Output Feedback Tracking Control of an Inverted Pendulum using Polynomial Fuzzy Modeling

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Abstract— **An inverted pendulum is a non-linear, open-loop unstable system. The approach used in this paper is to model a non linear system using polynomial fuzzy modeling and output feedback tracking for its control. The polynomial fuzzy model and the output-feedback polynomial fuzzy controller connected in a closed loop forms the control system. The output-feedback polynomial fuzzy controller then drives the system states of the nonlinear plant to follow those of a stable reference model. Based on the Lyapunov stability theory, sum-of-squares-based stability conditions are used to determine the system stability. Using the third-party Matlab toolbox SOSTOOLS, a feasible solution can be found numerically.**

***Keywords* – polynomial fuzzy model, output feedback, sum of squares, inverted pendulum, stability analysis**

I. INTRODUCTION

Over the past decades, many advances have been made in the field of control theory. Analysis and control of nonlinear systems are among the most challenging problems in systems and control theory. The balancing act of an inverted pendulum is a popular problem in the area on nonlinear control systems, and is widely used to demonstrate the merits of a control system.

A fuzzy control system is a control system based on fuzzy logic, and is effectively used for the control of nonlinear systems. Fuzzy-model-based control systems offer an effective way of handling control problems for nonlinear control systems. Model based fuzzy control has been successfully applied to systems that are mathematically poorly modeled, and where the knowledge and the experience of operators can achieve the control object well.

With the T-S fuzzy model [1], a nonlinear plant can be represented in a general form as a weighted sum of linear sub-systems which locally describe the system dynamics. Based on the local linear sub-system, a linear sub-controller can be designed. A set of fuzzy rules is then employed to combine these linear sub-controllers to form a fuzzy controller. An FMB control system is formed by connecting the nonlinear plant represented by the T-S fuzzy model and a fuzzy controller connected in a closed loop.

Although the T-S fuzzy modeling and control approach has been applied to some control problems with some success, the method is subject to some limitations. One limitation is that when the T-S fuzzy modeling method is used to approximate a nonlinear system that cannot be linearized, the characteristic of the nonlinear system might be distorted. Indeed, for a highly nonlinear system, the T-S fuzzy model with linear subsystems may result in either imprecise approximation in the representation or a large number of fuzzy rules. The other is that the controller synthesis is similar to fuzzy blending, which may lead to a complex fuzzy controller.

Recently, Tanaka et al [2] proposed a polynomial fuzzy modeling and control framework that is a generalization of the T–S fuzzy model and is more effective in representing nonlinear control systems. The stability of the fuzzy polynomial systems is derived based on polynomial Lyapunov functions that contain quadratic Lyapunov functions as a special case. They presented a sum of squares (SOS) approach [3] for modeling and control of nonlinear systems using polynomial fuzzy systems. This model allows polynomial variables appearing in the sub-systems to represent a wider class of nonlinear plants. A feasible solution to the SOS-based stability conditions can be found numerically using the third-party Matlab toolbox SOSTOOLS [4].

Sum-of-squares (SOS) decomposition for multivariate polynomials plays an important role to determine positivity of a polynomial function [5]. If a polynomial function can be represented in a form of SOS, the polynomial function can be shown to be positive. This concept is widely used in the stability analysis using the Lyapunov stability theory. Based on the SOS decomposition techniques, construction of polynomial Lyapunov function was formulated as semi-definite programming [3], which is shown to be an effective technique investigating the stability analysis and control synthesis problems for control systems.

This paper studies the tracking control of the inverted pendulum using polynomial fuzzy modeling under the SOS based methods.

II. SUM OF SQUARES DECOMPOSITION

The main computational method used in this paper is based on sum of squares (SOS) decomposition of multivariate polynomials. In the general multivariate case, however, $p(x) \geq 0$ in the usual sense does not necessarily imply that $p(x)$ is SOS. Notwithstanding this fact, the crucial thing to keep in mind is that, while being stricter, the condition that *p(x)* is SOS is much more computationally tractable than nonnegativity. At the same time, practical experience indicates that replacing nonnegativity with the SOS property in many cases leads to the exact solution.

A multivariate polynomial $p(x) = p(x_1, \ldots, x_n)$ is a SOS, if there exists polynomials $f_1(x), \ldots, f_m(x)$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2 x(t) \tag{1}$$

The existence of sum of squares representation for *p(x)* is sufficient condition for its non-negativity by the following proposition.

