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PRIMITIVE PERMUTATION GROUPS FOR THE TWO ORBITAL EQUATIONS

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ABSTRACT

In this paper, the orbital and orbit are basic ideas in group theory. Xu Changliang began to investigate the orbital equations in the primitive permutation groups. There are two equations of orbital to be solved. The aim of this paper is to interpret the solution to the first orbital equation $P^n = P^m$ and the second orbital equation $P^m = P \cup \overline{P}$, where m ≥ 3. Some general theory of permutation group is needed for the statement and proof of the result in this paper.

Keywords: Permutation groups , orbital , sequence , monoid , invariant , non-trivial and prime numbers.

1. INTRODUCTION

We first introduce the basic idea under lying a permutation group form a monoid, the binary relation invariant Neumann, P.M. and Praeger, C.E. [1] discussed the orbital equation $\mathrm{P} \circ \mathrm{P} = \overline{\mathrm{P}}$ in their study of three star permutation groups. The aim of this paper is to complete the discussion of two equation of orbital $P^n = P^m$ and $P^m = P \bigcup \overline{P}$, where m ≥ 3, Neumann, P.M.[3] began the investigation of the influence of some equations in this monoid.

In this paper, suppose that S is a non-empty set and let G is a transitive subgroup of S. Consider that the group G acting on a set S, the orbitals are the orbits of G in S x S. The suborbits are orbits of G_t for t ∈ S; G is transitive.

The relation

$$\mathrm{P(t)} = \{\mathrm{x} \in \mathrm{S} \,|\, (t, x) \in P\} \qquad (1.1).$$

This shows that a one to one corresponding subdegree of G on S. The number of suborbits or orbitals is said to be rank of G.

The paired orbital $\overline{P}$ of the orbital P is defined as

$$\overline{\mathrm{P}} = \{(\mathrm{x}_1, \mathrm{x}_2) \,|\, (\mathrm{x}_2, \mathrm{x}_1) \in \mathrm{P}\} \qquad (1.2)$$

The orbital P is known as the self-paired if $P = \overline{\mathrm{P}}$.

The subsets P and T defined as;

$$\mathrm{P} \circ \mathrm{T} = \{(\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{S}^2 \,|\, (\exists\, x \in S) : (x_1, x) \in P\} \qquad (1.3)$$

and $$(\mathrm{x}_1, \mathrm{x}_2) \in \mathrm{T},$$

Since if P and T are orbitals, then $P \circ T$ will be a union of orbitals i.e $P \bigcup T$.

Therefore, for three binary relations P, T and Q

$$(\mathrm{P} \circ \mathrm{T}) \circ \mathrm{Q} = \mathrm{P} \circ (\mathrm{T} \circ \mathrm{Q})$$

This is associated with orbitals.

Notice that if P and T are invariant of G then $\mathrm{P} \circ \mathrm{T}$ will also be invariant of G. The equality operator E is an identity element and the relation P as being a non-trivial if $\mathrm{P} \neq \mathrm{E}$.

In this sense the empty relation Φ is an element of zero, such that $\mathrm{P} \circ \Phi = \Phi = \Phi \circ \mathrm{P}$, for all binary relations.

Let the group G is transitive on a set S and the universal set U satisfies $\mathrm{P} \circ \mathrm{U} = \mathrm{U} = \mathrm{U} \circ \mathrm{P}$ for any non-zero invariant of group G binary relation P.

In this section, Neumann, P..M.[3] focuses on a discussion of the equations
$P^2 = P \bigcup \overline{P}, P^i = \overline{P}, \ P^3 = P^2, P^i = P$ and $P^i = E$.

Changliang, X. [2] proposed the two equations of orbitals $P^n = P^m$ and $P^m = P \bigcup \overline{P}$, where $m \geq 3$. The other section contains the solution of these two orbital equations.

