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# p-Valently Meromorphic Functions with Fixed First Coefficient

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***Abstract*** **: In this paper, we have considered the class of the functions of the form** $f(z)=\frac{1}{z^p}+\left(\frac{p-\alpha}{\alpha}\right)e^{i\theta}+\sum_{n=p+1}^{\infty} a_{n-p}z^{n-p}$ $,(p\in N)$ $a_{n-p}\geq 0$ **and** $\frac{p}{2}\leq\alpha<p$. **Coefficient inequalities, closure theorems and radius of convexity for this class are determined.**

## 1. INTRODUCTION

Let $M_p$ denote the class of functions of the form

(1.1) $$f(z)=\frac{1}{z^p}+\sum_{n=p}^{\infty} a_{n-p}z^{n-p} \quad , \qquad p\in N$$

which are analytic in the punctured disk U* = {z∈¢ : 0 < |z| < 1}. A function $f(z)\in M_p$ is said to be starlike of order $\alpha$ if it satisfies the inequality

(1.2) $$\text{Re}\left(-\frac{zf'(z)}{f(z)}\right)>\alpha \quad , \qquad (z\in U=U^*U\{0\})$$

for $\alpha$ ($0\leq\alpha<p$). We say that f(z) is in the class $S^*(p,\alpha)$ for such functions.

We have obtained [5] the following result :

**Result 1 :** Let the function $f(z)\in M_p$ analytic in $D_r$ = {z∈¢ : 0 < |z| < r ≤ 1} be given by (1.1) with $a_{n-p}\geq 0$, then $f(z)\in S^*(p,r;\alpha)$ if and only if

(1.3) $$\sum_{n=p}^{\infty} (n+\alpha-p)a_{n-p}r^n \leq p-\alpha$$

for some $\alpha\left(\frac{p}{2}\leq\alpha<p\right)$.

Letting $r\to 1$ in above result – 1, we get the following result 2.

**Result 2 :** If $f(z)\in M_p$ defined on U* = {z∈¢ : 0 < |z| < 1} satisfies

(1.4) $$\sum_{n=p}^{\infty} (n+\alpha-p)a_{n-p} \leq p-\alpha \qquad a_{n-p}\geq 0$$

for some $\alpha\left(\frac{p}{2} \le \alpha < p\right)$, then $f(z)\in S^*(p, \alpha)$

In view of result 2, a function of the form (1.1) belonging to the class $S^*(p, \alpha)$ must satisfy the coefficient inequality

(1.5) $$a_{n-p} \le \frac{p-\alpha}{n+\alpha-p} \qquad (n \ge p)$$

hence, we write

$$a_0 \le \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta}$$

by fixing the first coefficient $a_0$, we introduce a new subclass $S^*(p,\theta;\alpha)$ of $S^*(p, \alpha)$ consisting of functions of the form

(1.6) $$f(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} a_{n-p} z^{n-p}, \qquad p \in N$$

and $\frac{p}{2} \le \alpha < p$.

In this paper, we obtain coefficient inequality, closure theorems and radius of convexity for the class $S^*(p,\theta;\alpha)$. Techniques used are similar to those of Silverman and Silvia [2], Uralegaddi [3], Owa and Srivastava [1] and M.K. Aouf and H.E. Darwish [4].

## 2. COEFFICIENT INEQUALITY :

**Lemma 2.1 :** Let function f(z) defined by (1.6) is in the class $S^*(p,\theta;\alpha)$ if and only if

(2.1) $$\sum_{n=p+1}^{\infty} (n+\alpha-p)a_{n-p} \le (p-\alpha)(1-e^{i\theta}), \qquad (a_{n-p} \ge 0)$$

for $\frac{p}{2} \le \alpha < p$. The result is sharp.

**Proof :** Putting $a_0 = \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta}$ in (1.4), we get

$$(p-\alpha)e^{i\theta} + \sum_{n=p+1}^{\infty} (n+\alpha-p)a_{n-p} \le (p-\alpha)$$

$$\sum_{n=p+1}^{\infty} (n+\alpha-p)a_{n-p} \le (p-\alpha)(1-e^{i\theta})$$

Further, by taking the function f(z) of the form

(2.2) $$f(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \frac{(p-\alpha)(1-e^{i\theta})}{(n+\alpha-p)} z^{n-p}$$

for $n \ge p + 1$, we can see that results (2.1) is sharp.

