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Positive Stationary Solutions of Cross-Diffusive Competition Model with A Protection Zone

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***Abstract* - In this paper, positive stationary solutions of a cross-diffusive competition model with a protection zone for the weak competitor are examined. The asymptotic behaviour of positive stationary solutions is obtained for any birth rate as the cross-diffusion coefficient tends to infinity.**

Keywords - Cross-diffusion; heterogeneous environment; stationary solution

INTRODUCTION

Competition is one of the most essential mechanisms in ecology, shaping the distribution and abundance of biological species. When two species utilize similar resources, their interaction may lead to coexistence, competitive exclusion, or complex spatial-temporal dynamics depending on environmental conditions and intrinsic population traits. To analyze such processes rigorously, mathematical models, especially those based on systems of partial differential equations, have become indispensable tools.

The effects of environmental heterogeneity with large cross-diffusion are studied. In the following, we consider Lotka-Volterra cross-diffusive competition model with a protection
zone

$$\left.\begin{array}{ll} u_t = \Delta\left[\left(1+k\rho(x)v\right)u\right]+u\left(\lambda-u-\eta b(x)v\right), & x\in\Omega, \quad t>0,\\ \tau v_t = \Delta v+v\left(\mu-v-du\right), & x\in\Omega\setminus\bar{\Omega}_1, t>0,\\ \dfrac{\partial u}{\partial n}=0, & x\in\partial\Omega, \quad t>0,\\ \dfrac{\partial v}{\partial n}=0, & x\in\partial\left(\Omega\setminus\bar{\Omega}_1\right), t>0,\\ u(x,0)=u_0(x)\ge 0, & x\in\Omega,\\ v(x,0)=v_0(x)\ge 0, & x\in\Omega\setminus\bar{\Omega}_1, \end{array}\right\} \tag{1}$$

where $\Omega\subseteq\mathbb{R}^N\ (N\ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, Ω_1 is a subdomain of Ω with smooth boundary $\partial\Omega_1$ and $\bar{\Omega}_1\subset\Omega$; n is the outward unit normal vector on the boundary; positive constants λ and μ are the intrinsic growth rates of the respective species; $\eta b(x)$ and $d>0$ are the interspecific competitive pressure on u and v, respectively; $\rho(x)$ and $b(x)$ are spatially heterogeneous, and satisfies $\rho(x)\equiv b(x)\equiv 0$ in $\bar{\Omega}_1$ and $\rho(x)\equiv 1>0$ and $b(x)\equiv b>0$ in $\Omega\setminus\bar{\Omega}_1$; $\tau>0$ is a constant; Δ denotes the Laplacian operator on the space variable x; $u(x,t)$ and $v(x,t)$ represent the population densities of the respective competing species.

In the model, u lives in the larger habitat Ω, and Ω_1 is its protection zone, where u can leave and enter the protection zone freely, while v can only live outside Ω_1. Thus we impose a no-flux boundary condition on $\partial\Omega_1$ for v. On $\partial\Omega$, a no-flux boundary condition is also assumed for both species, and no individuals cross the boundary $\partial\Omega$. Throughout the paper, we write $\Omega^*=\Omega\setminus\bar{\Omega}_1$.

It should be noted that $k\Delta\left[\rho(x)vu\right]$ is the cross-diffusion term to model the habitat segregation phenomena between two competing species. From the cross-diffusion term, u diffuses to low density regions of v in their common living habitat Ω^*, and the coefficient k denotes the sensitivity of the competitor u to the population pressure from the other competitor v.

The corresponding stationary problem is

$$\left.\begin{array}{ll} \Delta\left[\left(1+k\rho(x)v\right)u\right]+u\left(\lambda-u-\eta b(x)v\right)=0, & x\in\Omega,\\ \Delta v+v\left(\mu-v-du\right)=0, & x\in\Omega^*,\\ \dfrac{\partial u}{\partial n}=0, & x\in\partial\Omega,\\ \dfrac{\partial v}{\partial n}=0, & x\in\partial\Omega^*. \end{array}\right\} \tag{2}$$

We denote by $\lambda_1^D(\phi,U)$ and $\lambda_1^N(\phi,U)$ the first eigenvalue of $-\Delta+\phi$ over the bounded domain U with Dirichlet and Neumann boundary conditions, respectively. We usually omit U in the notation if $U=\Omega$. If the potential function ϕ is omitted, then we understand $\phi=0$. It is well-known that the following properties hold:

(i) the mapping $q\mapsto\lambda_1^B(q,U):L^\infty(U)\to\mathbb{R}$ is continuous with $B=D$ or $B=N$;

(ii) $\lambda_1^B(q_1,U)>\lambda_1^B(q_2,U)$ if $q_1\ge q_2$ and $q_1\not\equiv q_2$ with

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$B = D$ or $B = N$;

(iii) $\lambda_1^D(q,U_1) \geq \lambda_1^D(q,U_2)$ if $U_1 \subset U_2$, and

$\lambda_1^N(0,U) = 0$.

The usual norm of the space $L^p(U)$ for $p \in [1,\infty)$ is defined by

$$\|u\|_{p,U} = \left(\int_U u(x)^p\, dx\right)^{1/p} \text{ and } \|u\|_{\infty,U} = \max_U |u(x)|.$$

2. Properties of Stability and A Priori Estimates

In this section, we will show the stability of semi-trivial solutions and a priori estimates for any positive solution of (2).

It is clear that the steady-state problem (2) admits two semi-trivial solutions $(\lambda,0)$ and $(0,\mu)$ in addition to the trivial solution $(0,0)$. The stabilities of such trivial and semi-trivial solutions are shown in the following lemma.

2.1 Lemma

We have the following stability results:

(i) the trivial solution $(0,0)$ is always unstable;

(ii) the semi-trivial solution $(\lambda,0)$ is asymptotically stable if $\mu < d\lambda$, while it is unstable if $\mu > d\lambda$;

(iii) the semi-trivial solution $(0,\mu)$ is asymptotically stable if $\lambda_1^N\left(\frac{\mu\eta b(x)-\lambda}{1+\mu k\rho(x)},\Omega\right) > 0$, while it is unstable if $\lambda_1^N\left(\frac{\mu\eta b(x)-\lambda}{1+\mu k\rho(x)},\Omega\right) < 0$.

Proof:

The proof of (i) is clear. Therefore, we start proof of (ii).

The linearized parabolic system of (1) at $(\lambda,0)$ is

$$\begin{cases} u_t = \Delta u + \Delta[\lambda k\rho(x)v] - \lambda u - \lambda\eta b(x)v, & x \in \Omega\times(0,\infty),\\ \tau v_t = \Delta v + (\mu - d\lambda)v, & x \in \Omega^*\times(0,\infty),\\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega\times(0,\infty),\\ \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega^*\times(0,\infty), \end{cases}$$

thus, the corresponding spectral problem is

$$\left.\begin{array}{ll} -\Delta\phi - \Delta[\lambda k\rho(x)\psi] + \lambda\phi + \lambda\eta b(x)\psi = \sigma\phi, & x \in \Omega,\\ -\Delta\psi - (\mu - d\lambda)\psi = \sigma\tau\psi, & x \in \Omega^*,\\ \frac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega,\\ \frac{\partial\psi}{\partial n} = 0, & x \in \partial\Omega^*. \end{array}\right\} \quad (3)$$

If $\psi \equiv 0$, $x \in \Omega^*$, then $\phi \not\equiv 0$, $x \in \Omega$, and ϕ satisfies

$$\left.\begin{array}{ll} -\Delta\phi + \lambda\phi = \sigma\phi, & x \in \Omega,\\ \frac{\partial\phi}{\partial n} = 0, & x \in \partial\Omega. \end{array}\right\} \quad (4)$$

Then, it follows that σ is an eigenvalue of (4), and satisfies $\mathrm{Re}\,\sigma \geq \lambda_1^N(\lambda,\Omega) = \lambda > 0$. So, the eigenvalues with associated eigenfunctions of the form $(\phi,0)$ possess positive real parts.

On the other hand, if $\psi \not\equiv 0$, $x \in \Omega^*$, then σ is an eigenvalue of the following problem:

$$\left.\begin{array}{ll} -\Delta\psi - (\mu - d\lambda)\psi = \sigma\tau\psi, & x \in \Omega^*,\\ \frac{\partial\psi}{\partial n} = 0, & x \in \partial\Omega^*. \end{array}\right\} \quad (5)$$

Then, if $\mu < d\lambda$, we see that any eigenvalue σ of (5) satisfies $\tau\,\mathrm{Re}\,\sigma \geq \lambda_1^N(d\lambda-\mu,\Omega^*) = d\lambda - \mu > 0$.