2. SOLUTION OF THE ORBITAL EQUATION $P^n = P^m$

Let us consider that S is a non-empty set and the group G is a transitive subgroup of S. Some general theory of the permutation group is needed for the statement and proof of the results in this section. For the equation of orbital $P^n = P^m$, if G is strongly primitive on S.
By definition, if a strongly primitive permutation group leaves trivial proper transitive binary relation invariant and we get the following statement.

If G is strongly primitive group on S and G has a non- trivial orbital P for which $P^n = P^m$, where $n > m$; then P^m is the universal relation on a set S. i.e.

$$P^m = U \text{ or } P^{n-m} = E$$

Such that $|G| = |S| = p$ and $n \equiv m \pmod{p}$, where p is a prime number.
To prove our main result we need a discussion of the orbital equation $P^n = P^m$ [2]. Therefore, we restate the theorems for ready reference.

Theorem 2.1: Let G is primitive on S and let T is a non-trivial invariant relation of G satisfying $T \circ T = T$. Then T is a non-trivial invariant partial-order relation of G on S i.e. T = E or T is the universal relation U on S.
Proof : We have

$$T \circ T = T \qquad (2.1)$$

Since T is a transitive relation. It has a corresponding equivalence relation $E \bigcup (T \bigcap \overline{T})$, (say).
Because the group G is primitive, this must be E or universal relation U.
Therefore, there are some possibilities for $T \bigcap \overline{T} = \Phi$, E, U and U/E. If $T \bigcap \overline{T} = \Phi$, then T is a strict partial-order relation on S. If $T \bigcap \overline{T} = E$, then T is a non-strict partial-order relation or T = E.
If $U/E \subseteq T \bigcap \overline{T}$, then $E \subseteq T \circ \overline{T}$. Since $E \subseteq T$ and so that $E \subseteq T \bigcap \overline{T}$.
Therefore, $T \bigcap \overline{T} = U$ i.e. T = U.
Theorem 2.2: Let G is a primitive on S and G has a non-trivial orbital P, Such that $P^n = P^m$, where $n > m$. Define $q = n - m$ and the natural number N so that $q(N-1) < m \leq qN$. Then P^{qN} is a non-trivial invariant partial order relation of G on S i.e. $P^q = E$ or P^m is the universal relation U on S .
Proof: Given as $P^n = P^m$ and $P^r = P^{r+q}$ for any natural number r with $r \geq m$.
Therefore, $P^r = P^{r+q\delta}$ for natural number s and r with $r \geq m$.
Because $qN \geq m$ and $P^{qN} = P^{qN+qs}$ (2.1)
In particular case, when s = N, we get from equation (2.1),then

$$P^{qN} = P^{2qN} \qquad (2.2)$$

Now we define $T = P^{qN}$ (2.3)
Since P^{qN} is a non-trivial invariant partial order relation of G on S i.e. T = E. Hence T is the universal relation U on S.
When T = E, then it is easy to prove that $P^q = E$.
Therefore, T is the universal relation U on S, because $n > qN$, we have

$$P^n = U \qquad (2.4)$$

If P is a orbital of G and $P^m = U$ then $P^n = U$ for any n > m. Hence the equation $P^m = P^n$, if G is strongly primitive on S, then the result is clear $P^m = U$ or $P^{n-m} = E$.

3. SOLUTION OF THE ORBITAL EQUATION $P^m = P \cup \overline{P}$

Neumann, P.M.[3], the orbital equations in the primitive permutation groups, If G is finite and $P^2 = P \cup \overline{P}$,then G is two-homogeneous but not two transitive. The finiteness of the S turns out to be significant for the analysis of the equation $P^2 = P \cup \overline{P}$. The result for the orbital equation $P^m = P \cup \overline{P}$, where $m \geq 3$, applies in both the finite and infinite conditions : m is odd and $|G| = |S| = 2$.

This section focuses on a discussion of the second orbital equation $P^m = P \cup \overline{P}$, where $m \geq 3$. Let G is a primitive on S and let P be a non-trivial orbital. Consider that $P^3 = P \cup \overline{P}$ and $|G| = |S| = 2$.