**Corollary 2.2 :** Let the function defined by (1.6) be in the class $S^*(p,\theta;\alpha)$, then

(2.3) $$a_{n-p} \le (p-\alpha)\,(1-e^{i\theta})\,(n+\alpha-p)^{-1} \qquad (n \ge p+1)$$

The result (2.3) is sharp for the function f(z) is given by (2.2).

## 3. CLOSURE THEOREMS :

**Theorem 3.1 :** Let the function

(3.1) $$f_j(z) = \frac{1}{z} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} a_{n-p,j} z^{n-p} z$$

be in the class $S^*(p,\theta;\alpha)$ for j = 1, 2, …m, then the function F(z) defined by

(3.2) $$F(z) = \sum_{j=1}^{m} d_j F_j(z) \qquad (d_j \ge 0)$$

is also in the same class $S^*(p,\theta;\alpha)$ where

(3.3) $$\sum_{j=1}^{m} d_j = 1$$

**Proof :** Combining (3.1) and (3.2), we get

(3.4) $$F(z) = \sum_{j=1}^{m} d_j \left\{ \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} a_{n-p,j} z^{n-p} \right\}$$

$$F(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} \left( \sum_{j=1}^{m} d_j a_{n-p,j} \right) z^{n-p} \qquad \text{(using 3.3)}$$

Since $f_j(z) \in S^*(p,\theta;\alpha)$ for every j = 1, 2, …, m, therefore, theorem 2.1 yields

$$\sum_{n=p+1}^{\infty} (n+\alpha-p) a_{n-p,j} \le (p-\alpha)(1-e^{i\theta})$$

for j = 1, 2, …, m. Thus, we obtain

$$\sum_{n=p+1}^{\infty} (n+\alpha-p)\left( \sum_{j=1}^{m} d_j a_{n-p,j} \right) = \sum_{j=1}^{m} d_j \left( \sum_{n=p+1}^{\infty} (n+\alpha-p) a_{n-p,j} \right)$$

$$\le (p-\alpha)(1-e^{i\theta})$$

which implies $F(z) \in S^*(p,\theta;\alpha)$.

**Theorem 3.2 :** Let the function $f_j(z)$ be defined by (3.1). If $f_j(z) \in S^*(p,\theta;\alpha)$ for every j = 1, 2, …, m, then the function

(3.5) $$g(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} b_{n-p} z^{n-p}$$

is in the same class $S^*(p,\theta;\alpha)$, where

(3.6) $$b_{n-p} = \frac{1}{m} \sum_{j=1}^{m} a_{n-p,j}$$

**Proof :** Since $f_j(z) \in S^*(p,\theta;\alpha)$ it follows from theorem 2.1 that

$$\sum_{n=p+1}^{\infty} (n+\alpha-p)a_{n-p,j} \le (p-\alpha)(1-e^{i\theta})$$

hence

$$\sum_{n=p+1}^{\infty} (n+\alpha-p)b_{n-p,j} = \sum_{n=p+1}^{\infty} (n+\alpha-p)\left(\frac{1}{m}\sum_{j=1}^{m} a_{n-p,j}\right)$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=p+1}^{\infty}(n+\alpha-p)a_{n-p,j}\right)$$

$$\le \frac{1}{m}\sum_{j=1}^{m}(p-\alpha)(1-e^{i\theta})$$

$$\le (p-\alpha)(1-e^{i\theta})$$

which (in view of theorem 2.1) implies $g(z) \in S^*(p,\theta;\alpha)$. This completes the proof of theorem.

**Theorem 3.3 :** The class $S^*(p,\theta;\alpha)$ is closed under convex linear combination.

**Proof :** Let the function $f_j(j = 1, 2)$ defined by (3.1) be in the class $S^*(p,\theta;\alpha)$. It is sufficient to prove that the function H(z) defined by

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \qquad (0 \le \lambda \le 1)$$

is also in the class $S^*(p,\theta;\alpha)$.

$$H(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty}\{\lambda a_{n-p,1} + (1-\lambda)a_{n-p,2}\}z^{n-p}$$

Since $f_1(z)$ and $f_2(z)$ belong to the class $S_p^*(\theta;\alpha)$.