Proposition 1 [5]: Let *p(x)* be a polynomial in $x \in R^n$ of degree $2d$. In addition, let $x^T(x(t))$ be a column vector, which is a properly chosen vector of monomials in x with polynomials of degree no greater than d . Then *p(x)* is a sum of squares iff there exists a positive symmetric semidefinite matrix such that P such that

$$p(x(t)) = x^T(x(t))\, P\, x(x(t)) \tag{2}$$

When *p(x)* is a sum of squares, matrix $P \geq 0$ can be found using semi-definite programming (SDP) techniques.

III. Modeling of an Inverted Pendulum

An inverted pendulum system is unstable without control, that is, the pendulum wills simply fall over if the cart isn't moved to balance it and naturally falls downward because of gravity. Thus, the inverted pendulum system is an inherently unstable. In order to keep it upright, or stabilize the system, it requires a continuous correction mechanism to stay upright since the system is unstable and non-linear.

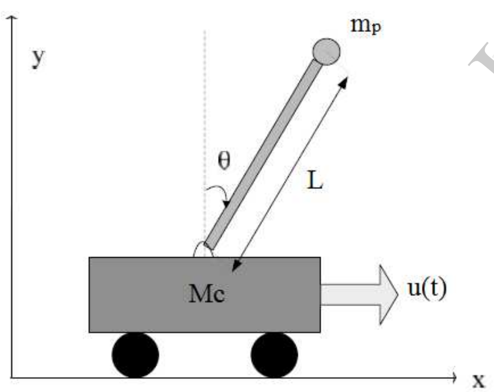

Figure 1. Inverted Pendulum on a cart

$\Theta(t)$ = angular displacement of the pendulum

g = acceleration due to gravity

m_p = mass of the pendulum

M_c = mass of the cart

L = length of the pendulum

u(t) = force applied to the cart

g = 9.8 m/s^2

A. Polynomial Fuzzy Model

Based on [6], let p be the number of fuzzy rules describing the behavior of the nonlinear plant. The i-th rule is of the following format:

Rule *i:* IF $f_1(y(t))$ is M^i_1 AND . . . AND $f_\psi(y(t))$ is M^i_ψ
THEN $\dot{x}(t) = A_i(x(t))\, x(x(t)) + B_i(x(t))\, u(t)$ (3)

The system dynamics and output are defined as

$$\dot{x}(t) = \sum_{i=1}^{p} w_i(y(t))\,(A_i(x(t))\, x(x(t)) + B_i(x(t))\, u(t)) \tag{4}$$

$$y(t) = C\, x(x(t)) \tag{5}$$

$$\sum_{i=1}^{p} w_i(y(t)) = 1, \quad w_i(y(t)) \geq 1 \tag{6}$$

$x(t) \in R^n$ is the system state vector; $y(t) \in R^l$ is the output vector; $A_i(x(t)) \in R^{n \times N}$ and $B_i(x(t)) \in R^{n \times m}$ are the known polynomial system and input matrices, respectively; $x(x(t)) \in R^N$ is a vector of monomials in x(t); $u(t) \in R^m$ is the input vector. It is assumed that x(x(t)) = 0 if and only if x(t) = 0.

$w_i(y(t))$ is the normalized grade of membership

The system dynamics of the system is as follows.

$$\ddot{\Theta}(t) = \frac{g \sin(\Theta(t)) - a\, m_p L\, \Theta(t)^2 \sin(\Theta(t)) \cos(\Theta(t))}{4L/3 - a\, m_p L \cos^2(\Theta(t))} - \frac{a \cos(\Theta(t))\, u(t)}{4L/3 - a\, m_p L \cos^2(\Theta(t))} \tag{7}$$

Let

$x_1(t) = \Theta(t)$

$x_2(t) = \Theta(t) = \dot{x}_1(t)$

$\dot{x}_2(t) = \ddot{\Theta}(t)$

Substituting these in (1),

$$\dot{x}_1(t) = x_2(t) \tag{8}$$

$$\dot{x}_2(t) = \frac{g \sin(x_1(t)) - a\, m_p L\, x_2(t)^2 \sin(x_1(t)) \cos(x_1(t))}{4L/3 - a\, m_p L \cos^2(x_1(t))} - \frac{a \cos(x_1(t))\, u(t)}{4L/3 - a\, m_p L \cos^2(x_1(t))} \tag{9}$$

B. Reference Model

A stable reference model is defined as follows

$$\dot{x}_r(t) = A_r\, x_r(x_r(t)) + B_r\, r(t)) \quad (10)$$

$$y_r(t) = C\, x_r(x_r(t)) \quad (11)$$

$Ar \in R^{n\times N}$ and $B_r \in R^{n\times m}$ are the constant system and input matrices, respectively, $r(t) \in R^m$ is the reference input vector, $yr(t) \in R^l$ is the output vector of the reference model. The system and input matrices A_r and B_r are not limited to be constant matrices. If they are function of the system states, we have polynomial matrices $A_r(x_r(t))$ and $B_r(x_r(t))$. However, it is required to make sure that the reference model is stable.