Theorem3.1: Let G is primitive on S and let P is non-trivial orbital. Consider that

$$P^m = P \cup \overline{P}, \quad \text{where } m \geq 3.$$

Suppose that the points $x_{1,} x_{2,} x_{3,}$ ...., $x_{2m-2} \in$ S, such that $x_1 \neq x_m$ and $(x_i, x_{i+1}) \in$ P for i=1,2,3,……,2m-3 and $(x_{2m-2}, x_1) \in$ P. Then $P = \overline{P}$.

Proof: The solution of the equation

$$P^m = P \cup \overline{P} \tag{3.1}$$

where $m \geq 3$

Consider that $P \neq \overline{P}$, so that $P \cap \overline{P} = \Phi$.

We define

$$\psi(x) = P(x) \cup \overline{P}(x) \tag{3.2}$$

and
$$\phi(x) = P(x) \cup \overline{P}(x) \cup \{x\}. \tag{3.3}$$

The first result, it is easy to show that either $(x_1, x_m) \in P$ or $(x_m, x_1) \in P$. Consider that $(x_1, x_m) \notin P$ and $(x_m, x_1) \notin P$.

We show that the following equation :

$$\psi(x_1) = \psi(x_m) \tag{3.4}$$

Given that any value λ in P(x_1), because $(x_1, x_m) \notin P$ and $\lambda \neq x_m$.

But $(x_m, x_{m+1}), (x_{m+!}, x_{m+2}), (x_{m+2}, x_{m+3})$,........ ......, $(x_{2m-3}, x_{2m-2}), (x_{2m-2}, x_1)$ and (x_1, λ) are in P and the equation (3.1) holds, we obtain that either $(\lambda, x_m) \notin P$ or $(x_m, \lambda) \in P$ and so that $\lambda \in \psi(x_m)$.

Therefore

$$\psi(x_1) \subseteq \psi(x_m) \tag{3.5}$$

In the same way for the sequence of the points x_1, x_2, x_3, ……..,x_{m-1}, x_m , then we obtain

$$\psi(x_m) \subseteq \psi(x_1) \tag{3.6}$$

From equations (3.5) and (3.6), then we get

$$\psi(x_1) = \psi(x_m)$$

Hence the result (3.4) is proved correct.

Now we define equivalence relation ($\approx$) on S:

$\alpha \approx \beta$ if and only if

$$\psi(\alpha) = \psi(\beta) \qquad (3.7)$$

Since G is primitive and the relation $\approx$ is obviously an invariant equivalence relation of G. the relation $\approx$ is universal and trivial. Therefore the relation $\psi(x_1) = \psi(x_m)$ holds and $x_1 \neq x_m$,then the relation $\approx$ is not trivial and so that it must be universal.

Since $\psi(x_1) = \phi(x_2)$ and then $x_1 \notin \psi(x_2)$.

Hence $(x_1, x_2) \notin P$, this is a contradiction, so that either $(x_1, x_m) \in P$ or $(x_m, x_1) \in P$.

The first result is complete.

The second result, it is easy to prove that $P \bigcup \overline{P} = U \mid E$.

In the same way exactly the proof $\psi(x_1) = \psi(x_m)$, but considering that either $(x_1, x_m) \in P$ or $(x_m, x_1) \in P$,we obtain the following result :

$$P(x_1) \bigcup \overline{P}(x_1) \bigcup \{x_1\} = P(x_m) \bigcup \overline{P}(x_m) \bigcup \{x_m\} \qquad (3.8)$$

i.e. $$\phi(x_1) = \phi(x_m) \qquad (3.9)$$

Now, a new relation ~ on S is define by

$\alpha \sim \beta$ if and only if

$$\phi(\alpha) = \phi(\beta) \qquad (3.10)$$

Since G is the primitive and the relation ~ is obviously a invariant equivalence relation of G and the relation ~ is universal or trivial. Therefore $\phi(x_1) = \phi(x_m)$, then the relation ~ is not trivial and so that it is universal.