Thus, the real parts of any eigenvalue σ of (3) are positive, $(\lambda,0)$ is asymptotically stable.

If $\mu > d\lambda$, then $\sigma^* = (d\lambda-\mu)/\tau < 0$ is an eigenvalue to the second equation of (3) with a unique positive eigenfunction $\psi^* = |\Omega^*|^{-\frac{1}{2}}$ normalized as $\|\psi^*\|_{2,\Omega^*} = 1$. Then, for ϕ^* denote by

$$\phi^* = (-\Delta + \lambda - \sigma^*)_\Omega^{-1}\left[\Delta[\lambda k\rho(x)\psi^*] - \lambda\eta b(x)\psi^*\right],$$

we know that $\sigma^* < 0$ is an eigenvalue of (3) with an eigenfunction (ϕ^*,ψ^*), $(\lambda,0)$ is unstable.

The proof of (iii) is rather similar to that of [7]. □

2.2 Lemma

Assume spatial dimension $N \leq 3$ and $U = (1+k\rho(x)v)u$, there exists a positive constant C independent of k such that any positive solution (u,v) of (2) satisfies

$$0 \leq \|u\|_{\infty,\Omega} \leq \|U\|_{\infty,\Omega} \leq C, \quad 0 \leq \|v\|_{\infty,\Omega^*} \leq \mu, \text{ and}$$

$$\|U\|_{\infty,\Omega} \leq C\min_{\Omega} U, \quad \|v\|_{\infty,\Omega^*} \leq C\min_{\Omega^*} v.$$

Proof:

The proof of lemma is the same as in [6]. So, we omit it. □

3. Asymptotic Behavior of Positive Solutions as $k \uparrow\infty$

In this section, we study the asymptotic behavior of positive solutions of (2) for any $\lambda,\mu > 0$ as $k \to \infty$, and show the structure of the positive solution set of the limiting system.

For the asymptotic behavior, we have the following theorem.

3.1 Theorem

Assume spatial dimension $N \le 3$, and $\mu \ne d\lambda$. Let (u_i, v_i) be positive solutions of (2) with $k = k_i$ and $k_i \to \infty$. Then by passing to a subsequence if necessary, the following conclusions hold.

(i) If $\lim_{i\to\infty} k_i \|v_i\|_{\infty,\Omega^*} = \ell \in (0,\infty)$, $\|v_i\|_{\infty,\Omega^*} \to 0$, then $(u_i, k_i v_i) \to (u, w)$ uniformly in $\bar{\Omega} \times \bar{\Omega}^*$, where (u, w) is a positive solution of

$$\left.\begin{aligned} &\Delta\left[\left(1+\rho(x)w\right)u\right] + u(\lambda - u) = 0, && x \in \Omega, \\ &\Delta w + w(\mu - du) = 0, && x \in \Omega^*, \\ &\frac{\partial u}{\partial n} = 0, && x \in \partial\Omega, \\ &\frac{\partial w}{\partial n} = 0, && x \in \partial\Omega^*. \end{aligned}\right\} \quad (6)$$

(ii) If $\lim_{i\to\infty} k_i \|v_i\|_{\infty,\Omega^*} = \infty$, then

$$\lim_{i\to\infty} (u_i, u_i, v_i) = (\lambda, 0, \mu) \text{ in } C^1(\Omega_0) \times C^1(\bar{\Omega}^*) \times C^1(\bar{\Omega}^*).$$

Proof:

Let $\{(u_i, v_i, k_i)\}_{i=1}^{\infty}$ be any sequence such that (u_i, v_i) is a positive solution of (2) with $k = k_i$ and $k_i \to \infty$, we further set $U_i = (1 + k_i\rho(x)v_i)u_i$.

Since $\|U_i\|_{\infty,\Omega}$ and $\|v_i\|_{\infty,\Omega^*}$ are uniformly bounded by virtue of Lemma 2.2, $\|U_i\|_{W^{2,p}(\Omega)}$ and $\|v_i\|_{W^{2,p}(\Omega^*)}$ are also uniformly bounded for $p > N$, we deduce that there exists a subsequence of $\{k_i\}_{i=1}^{\infty}$, still denoted by $\{k_i\}_{i=1}^{\infty}$, such that

$$\lim_{i\to\infty} (U_i, v_i) = (\bar{U}, \bar{v}) \text{ in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}^*)$$

for some nonnegative function $(\bar{U}, \bar{v}) \in C^1(\bar{\Omega}) \times C^1(\bar{\Omega}^*)$.

