq i \leq n$
$f(e_i) = n + i$ for $1 \leq i \leq 2n$




Magic number, K = 45

The complete graphs is labeled with vertex magic, then the vertices in the complete graph is coloured with different colours by proper colouring and the number of colours used for colouring the complete graph is 5.

Therefore, $\psi(G) = 5$

Therefore, $\dfrac{K - 1}{v + \in} \leq \psi(G)$ . . . (7)

The general condition of r-regular graph is denoted as $\dfrac{1}{2}nr$.

Therefore, $\psi(G) \leq \dfrac{1}{2}nr$ . . . (8)

from equation (7) & (8)
It is easily verified that

$$\frac{K - 1}{v + \in} \leq \psi(G) \leq \frac{1}{2}nr.$$

Therefore, the complete graph satisfies the condition for colourings in vertex magic labelings for 4-regular graphs

## 2. COLOURINGS IN ANTI-MAGIC GRAPHS

The following are some of the basic definitions which are to be referred and has established some results.

**Definition 2.1**

The weight of a vertex $x \in V$, under the labeling $\alpha$, is $wt(x) = \alpha(x) + \sum_{y \in N(x)} \alpha(xy)$, where for every $x \in V$, $\alpha(x) = 0$ under an edge labeling and $\alpha(x) \neq 0$ under a total labeling.

**Theorem 2.1**

If G is a vertex anti-magic graphs then the chromatic number satisfies double inequality.

$$\frac{t_1}{t_0}-1 \leq \chi(G) \leq \left|\frac{a - d'}{n}\right| + d$$

**Proof :**

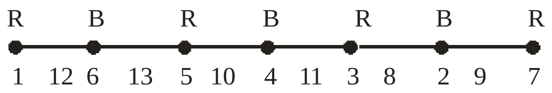
**Case i**

Let $P_n$ be the path and 'n' be an odd integer.
Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices and $\{e_1, e_2, \ldots, e_n\}$ be the edges of the path $P_n$ receive the following labels.

$f(v_1) = 1$
$f(v_n) = n$
$f(v_i) = n - i + 1$ for $2 \leq i \leq n-1$

and the edge receive the labels

$$f(v_iv_{i+1}) = \begin{cases} 2n-1-i & \text{for i is odd} \\ 2n+1-i & \text{for i is even} \end{cases}$$




Let $P_n$ be the path and with vertex antimagic labeling, then the vertices $v_1$, $v_2$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$ are coloured with different colours by proper colourings and the number of colours used for colouring this graph is 2.
Therefore, $\chi(G) = 2$
Let $t_1 = \max\{13, 16, 19, 22, 25, 28, 31\}$
$\therefore t_1 = 31$
Let $t_0 = \min\{13, 16, 19, 22, 25, 28, 31\}$
$\therefore t_0 = 13$

$$\therefore \frac{t_1}{t_0}-1 \leq \chi(G) \qquad \ldots (1)$$

$$\left|\frac{a - d'}{n}\right| + d \geq \chi(G)$$

$$\therefore \chi(G) \leq \left|\frac{a - d'}{n}\right| + d \qquad \ldots (2)$$

From (1) and (2)
It is easily verified that

$$\frac{t_1}{t_0} - 1 \leq \chi(G) \leq \left|\frac{a - d'}{n}\right| + d$$

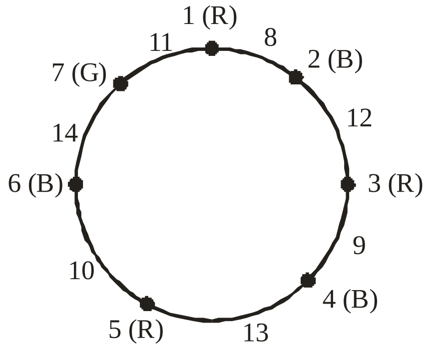
Then the vertex weights form an arithmetic progression with difference two, namely $2n - 1, 2n + 2, 2n + 3, 2n + 5, \ldots$

**Case ii**

Let $C_n$ be the cycle and 'n' be the number of vertices. Label the vertices and the edges of $C_n$ by

$f(v_i) = n - i + 1$ for $1 \leq i \leq n$

$$f(v_iv_{i+1}) = \begin{cases} n+\dfrac{i+1}{2} & \text{for i is odd} \\ 2n-2+\dfrac{i}{2} & \text{for i is even} \end{cases}$$




Let $C_n$ be the cycle and with vertex antimagic labeling, then the vertices $v_1$, $v_2$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$ are coloured with different colours by proper colourings and the number of colours used for colouring this graph is 3.
Therefore, $\chi(G) = 3$
Let $t_1 = \max\{20,22,24,26,28,30,32\}$
$\therefore t_1 = 32$
Let $t_0 = \min\{20,22,24,26,28,30,32\}$
$\therefore t_0 = 20$

$$\therefore \frac{t_1}{t_0}-1 \leq \chi(G) \qquad \ldots (3)$$

$$\therefore \chi(G) \leq \left|\frac{a - d'}{n}\right| + d \qquad \ldots (4)$$

From (3) and (4)
It is easily verified that

$$\frac{t_1}{t_0}-1 \leq \chi(G) \leq \left|\frac{a - d'}{n}\right| + d$$

Then the vertex weights form an arithmetic progression with difference two, namely $\frac{5n+5}{2}, \frac{5n+9}{2}, \ldots$

**Conclusion**

In this Paper,a new double inequalities has been established. Further,it has been verified for magic and antimagic graphs. Finally,we conclude that new inequalities on chromatic number related with magic and anti-magic graphs are satisfied.



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