Partial Edge Incidence Matrix of Semigraph over GF(2^2)

Prabhakar R. Hampiholi,
Department of Master of Computer Applications,
Gogte Institute of Technology,
Belgaum, India.

Jotiba P. Kitturkar,
Department of Mathematics,
Maratha Mandal Engineering College,
Belgaum, India.

***Abstract*—The notion of a Semigraph is a new concept introduced by E. Sampathkumar, generalizing the concept of a graph. The edges of semigraph contain atleast two vertices and are classified as full edge, subedge and partial edge. In this paper partial edge incidence matrix of semigraph over GF(2^2) is defined. Also the ranks of two types of semigraphs are obtained.**

Keywords*—*Partial edge incidence matrix; Rank of semigraph.

I. Introduction

A matrix is often an eloquent and efficient way of representing a graph for analysis. There is a relationship between many graph-theoretical properties and matrix properties, which makes the problem easier to visualize and solve. The authors [2], [4] have studied properties of semigraph matrices. The author [2] defines the incidence matrix and consecutive adjacent matrix of semigraph, but these matrices alone cannot represent semigraph uniquely. As specialty of semigraphs lie in the varieties of definitions and concepts, in this paper partial edge incidence matrix of semigraph over *GF(2^2)* is defined, which represent semigraph uniquely. The results of incidence matrix of graph *G* [3] are generalized in this paper.

II. Preliminaries

Definition 2.1 [2]: Semigraph

A semigraph *G* is an ordered pair *(V, X)* where *V* is a non-empty set, whose elements are called *vertices* of *G* and a set *X* is a set of *n*–tuples, called *edges* of *G*, of distinct vertices, for various $n \geq 2$, with the following conditions :

SG1: Any two edges have at most one vertex in common.

SG2: Two edges $(u_1, u_2, \dots, u_n)$ and $(v_1, v_2, \dots, v_m)$

are equal if and only if

i) $m = n$ and

ii) either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $i = 1, 2, 3, \dots, n$.

Thus the edge $(u_1, u_2, \dots, u_n)$ is the same as the edge $(u_n, u_{n-1}, \dots, u_1)$

Let $G = (V, X)$ be semigraph and $E = (v_1, v_2, \dots, v_{n-1}, v_n)$ is an edge of *G*. Then the vertices v_1 and v_n are called the *end vertices* of *E*, represented by thick dots, the vertices $v_2, \dots, v_{n-1}$ are called the *middle vertices or m–vertices* of *E* , represented by small hollow circles. A vertex *v* in *G* which appears as end vertex of one edge and middle vertex of the other edge is known as the *middle–cum–end vertex or ((m,e)) vertex*, represented by a small tangent to the hollow circle of middle vertex.

Example 2.2:

Let $G = (V, X)$ be a semigraph (Figure 1) , where $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $X = \{(1, 2), (2, 3, 4, 5), (5, 6), (2, 7, 6), (1, 7), (5, 7)\}$ In *G*, 1, 2, 5, 6 are end vertices, 3 and 4 are middle vertices , 7 is middle–cum–end vertex and 8 is isolated vertex.

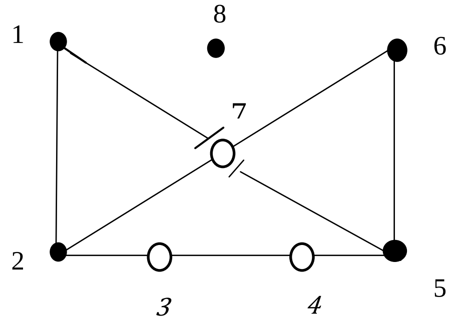

Fig.1 Semigraph *G*

In a semigraph, two edges are *adjacent* if they have a vertex in common. Any two vertices in semigraph are *adjacent* if they belong to the same edge. In addition if they are consecutive in order then are called as *consecutive adjacent vertices.*

Definition 2.3 [2]: Subedge

A *subedge* of an edge $E = (v_1, v_2, \dots v_n)$ is a *k*–tuple $E' = (v_{i_1}, v_{i_2}, \dots v_{i_k})$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ or $1 \leq i_k < i_{k-1} < \cdots < i_1 \leq n$.