Set $\phi_i = \left(\frac{1}{k_i} + \rho(x)v_i\right)u_i$, then

$$\begin{cases} k_i\Delta\phi_i + u_i(\lambda - u_i - \eta b(x)v_i) = 0, & x \in \Omega, \\ \frac{\partial \phi_i}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

By L^p estimates and the Sobolev embedding theorem, and by the fact that $\|u_i\|_{\infty,\Omega}$, $\|v_i\|_{\infty,\Omega^*}$ are uniformly bounded, we can show that subject to a subsequence, ϕ_i converges uniformly to some nonnegative constant C_1, then $\rho(x)u_i v_i \to C_1$ uniformly. As $\rho(x) = 0$ in Ω_0, we know that this constant C_1 must be zero, i.e., $u_i v_i \to 0$ uniformly in Ω^*. Furthermore, as $i \to \infty$, $v_i \to \bar{v}$ in $C^1(\bar{\Omega})$, and $\bar{v}$ is a nonnegative weak solution of

$$\begin{cases} \Delta\bar{v} + \bar{v}(\mu - \bar{v}) = 0, & x \in \Omega^*, \\ \frac{\partial \bar{v}}{\partial n} = 0, & x \in \partial\Omega^*. \end{cases}$$

It follows that $\bar{v} \equiv 0$ or $\bar{v} \equiv \mu$, i.e., $v_i \to 0$ or $v_i \to \mu$ in $\bar{\Omega}^*$.

In the following, we discuss the cases $v_i \to 0$ and $v_i \to \mu$ in $\bar{\Omega}^*$, respectively.

(i) $v_i \to 0$ in $C^1(\bar{\Omega}^*)$, in this case we first show that $k_i \|v_i\|_{\infty,\Omega^*} \to \ell \in (0,\infty)$.

If $k_i \|v_i\|_{\infty,\Omega^*} \to \infty$, set

$$\tilde{v}_i = \frac{v_i}{\|v_i\|_{\infty,\Omega^*}}, \quad \phi_i = \left(\frac{1}{k_i \|v_i\|_{\infty,\Omega^*}} + \rho(x)\tilde{v}_i\right)u_i,$$

then

$$\begin{cases} k_i \|v_i\|_{\infty,\Omega^*} \Delta\phi_i + u_i(\lambda - u_i - \eta b(x)v_i) = 0, & x \in \Omega, \\ \frac{\partial \phi_i}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

So, we see that ϕ_i converges uniformly to some nonnegative constant C_2, i.e., $\rho(x)\tilde{v}_i u_i \to C_2$, thus $\tilde{v}_i u_i \to 0$ uniformly in Ω^*. Since $-\Delta\tilde{v}_i = \tilde{v}_i(\mu - v_i - du_i)$, letting $i \to \infty$, we see that $\tilde{v}_i \to \tilde{v}$ and $\tilde{v}$ is a nonnegative weak solution of

$$\begin{cases} -\Delta\tilde{v} = \mu\tilde{v}, & x \in \Omega^*, \\ \frac{\partial \tilde{v}}{\partial n} = 0, & x \in \partial\Omega^*. \end{cases}$$

While $\|\tilde{v}\|_{\infty,\Omega^*} = 1$, it is clear that $\tilde{v} > 0$ in Ω^*, thus $\mu = 0$, it is a contradiction.

If $k_i \|v_i\|_{\infty,\Omega^*} \to 0$, we also set $\tilde{v}_i = \frac{v_i}{\|v_i\|_{\infty,\Omega^*}}$, $\phi_i = \left(1 + k_i\rho(x)\|v_i\|_{\infty,\Omega^*} \tilde{v}_i\right)u_i$, then

$$\begin{cases} \Delta\phi_i + u_i(\lambda - u_i - \eta b(x)v_i) = 0, & x \in \Omega, \\ \frac{\partial \phi_i}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Thus, the L^p estimates and the Sobolev embedding theorem deduce that

$$\phi_i \to \phi \text{ in } C^1(\bar{\Omega}).$$

Thus, u_i also converges uniformly to ϕ, and ϕ is a nonnegative weak solution of the equation

$$\begin{cases} \Delta\phi+\phi(\lambda-\phi)=0, & x\in\Omega, \\ \dfrac{\partial\phi}{\partial n}=0, & x\in\partial\Omega. \end{cases}$$