For any value X, ψ in S, so either $(X, \psi) \in P$ or $(\psi, X) \in P$ and $X = \psi$. So that $U \mid E \subseteq P \bigcup \overline{P}$.

Therefore $(P \bigcup \overline{P}) \bigcap E = \Phi$ and $P \bigcup \overline{P} = U \mid E$.

Hence the second result is complete.

Suppose that any three point x_1, x_2 and x_3, such that

$(x_1, x_2) \in P$ and $(x_2, x_3) \in P$. Because $P \neq \overline{P}$, we have $x_1 \neq x_3$,

then $(x_1, x_3) \in U / E$.

Since it follows that $(x_1, x_3) \in P \bigcup \overline{P}$,

therefore $P^2 \subseteq P \bigcup \overline{P}$.

If $P^2 = P$ then $P^m = P^2 = P$, which is not true. If $P^2 = P \bigcup \overline{P}$ then using the mathematical induction on k, we see that $P \bigcup \overline{P} \subseteq P^k$ for all $k \geq 2$.

In particular, $P \bigcup \overline{P} \subseteq P^{m-1}$ then $E \subseteq P^m$, which is not true.

If $P^2 = \overline{P}$ then $\overline{P}^2 = P$, so that $P^4 = P$.

Therefore, $P^m = P$, if n=1 (mod.3)

or $P^m = \overline{P}$, if n=2 (mod.3)

or $E \subseteq P^m$, if n=0 (mod.3)

which is not true.

Theorem3.2: Let G is primitive on S and P be a non-trivial orbital. Consider that $P^m = P \bigcup \overline{P}$ and $m \geq 4$.

Let $P \subseteq P^3$ and $P \bigcup \overline{P} = \Phi$,then there are points $x_1, x_2, \ldots, x_{m+1} \in S$, such that $x_1 \neq x_m$, $(x_i, x_{i+1}) \in P$ for i =1, 2, 3,………,m and $(x_1, x_{m+1}) \in P$.

Proof: Suppose that an action expansion on a sequence of the point x_{j1}, x_{j2}, ………,x_{jk} of S which satisfy $(x_{ji}, x_{ji+1}) \in S$ for i =1,2,3,……..,k-1 and $(x_{j1}, x_{jk}) \in P$. The action expansion is defined to add the two points in the following way, for the edge (x_{j1}, x_{j2}), because $P \subset P^3$.

We obtain x_{j11} and x_{j12}, which satisfy $(x_{j1}, x_{j11}) \in P$, $(x_{j11}, x_{j12}) \in P$ and $(x_{j12}, x_{j2}) \in P$.

Substituting the point x_{j11} and x_{j12} to the sequence between x_{j1} and x_{j2}, we obtain a new sequence of the points x_{j1}, x_{j11},x_{j12},……..,x_{jk} .

Consider that m is even, we deal with the problem by some applications of action expansion. In the case when n is odd, given an edge (y_1,y_{m+2}), we obtain a sequence of four points y_1,y_m,y_{m+1},y_{m+2} which satisfy $(y_1, y_m) \in P$, $(y_m, y_{m+1}) \in P$ and $(y_{m+1}, y_{m+2}) \in P$.

Step by step we obtain m+2 points y_1, y_2, y_3, ………. , y_{m+2}.

Therefore $(y_i, y_{i+1}) \in P, \forall i = 1,2,3,4,......., m$ and $P^m = P \bigcup \overline{P}$.

We obtain that either $(y_1, y_{m+1}) \in P$ or $(y_{m+1}, y_1) \in P$, since $(y_1, y_m) \in P$, we obtain $y_1 \neq y_m$.

If $(y_1, y_{m+1}) \in P$ then y_1, y_2,......., y_{m+1} are the points which satisfy the assertion of the theorem.

If $(y_{m+1}, y_1) \in P$, we obtain a sequence of three points y_{m+1},y_1 and y_{m+2}, step by step we get a sequence of m+1, when n is odd ,we deal with the problem by some application of the expansion.