Therefore,

$$\sum_{n=p+1}^{\infty} |a_{n-p,1}|(n+\alpha-p) \le (p-\alpha)(1-e^{i\theta})$$

and

$$\sum_{n=p+1}^{\infty} |a_{n-p,2}|(n+\alpha-p) \le (p-\alpha)(1-e^{i\theta})$$

Now, we observe that

$$\sum_{n=p+1}^{\infty} |\lambda a_{n-p,1} + (1-\lambda)a_{n-p,2}|(n+\alpha-p) \le (p-\alpha)(1-e^{i\theta})$$

hence, in view of theorem 2.1, we get $H(z) \in S_p^*(\theta;\alpha)$

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**Theorem 3.4 :** Let

(3.7) $$f_p(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta}$$

and

(3.8) $$f_n(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \frac{(p-\alpha)(1-e^{i\theta})}{(n+\alpha-p)} z^{n-p} \qquad (n \geq p+1)$$

Then f(z) is in the class S*(p,θ;α) if and only if it can be expressed in the form

(3.9) $$f(z) = \sum_{n=p}^{\infty} \lambda_n f_n(z) \qquad \text{where} \lambda_n \geq 0$$

and

(3.10) $$\sum_{n=p}^{\infty} \lambda_n = 1.$$

**Proof :** We suppose that f(z) can be expressed in the form (3.9) then it follows from (3.8), (3.9) and (3.10) that

$$f(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} \frac{(p-\alpha)(1-e^{i\theta})}{(n+\alpha-p)} \lambda_n z^{n-p}$$

Note that

$$\sum_{n=p+1}^{\infty} \frac{(p-\alpha)(1-e^{i\theta})}{(n+\alpha-p)} \lambda_n \frac{(n+\alpha-p)}{(p-\alpha)(1-e^{i\theta})} = \sum_{n=p+1}^{\infty} \lambda_n$$

$$= 1 - \lambda_p \leq 1.$$

hence f(z) ∈S*(p,θ;α)

for the converse assume that the function f(z) of the form (1.6) belongs to the class S*(p,θ;α). Since f(z) satisfies (2.3) for $n \geq p + 1$, we may set

$$\lambda_n = \frac{(n+\alpha-p)}{(p-\alpha)(1-e^{i\theta})} = a_{n-p} \quad , \quad n \geq p+1$$

and

$$\lambda_p = 1 - \sum_{n=p+1}^{\infty} \lambda_n$$

Then

$$f(z) = \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} \frac{(p-\alpha)(1-e^{i\theta})}{(n+\alpha-p)} \lambda_n z^{n-p}$$

$$= \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} + \sum_{n=p+1}^{\infty} \lambda_n \left\{ f_n(z) - \frac{1}{z^p} - \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} \right\}$$

$$= \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right)e^{i\theta} \left\{ 1 - \sum_{n=p+1}^{\infty} \lambda_n \right\} + \sum_{n=p+1}^{\infty} \lambda_n \left\{ f_n(z) - \frac{1}{z^p} \right\}$$

$$= \left\{1 - \sum_{n=p+1}^{\infty} \lambda_n\right\} \frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right) e^{i\theta} \lambda_p + \sum_{n=p+1}^{\infty} \lambda_n f_n(z)$$

$$= \lambda_p \left[\frac{1}{z^p} + \left(\frac{p-\alpha}{\alpha}\right) e^{i\theta}\right] + \sum_{n=p+1}^{\infty} \lambda_n f_n(z)$$

$$= \lambda_p f_p(z) + \sum_{n=p+1}^{\infty} \lambda_n f_n(z)$$

$$= \sum_{n=p}^{\infty} \lambda_n f_n(z)$$

This complete the proof of the theorem.