Hence, $\phi\equiv 0$ or $\phi\equiv\lambda$, i.e., $u_i\to 0$ or $u_i\to\lambda$ uniformly in Ω. If $u_i\to 0$, since $\int_{\Omega^*} v_i(\mu-v_i-du_i)dx=0$, we see that for sufficiently large i, $\mu-v_i-du_i>0$, we derive it is a contradiction; if $u_i\to\lambda$, then $\tilde{v}_i\to\tilde{v}$, and $\tilde{v}$ is a nonnegative weak solution of

$$\begin{cases} \Delta\tilde{v}+\tilde{v}(\mu-d\lambda)=0, & x\in\Omega^*, \\ \dfrac{\partial\tilde{v}}{\partial n}=0, & x\in\partial\Omega^*. \end{cases}$$

By virtue of $\|\tilde{v}\|_{\infty,\Omega^*}=1$, we know that $\tilde{v}\ge 0$ and $\tilde{v}\not\equiv 0$ in Ω^*, thus $\tilde{v}>0$ in $\bar{\Omega}^*$. So $\mu=d\lambda$, it is a contradiction.

Therefore, we see that as $v_i\to 0$, $k_i\|v_i\|_{\infty,\Omega^*}\to\ell\in(0,\infty)$ by passing to a subsequence if necessary. Set $w_i=k_iv_i$, then

$$\begin{cases} \Delta\left[(1+\rho(x)w_i)u_i\right]+u_i\left(\lambda-u_i-\dfrac{\eta b(x)}{k_i}w_i\right)=0, & x\in\Omega, \\ \Delta w_i+w_i\left(\mu-\dfrac{w_i}{k_i}-du_i\right)=0, & x\in\Omega^*, \\ \dfrac{\partial u_i}{\partial n}=0, & x\in\partial\Omega, \\ \dfrac{\partial w_i}{\partial n}=0, & x\in\partial\Omega^*. \end{cases}$$

Since $\|w_i\|_{\infty,\Omega^*}\to\ell>0$, by Lemma 2.2 we can know that $\min_{\bar{\Omega}^*} w_i\ge C\|w_i\|_{\infty,\Omega^*}\ge\frac{C\ell}{2}>0$ for large i. Furthermore, since $\int_{\Omega} u_i(\lambda-u_i-\eta b(x)v_i)dx=0$, some calculations deduce that $\int_{\Omega} u_i(\lambda-u_i)dx=b\eta\int_{\Omega^*}u_iv_i\,dx$. Then,

$$\left(\lambda-\|u_i\|_{\infty,\Omega}\right)\int_{\Omega}u_i\,dx\le b\eta\int_{\Omega^*}u_iv_i\,dx$$
$$\le b\eta\|v_i\|_{\infty,\Omega^*}\int_{\Omega^*}u_i\,dx\le b\eta\|v_i\|_{\infty,\Omega^*}\int_{\Omega}u_i\,dx,$$

which means that $\lambda-\|u_i\|_{\infty,\Omega}-b\eta\|v_i\|_{\infty,\Omega^*}\le 0$. As $\|v_i\|_{\infty,\Omega^*}\to 0$, we see that for large i, $\|u_i\|_{\infty,\Omega}\ge\lambda/2$. For $\phi_i=(1+\rho(x)w_i)u_i$, we see that $\min_{\bar{\Omega}} u_i\ge\dfrac{\min_{\bar{\Omega}}\phi_i}{1+\max_{\bar{\Omega}^*}w_i}$, and $\|w_i\|_{\infty,\Omega^*}$ is uniformly bounded, thus we know that

$$\min_{\bar{\Omega}}u_i\ge C\min_{\bar{\Omega}}\phi_i\ge C\max_{\bar{\Omega}}\phi_i\ge C\max_{\bar{\Omega}}u_i\ge\frac{C\lambda}{2}.$$

Furthermore, since $\|u_i\|_{\infty,\Omega}$ and $\|w_i\|_{\infty,\Omega^*}$ are uniformly bounded, the standard elliptic regularity deduces that $(u_i,w_i)\to(u,w)$, where (u,w) is a positive smooth solution of (6).