Given any two points x_1 and x_{m+1} which satisfy $(x_1, x_{m+1}) \in P$ by applying the expansion, we obtain that the points x_{m-1} and $x_m \in S$ which satisfy $(x_1, x_{m-1}) \in P$ and $(x_m, x_{m+1}) \in P$ .It is clear that $x_1 \neq x_m$.

Now, we get a sequence of the four points x_1, x_{m-1}, x_m and x_{m+1}, since n is odd, step by step we obtain a sequence m+1 points x_1,x_2,x_3,………,x_{m+1}, such that $x_1 \neq x_m$, then $(x_i, x_{i+1}) \in P$ and $(x_1, x_{m+1}) \in P$ for i =1, 2, 3, ……., m. Hence we get a sequence of m+1 points as required for the theorem.

Theorem3.2: Let G is primitive on S and let P be a non-trivial orbital, consider that $P^m = P \bigcup \overline{P}$ and $m \geq 4$, then n is odd and $|G| = 2 = |S|$.

Proof: Suppose that $P = \overline{P}$,we have $P^m = P$ and $P^m = \overline{P}$, by the theorem of Neumann [3]; $|G| = p = |S|$ and m=1(mod. p).

Also $|G| = |S| = p$ and m= -1(mod.p) for some prime number p. So that p = 2 and n is odd, $P \neq \overline{P}$.

First, we show that $P \bigcap P^3 \neq \Phi$ (3.1)

Consider that $P \bigcap P^3 = \Phi$, since $\overline{P} \subseteq P^m$, we choose x_1,x_2,x_3,…….,x_{m+1} from S, such that $(x_i, x_{i+1}) \in P$ for i = 1, 2, 3, …….,m and $(x_{m+1}, x_1) \in P$.

Therefore $P \subseteq P^m$, we obtain x_{m+2}, x_{m+3},……., x_{2m} which satisfy $(x_i, x_{i+1}) \in P$ for i = m+1, m+2, ……,2m-1 and $(x_{2m}, x_1) \in P$.

Now the points x_1, x_2, x_3, ……, x_{2m}, make a cycle of *2m* points which is degenerate. For the (n+1) points x_2, x_3, ...,x_{m+2}, because $(x_i, x_{i+1}) \in P$, for i=2, 3,……., m+1 and $P^m = P \bigcup \overline{P}$, either $(x_2, x_{m+2}) \in P$ or $(x_{m+2}, x_2) \in P$.

If $(x_2, x_{m+2}) \in P$ and the four points x_{m+1}, x_1, x_2, x_{m+2} make it impossible that $P \bigcap P^3 = \Phi$. So that $(x_2, x_{m+2}) \notin P$ and therefore $(x_{m+2}, x_2) \in P$. By the same argument, we obtain the following results one by one $(x_{m+3}, x_3) \in P$, $(x_{m+4}, x_4) \in P$, ……., $(x_{2m}, x_m) \in P$, and $(x_1, x_{m+1}) \in P$. Since $(x_1, x_{m+1}) \in P$ and $(x_{m+1}, x_1) \in P$ are both true .

Hence this contradiction complete the proof and the equation (3.1) is proved correct.
Secondly, it is easy to show that

$$P^{m-2} \subseteq P \cup \overline{P} \qquad (3.2)$$

Since P^3 is a union of the orbitals, the equation (3.1) gives

$$P \subseteq P^3 \qquad (3.3)$$

Given a sequence of (m-1) points $z_1, z_2, z_3, \ldots, z_{m-1}$ of S ,which satisfy (z_i,z_{i+1}) for i =1, 2, 3, ……, m-2 . The action expansion is defined to add the two points in the following way for the edge (z_1, z_2), because the equation (3.3) holds. There are two points ϕ and ψ, which satisfy $(z_1,\phi),(\phi,\psi),(\psi,z_2) \in P$.