Definition 2.4 [2]: Partial Edge

A *partial edge* of $E = (v_1, v_2, \dots v_n)$ is a $(j - i + 1)$–tuple $E'(v_i, v_j) = (v_i, v_{i+1}, \dots, v_j)$, where $1 \leq i \leq n$.

Definition 2.5 [2]: fs–edge and fp–edge

fs–edge is an edge which is either a full edge or a subedge and *fp–edge* is an edge which is either a full edge or a partial edge.

Definition 2.6 [5]: Consecutive subedges and consecutive *partial edges*

Let $E = (v_{i_1}, v_{i_2}, \ . \ . \ . \ . v_{i_n})$ be an edge of a semigraph G. Two subedges $S_j = (v_{i_{j1}}, v_{i_{j2}}, \ . \ . \ . \ . v_{i_{jl}})$ where $1 \leq j_1 < j_2 < . \ . \ . < j_n$ and $S_k = (v_{i_{k1}}, v_{i_{k2}}, \ . \ . \ . \ . v_{i_{km}})$ where $1 < k_1 < \ . \ . \ . < k_m \leq n$ of E are said to be consecutive subedges if $v_{i_{jl}} = v_{i_{k1}}$.

Two partial edges $P_j = (v_{i_{j1}}, v_{i_{j1+1}}, \ . \ . \ . \ . v_{i_{jl}})$ and $P_k = (v_{i_{k1}}, v_{i_{k1+1}}, \ . \ . \ . \ . v_{i_{km}})$ of E are said to be consecutive partial edges if $v_{i_{jl}} = v_{i_{k1}}$.

An edge $E = (v_{i_1}, v_{i_2}, \ . \ . \ . \ . v_{i_n})$ has n - 1 partial edges of cardinality two namely $P_1 = (v_{i_1}, v_{i_2})$; $P_2 = (v_{i_2}, v_{i_3})$ . . . ; $P_{n-1} = (v_{i_{n-1}}, v_{i_n})$ such that P_i and P_{i+1} are consecutive partial edges for $i = 1, 2, \ . \ . \ . \ , n-2$.

The partial edge $P_1 = (v_{i_1}, v_{i_2})$ is *e – partial edge* if both v_{i_1} and v_{i_2} are end vertices and forms an edge. It is *mm – partial edge* if both v_{i_1} and v_{i_2} are middle vertices and *me – partial edge* if one vertex is middle and other is end.

Definition 2.7 [2]: Dendroid

A *dendroid* is a connected semigraph without strong cycles. (all edges of strong cycle are fp – edges).

Definition 2.8 [6] [7] [8]: Galois Field of prime power GF(2^2)

Galois Field of prime power GF(2^2) is the field of polynomials of degree less than 2 over *GF(2)* modulo $(\alpha^2 + \alpha + 1)$ contains four elements $\{ 0, 1, \alpha, \alpha^2 = \alpha + 1 \}$ where α is a root of the polynomial $x^2 + x + 1$ (with coefficients in *GF(2)*). The addition and multiplication operation on *GF(2^2)* are as shown in the Table 1.

+	0	1	α	α^2
0	0	1	α	α^2
1	1	0	α^2	α
α	α	α^2	0	1
α^2	α^2	α	1	0

×	0	1	α	α^2
0	0	0	0	0
1	0	1	α	α^2
α	0	α	α^2	1
α^2	0	α^2	1	α

TABLE 1

III. MAIN RESULTS

Now we define the partial edge incidence matrix representation of semigraph.

Definition 3.1: Partial Edge Incidence Matrix of a Semigraph

The partial edge incidence matrix B of a semigraph G is a matrix of order $n \times m$, where n is number of vertices and m is number of consecutive partial edges P_i of cardinality 2 of semigraph G, is defined as

b_{ij} = 1, if e – partial edge or me – partial edge P_j is incident on end vertex v_i

= α, if me – partial edge P_j is incident on middle vertex v_i

= α^2, if mm – partial edge P_j is incident on middle vertex v_i

= 0, otherwise

The above definition is illustrated in example 3.2

Example 3.2:

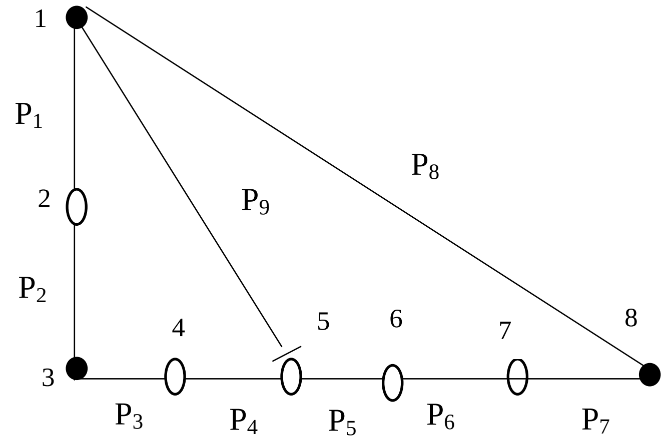

Fig. 2 Semigraph G

For the semigraph G (Figure 2), the partial edge incidence matrix $B(G)$ is

$$B(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \alpha & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^2 & \alpha^2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha^2 & \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha^2 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Observations 3.3:

In case of partial edge incidence matrix,

1) The matrix is of order $n \times m$.
2) Each column corresponds to a consecutive partial edge of cardinality 2 and therefore each column has two non-zero entries as 1 and 1 or α^2 and α^2 or 1 and α.
3) The row sums for e–partial edge, mm–partial edge and me–partial edge respective are 0, 0 and α^2 with respect to addition defined on *GF(2^2)* .
4) $deg(v_i)$ = Number of edges having v_i as an end vertex = Number of 1's in R_i , the i^{th} row .
5) $deg_e(v_i)$ = Number of edges containing v_i = { No. of 1's + ½(Number of α's and α^2's) } in R_i
6) $deg_{ca}(v_i)$ = Number of vertices which are consecutively adjacent to v_i = Number of 1's , α's and α^2's in R_i

7) The row corresponding to end vertex contain all non–zero entries as 1.

8) The row corresponding to middle vertex contains entries α or α^2 or both.

9) The row corresponding to middle–cum–end ((m, e)) vertex contains atleast one entry 1 and even number of other non-zero entries.

10) Semigraph can be redrawn using observations 1 to 9.

11) A row with all 0's, therefore, represents an isolated vertex.

12) If a semigraph G is disconnected and consists of two components G_1 and G_2, then the partial edge incidence matrix $B(G)$ of semigraph G can be written in a block-diagonal form as

$$B(G) = \begin{bmatrix} B(G_1) & \cdot & 0 \\ . & \cdot & . \\ 0 & \cdot & B(G_2) \end{bmatrix}$$

The following theorems characterize the partial edge incidence matrix of a semigraph.

Theorem 3.4:

If G is a semigraph with n vertices and not containing middle–cum–end vertices then the rank of partial edge incidence matrix $B(G)$ is $n - 1$.

Proof: Let $B(G)$ be the incidence matrix of a semigraph G, not containing the middle–cum–end vertex. Then each row of the incidence matrix $B(G)$ may be regarded as a vector over $GF(2^2)$. Let the vector in the first row be called B_1, in the second row B_2, and so on. Thus

$$B(G) = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ . \\ B_n \end{bmatrix}$$

In this case, each column of B consists of exactly two non-zero entries 1 and α or α and α^2 or α^2 and α^2. Clearly the linear combination $\alpha\times$(sum of the rows B_i's corresponding to end vertices) + (sum of the rows B_i's corresponding to middle vertices) or {(sum of the rows B_i's corresponding to end vertices) + $\alpha^2\times$(sum of the rows B_i's corresponding to middle vertices)} is zero with respect to $GF(2^2)$.

Thus the vectors $B_1, B_2, \ldots, B_n$ are not linearly independent.

Therefore, the rank of B is less than n;

that is rank $B(G) \leq n - 1$

Now consider the sum of any l of these n vectors ($l \leq n - 1$). If the semigraph is connected, $B(G)$ cannot be partitioned, such that $B(G_1)$ is with l rows and $B(G_2)$ with $n - l$ rows . In other words, no submatrix of $B(G)$ can be found , for $l \leq n - 1$, such that linear combination of those l rows is equal to zero. Therefore rank $B(G) \geq n - 1$.

Hence the rank of $B(G) = n - 1$.

Theorem 3.5:

If G is a dendroid with n vertices then the rank of partial edge incidence matrix $B(G)$ is $n - 1$.