(ii) If $k_i\|v_i\|_{\infty,\Omega^*}\to\infty$, we must have $v_i\to\mu$ in $\bar{\Omega}^*$, then

$$u_i=\frac{U_i}{1+k_iv_i}\to 0\quad\text{in } C^1(\bar{\Omega}^*).$$

From a similar argument to that of [11], we can deduce that

$$\int_{\Omega_0}(\lambda-\bar{U})dx\le 0,\ \text{and}\ \int_{\Omega_0}\bar{U}(\lambda-\bar{U})dx=0,$$

Thus $\int_{\Omega_0}(\lambda-\bar{U})^2dx=\lambda\int_{\Omega_0}(\lambda-\bar{U})dx-\int_{\Omega_0}\bar{U}(\lambda-\bar{U})dx\le 0.$

Then $\bar{U}\equiv\lambda$ in Ω_0, $u_i\to\lambda$ in $C^1(\Omega_0)$. The proof of the theorem completes. □

Finally, we give the positive solution set of the limiting system (6). Set $U=(1+\rho(x)w)u$, then (6) is equivalent to the following system

$$\begin{cases} \Delta U+\dfrac{U}{1+\rho(x)w}\left(\lambda-\dfrac{U}{1+\rho(x)w}\right)=0, & x\in\Omega, \\ \Delta w+w\left(\mu-\dfrac{dU}{1+w}\right)=0, & x\in\Omega^*, \\ \dfrac{\partial U}{\partial n}=0, & x\in\partial\Omega, \\ \dfrac{\partial w}{\partial n}=0, & x\in\partial\Omega^*. \end{cases}\tag{7}$$

By virtue of the local bifurcation theory and regarding μ as the bifurcation parameter, we give the following local bifurcation result.

3.2 Lemma

Positive solutions of (7) bifurcate from $\{(\lambda,0,\mu)\in X\times\mathbb{R}:\mu>0\}$ if and only if $\mu=\mu_*=d\lambda$. To be precise, all positive solutions of (7) near $(\lambda,0,d\lambda)\in X\times\mathbb{R}$ can be parameterized as

$$\Gamma_\delta=\left\{(U,w,\mu)=\left(\lambda+s(\phi_*+s\bar{U}(s)),s(1+s\bar{w}(s)),d\lambda+s\mu(s)\right):\ s\in(0,\delta)\right\}$$

for some $\delta>0$ and $\phi_*=(-\Delta+\lambda I)_{\Omega}^{-1}\lambda^2\rho(x)$. Furthermore, $(\bar{U}(s),\bar{w}(s),\mu(s))$ is smooth with respect to s and $\int_{\Omega^*}\bar{w}(s)dx=0.$

3.3 Theorem

Assume spatial dimension $N \le 3$, regarding μ as the bifurcation parameter, an unbounded branch Γ_p of positive solutions of (7) bifurcates from the semi-trivial solution curve $\{(\lambda,0,\mu) \in X \times \mathbb{R} : \mu > 0\}$ at $\mu = d\lambda$. Moreover,

$$(0,d\lambda) \subset \{\mu : (U,w,\mu) \in \Gamma_p\} \subset (0,C) \tag{8}$$

for a large positive number C independent of μ, $\left\{\max_{\bar{\Omega}} U_\mu\right\}_{(U_\mu,w_\mu,\mu)\in\Gamma_p}$ is bounded, and $\|w_\mu\|_{\Omega^*,\infty} \to \infty$ as $\mu \to 0$.

Proof:

Let $\Gamma_p \subset E \times \mathbb{R}$ be the maximal connected set of the local bifurcation branch Γ_δ stated in Lemma 3.2 satisfying $\Gamma_\delta \subset \Gamma_p \subset \{(U,w,\mu) \in (E\times\mathbb{R}) \setminus \{(\lambda,0,d\lambda)\}:$

(U,w) satisfies (7)$\}$ with $E = C_n^1(\bar{\Omega}) \times C_n^1(\bar{\Omega}^*)$,

$$C_n^1(\bar{\Omega}) = \left\{u \in C^1(\bar{\Omega}) : \frac{\partial u}{\partial n} = 0,\ x \in \partial\Omega\right\}.$$

Define $P_\Omega = \{u \in C_n^1(\bar{\Omega}) : u > 0,\ x \in \bar{\Omega}\}$, then we can show that $\Gamma_p \subset P_\Omega \times P_{\Omega^*} \times \mathbb{R}$.