Adding the two points ϕ and ψ, we obtain a sequence of m+1 points z_1, $\phi, \psi, z_2, z_3, \ldots, z_{m-1}$ which satisfy $(z_1,\phi),(\phi,\psi),(\psi,z_2),(z_2,z_3),\ldots\ldots,(z_{m-2},z_{m-1})$. This means that the equations

$$P^{m-2} \subseteq P^m \qquad (3.4)$$

and

$$P^m = P \cup \overline{P} \qquad (3.5)$$

Since the equation (3.2) is correct.

In the last, when $P^{m-2} \cap P \neq \Phi$, we have choose $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_{m-1}$ in S, which satisfy $(\lambda_i, \lambda_{i+1}) \in P$ for i=1,2,3,……,m-2, and $(\lambda_1, \lambda_{m-1})$, since

$$\overline{P} \subseteq P^m \qquad (3.6)$$

We obtain a sequence of the points $\mu_1, \mu_2, \mu_3, \ldots\ldots, \mu_{m-1}$ which satisfy $(\lambda_{m-1}, \mu_1) \in P$ and $(\mu_i, \mu_{i+1}) \in P$ for i = 1, 2, 3,………., m-2 and $(\mu_{m-1}, \lambda_1) \in P$.

Therefore, $(\lambda_1, \lambda_{m-1}) \in P$ and $(\lambda_{m-1}, \mu_1) \in P$, then we obtain

$$\lambda_1 \neq \mu_1 \qquad (3.7)$$

For the sequence of the points $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_{m-1}$ and $\mu_1, \mu_2, \mu_3, \ldots\ldots, \mu_{m-1}$, using the theorem (3.1) and we obtain

$$P = \overline{P} \qquad (3.8)$$

which is the contradiction.

When $P^{m-2} \cap P = \Phi$, from equation (3.2), we obtain

$$P^{m-2} = \overline{P} \qquad (3.9)$$

Now consider that a sequence of the points $w_1, w_2, w_3, \ldots\ldots, w_{m+1}$, which satisfy $w_1 \neq w_m, (w_i, w_{i+1}) \in P$ and $(w_1, w_{m+1}) \in P$ for i=1,2,3,………m. Therefore, $(w_1, w_{m+1}) \in P$, then we obtain $(w_{m+1}, w_1) \in \overline{P}$.

We have choose $w_{m+2}, w_{m+3}, \ldots\ldots, w_{2m-2}$,such that $(w_i, w_{i+1}) \in P$ and $(w_{2m-2}, w_1) \in P$ for i = m+1,m+2,m+3,………..,m-3.

So that we obtain a cycle of (2m-2) points $w_1, w_2, w_3, \ldots\ldots, w_{2m-2}$, which may be degenerate.

Therefore, $w_1 \neq w_m$, using the theorem (3.1),then we obtain

$$P = \overline{P}$$

which is the contradiction.

Corollary 3.1: If G is strongly primitive on S and has a non-trivial orbital *P*. Consider that $m \geq 3$ and

$$P^m = P \cup \overline{P}$$

Then m is odd and $|G| = 2 = |S|$.

The method of proof is elementary and purely combinatorial.

Theorem 3.4: Let G is primitive on S and let P be a non-trivial orbital. Consider that

$$P^3 = P \cup \overline{P}$$

Then $|G| = |S| = 2$.

Proof : When $P = \overline{P}$,

we have $P^3 = P$

and $P^3 = \overline{P}$

By Neumann, P. M.[3], theorem (2.2)

$$|G| = |S| = p$$

and $3 \equiv 1(\mathrm{mod}.p)$, for some prime number p.

Since if p = 2, then

$$|G| = |S| = 2.$$

When, $P \neq \overline{P}$ then we have

$$\overline{P} \subseteq P^3$$

If we choose a sequence of four points x_1,x_2,x_3 and x_4 such that (x_1,x_2), (x_2,x_3) and (x_3,x_4) and also $(x_1, x_4) \in \overline{P}$.

Such as $P \neq \overline{P}$,

we have

$$x_1 \neq x_3$$

Using the theorem (3.1) in the case when n = 3 and we obtain

$$P = \overline{P}$$

This show that $P = \overline{P}$ is not true.

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