## 4. RADIUS OF CONVEXITY :

**Theorem 4.1 :** Let the function f(z) defined by (1.6) be in the class S*(p,θ;α), then f(z) is p–valent meromorphically convex in $0 < |z| < r = r(\rho,\theta;\alpha)$ where

(4.1) $$r(p,\theta,\alpha) = \inf_{n\geq p} \left\{\frac{p^2(n+\alpha-p)}{(n^2-p^2)(p-\alpha)(1-e^{i\theta})}\right\}^{1/n}$$

**Proof :** If is suffices to show that

(4.1) $$\left|\frac{zf''(z)}{f'(z)} + 1 + p\right| < p$$

Consider

$$\left|\frac{zf''(z)+(p+1)f'(z)}{f'(z)}\right| = \left|\frac{\sum_{n=p+1}^{\infty} (n-p)na_{n-p}z^{n-p-1}}{-\frac{p}{z^{p+1}} + \sum_{n=p+1}^{\infty} (n-p)a_{n-p}z^{n-p-1}}\right|$$

$$= \left|\frac{\sum_{n=p+1}^{\infty} n(n-p)a_{n-p}z^{n}}{-p + \sum_{n=p+1}^{\infty} (n-p)a_{n-p}z^{n}}\right|$$

Thus, the result follows if

$$\sum_{n=p+1}^{\infty} n(n-p)a_{n-p}|z|^n \leq p\left[p - \sum_{n=p+1}^{\infty} (n-p)a_{n-p}r^n\right]$$

or

(4.3) $$\sum_{n=p+1}^{\infty} (n^2-p^2)a_{n-p}|z|^n \leq p^2$$

But by theorem 2.1, we have

(4.4) $$\sum_{n=p+1}^{\infty}(n+\alpha-p)a_{n-p}\le(p-\alpha)(1-e^{i\theta})$$

Hence, (4.3) holds if and only if

$$\frac{(n^2-p^2)}{p^2}|z|^n\le\frac{(n+\alpha-p)}{(p-\alpha)(1-e^{i\theta})}$$

or

$$|z|\le\left[\frac{p^2(n+\alpha-p)}{(n^2-p^2)(p-\alpha)(1-e^{i\theta})}\right]^{1/n}\qquad(n\ge p, p\in N)$$

This proves the theorem.

## 5. THE CLASS $\mathbf{S}_{\mathbf{m}}^{*}$ (p, θ; α) :

Instead of fixing just the first coefficient, we can fix finitely many coefficients. Let $S_m^*$ (p,θ; α) denote the class of functions of the form

(5.1) $$f(z)=\frac{1}{z^p}+\sum_{k=p}^{m}\frac{(p-\alpha)(1-e^{i\theta_k})}{(k+\alpha-p)}z^{k-p}+\sum_{n=m+1}^{\infty}a_{n-p}z^{n-p},\quad(m\ge p)$$

where $0\le e^{i\theta_k}\le 1$ and

$$0\le\sum_{k=p}^{m}e^{i\theta_k}=e^{i\theta}\le 1$$

Note that $S_p^*$ (p,θ; α) ≡ S*(p, θ; α).

**Theorem 5.1 :** The extreme points of the class $S_m^*$ (p,θ; α) are

(5.2) $$f_m(z)=\frac{1}{z^p}+\sum_{k=p}^{m}\frac{(p-\alpha)(1-e^{i\theta_k})}{(k+\alpha-p)}z^{k-p}$$

and

(5.3) $$f_n(z)=\frac{1}{z^p}+\sum_{k=p}^{m}\frac{(p-\alpha)(1-e^{i\theta_k})}{(k+\alpha-p)}z^{k-p}+\frac{(p-\alpha)\left(1-\sum_{k=p}^{m}e^{i\theta_k}\right)}{(n+\alpha-p)}z^{n-p}\quad(n\ge m+1)$$

**Theorem 5.2 :** Let the function f(z) defined by (5.1) be in the class $S_m^*$ (p,θ; α), then f(z) is p–valently meromorphic convex function in $0<|z|<r_m(p,\theta;\alpha)$ where $r_m(p,\theta;\alpha)$ in the largest value for which

$$\sum_{k=p}^{m}\frac{(p-\alpha)(1-e^{i\theta_k})(k^2-p^2)}{(k+\alpha-p)}r^k+\frac{(n^2-p^2)(p-\alpha)\left(1-\sum_{k=p}^{m}e^{i\theta_k}\right)}{(n+\alpha-p)}r^n\le p^2$$

for n ≥ m + 1, the result is sharp for the function given by (5.3).

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