Since $P_\delta \subset P_\Omega \times P_{\Omega^*} \times \mathbb{R}$, if not, there exists a sequence $\{(U_i,w_i,\mu_i)\} \in \Gamma_p \cap (P_\Omega \times P_{\Omega^*} \times \mathbb{R})$ such that $(U_i,w_i,\mu_i) \to (U_\infty,w_\infty,\mu_\infty)$ in $E\times\mathbb{R}$, where $(U_\infty,w_\infty,\mu_\infty) \in \Gamma_p \cap (\partial(P_\Omega \times P_{\Omega^*}) \times \mathbb{R})$ with $U_\infty \equiv 0$ or $w_\infty \equiv 0$. As $\int_\Omega \frac{U_i}{1+\rho(x)w_i}\left(\lambda - \frac{U_i}{1+\rho(x)w_i}\right)dx = 0$, if $U_\infty \equiv 0$, then for large i, $\lambda - \frac{U_i}{1+\rho(x)w_i} > 0$, thus

$\int_\Omega \frac{U_i}{1+\rho(x)w_i}\left(\lambda - \frac{U_i}{1+\rho(x)w_i}\right)dx > 0$, it is a contradiction. If $w_\infty \equiv 0$, then

$$\begin{cases} \Delta U_\infty + U_\infty(\lambda - U_\infty) = 0, & x \in \Omega, \\ \frac{\partial U_\infty}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Since $U_\infty \not\equiv 0$, we see that $U_\infty = \lambda$. While $(\lambda,0,d\lambda)$ is the only bifurcation point of positive solutions of (7) bifurcates from $(\lambda,0)$ with bifurcation parameter μ, we know that $\mu_\infty = d\lambda$, it is a contradiction. Thus, Γ_p is contained in the set of positive solutions of (7). By a similar argument to that of [6], we can further know that Γ_p is unbounded in $E\times\mathbb{R}$.

Furthermore, as the first equation of (7) is the same in [6], we can deduce that $\max_{\bar{\Omega}} U_\mu \le C_1\lambda\left(\frac{|\Omega|}{|\Omega_0|}\right)^{\frac{1}{2}}$ for a large number C_1 independent of μ.

So, from the second equation of (7), we see that

$\mu = \lambda_1^N\left(\frac{dU}{1+w},\Omega^*\right) < \lambda_1^N\left(d\|U\|_{\infty,\Omega},\Omega^*\right) \le C$, thus $0 < \mu < C$, which shows that μ is bounded.

Since $\|U_\mu\|_{\infty,\Omega}$ is bounded, the elliptic regularity theory and the Sobolev embedding theorem deduce that $\|U_\mu\|_{C^1(\bar{\Omega})}$ is bounded. Thus, we see that $\|w_\mu\|_{C^1(\bar{\Omega}^*)}$ must be unbounded. Then, $\max_{\bar{\Omega}^*} w_\mu$ is unbounded. Therefore, there exists a sequence $\mu_i \to \mu_\infty \in [0,C]$ such that $\|w_{\mu_i}\|_{\infty,\Omega^*} \to \infty$.

Setting $W_{\mu_i} = \frac{w_{\mu_i}}{\|w_{\mu_i}\|_{\infty,\Omega^*}}$, then

$$\begin{cases} \Delta W_{\mu_i} + W_{\mu_i}\left(\mu - \frac{dU_{\mu_i}}{1+w_{\mu_i}}\right) = 0, & x \in \Omega^*, \\ \frac{\partial W_{\mu_i}}{\partial n} = 0, & x \in \partial\Omega^*. \end{cases}$$

So, we know that $W_{\mu_i} \to W, W > 0, x \in \bar{\Omega}^*$ and satisfies

$$\begin{cases} \Delta W + \mu_\infty W = 0, & x \in \Omega^*, \\ \frac{\partial W}{\partial n} = 0, & x \in \partial\Omega^*. \end{cases}$$

It follows that $\mu_\infty = 0$, together with the local bifurcation result, we know that (8) holds, and the proof of the theorem completes. □

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