Proof: Let $B(G)$ be represented as in Theorem 3.4

Now to prove, the rank of $B(G) \leq n - 1$, we show that row vectors $B_1, B_2, \ldots, B_n$ of $B(G)$ are linearly dependent.

For this, consider linear combination, $b_1B_1 + b_2B_2 + b_3B_3 + \ldots + b_nB_n$ for $0 \neq b_i \in GF(2^2)$, for all $i = 1, 2, 3, \ldots n$. Now the b_i s are selected in the following ways.

Let E_r be any edge in dendroid G. The scalars b_i s for B_i s, corresponding to end vertices and middle vertex (vertices) of E_r are chosen in one of the 3 ways as shown in the Table 2.

Choice	Scalar Multiplier b_i of B_i , corresponding to **end vertices**	Scalar Multiplier b_i of B_i , corresponding to **middle vertex (vertices)**
1	1	α^2
2	α	1
3	α^2	α

TABLE 2

If E_r is the only edge in G then by using these selected b_i s for B_i s and Table 1, we see that

$$b_1B_1 + b_2B_2 + b_3B_3 + \ldots + b_nB_n = 0 .$$

Therefore, B_i s are linearly dependent.

Hence, the rank of $B(G) \leq n - 1$.

If E_r is not the only edge of G, suppose E_s is another edge adjacent to E_r , then the selection of scalars b_i s for rows B_i s corresponding to the vertices of E_s depend on common vertex of E_r , E_s and previously selected b_i s for rows B_i s corresponding to the vertices of E_r.

For the common vertex of E_r and E_s, one of the following cases occurs.

1) If the **common** vertex is **middle vertex** of both the edges E_r and E_s then pattern of scalar multipliers b_i 's for B_i's corresponding to vertices of E_s is same as selection for b_i 's for B_i's corresponding to vertices of E_r .

2) If the **common** vertex is **end vertex** of both the edges E_r and E_s then pattern of scalar multipliers b_i 's for B_i's corresponding to vertices of E_s is same as selection for b_i 's for B_i's corresponding to vertices of E_r.

If b_i of B_i , corresponding to **middle vertex of** $\boldsymbol{E_r}$ is	Then we select b_i of B_i , corresponding to **end vertices of** $\boldsymbol{E_s}$ as	and b_i of B_i , corresponding to **middle vertex (vertices) of** $\boldsymbol{E_s}$ as
1	1	α^2
α	α	1
α^2	α^2	α

TABLE 3

3) If the **common** vertex is **middle vertex** of E_r and **end vertex** of E_s and if B_i corresponding to middle vertex of E_r is multiplied by b_i then the pattern of b_i s for B_i s corresponding to vertices of E_s is as shown in Table 3.

4) If the **common** vertex is **end vertex** of E_r and **middle vertex** of E_s and if B_i corresponding to end vertex of E_r is multiplied by b_i then the pattern of b_i s for B_i s corresponding to vertices of E_s is as shown in Table 4.

If b_i of B_i , corresponding to **end vertex of E_r** is	Then we select b_i of B_i , corresponding to **middle vertices of E_s** as	and b_i of B_i , corresponding to **end vertex (vertices) of E_s** as
1	1	α^2
α	α	1
α^2	α^2	α

TABLE 4

If E_r and E_s are the only two edges in dendroid G then for these selected b_i 's for B_i's and using Table 1, we see that

$$b_1B_1 + b_2B_2 + b_3B_3 + \dots + b_nB_n = 0 .$$

Hence the rank of $B(G) \leq n - 1$.

If E_r and E_s are not the only edges of G, suppose E_t be one more edge then the above process can be repeated.

Suppose without loss of generality, if E_t is adjacent to E_r . Then selection of b_i s for B_i s of E_t depend on previously selected b_i's of B_i's for E_r. Continuing this process for all edges we see that

$b_1B_1 + b_2B_2 + b_3B_3 + \dots + b_nB_n = 0$, for any number of edges.

Therefore B_i s are linearly dependent.

Hence rank of $B(G) \leq n - 1$ in all possible cases.

As in Theorem 3.4, we can show that rank $B(G) \geq n - 1$

Therefore, rank $B(G) = n - 1$.

Remark 3.6:

As every graph is a semigraph without middle vertices, by the Theorem 3.4 its rank is $n - 1$, which is also proved by theorem 7-2 [3] page 140. Therefore Theorem 3.4 generalizes the graph theory result.

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