C. Output-Feedback Polynomial Fuzzy Controller

An output-feedback polynomial fuzzy controller is proposed to drive the system states of the nonlinear plant in the form of (4) to follow those of the stable reference model (10). Define the state error as

$$\hat{e}(t) = x(x(t)) - x_r(x_r(t)) \quad (12)$$

and the output error as

$$\begin{aligned} e_y(t) &= y(t) - y_r(t) \quad (13)\\ &= C\,x(x(t)) - C\,x_r(x_r(t))\\ &= C\,(x(x(t)) - x_r(x_r(t)))\\ e_y(t) &= C\,\hat{e}(t) \quad (14)\end{aligned}$$

The output-feedback polynomial fuzzy controller is described by the following p rules

Rule j: IF $f_1(y(t))$ is M^j_1 AND . . . AND $f_\psi(y(t))$ is M^j_ψ
THEN $F_j(h(t))\, e_y(t) + G_j(h(t))\, y_r(t)$ (15)

The output-feedback polynomial fuzzy controller is defined as

$$u(t) = \sum_{j=1}^{p} w_j(y(t))\,(F_j(h(t))\,\hat{e}(t) + G_j(h(t))\,C\,x_r(x_r(t)) \quad (16)$$

F_j and G_j are the polynomial feedback gains.

IV. SIMULATION RESULTS

An inverted pendulum on a cart as in () is considered. Mass of the pendulum m_p = 2 kg, mass of cart M_c = 8 kg, length L = 0.5m. Consider the operating domain of $x_1(t) \epsilon [(-5n/12), (5n/12)]$.

$$f_1(x_1(t)) = (\cos(x_1(t)) / 4L/3 - a\, mp\, L \cos^2(x_1(t)))$$

$$f_1(x_1(t)) \epsilon [f_{1min}, f_{1max}]$$

Approximating $\sin(x_1(t))$ and $\tan(x_1(t))$ by polynomials,

$$\sin(x_1(t)) = s_3(x_1(t))^3 + s_1 x_1(t)$$

$$\tan(x_1(t)) = t_3(x_1(t))^3 + t_1 x_1(t)$$

$s_3 = -0.1460$, $s_1 = 0.9897$

$t_3 = 1.0545$, $t_1 = 0.6469$

The inverted pendulum is described by a two-rule polynomial fuzzy model with the following system, input and output matrices

$$A_1(x) = \begin{bmatrix} 0 & 1 \\ a_1 & 0 \end{bmatrix}$$

$$A_2(x) = \begin{bmatrix} 0 & 1 \\ a_2 & 0 \end{bmatrix}$$

$$a_1 = f_{1min}\,(g(t_3(x_1(t))^2 + t_1) - a\, mp\, L\, x_2(t)^2\,(s_3(x_1(t))^2 + s_1))$$

$$a_2 = f_{1max}\,(g(t_3(x_1(t))^2 + t_1) - a\, mp\, L\, x_2(t)^2\,(s_3(x_1(t))^2 + s_1))$$

$$B_1(x) = \begin{bmatrix} 0 \\ -f_{1min}\, a \end{bmatrix}$$

$$B_2(x) = \begin{bmatrix} 0 \\ -f_{1max}\, a \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The membership functions are defined as

$$w_1(x1(t)) = (f_1(x_1(t)) - f_{1max}) / (f_{1min} - f_{1max})$$

$$w_2(x_1(t)) = 1 - w_1(x_1(t))$$

The stable reference model is chosen as

$$A_r = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

$$B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$r(t) = 5 \sin(t)$$

Feedback gains F and G are computed using SOSTOOLS. The polynomial fuzzy controller is then employed to control the inverted pendulum subject to the initial condition

$x(0) = [5\pi/12\ 0]^T$

$x_r(0) = [5\pi/24\ 0.05]^T$

The systems responses are shown in fig. 2. It can be seen that the system states of the inverted pendulum are able to follow those of the stable reference model.

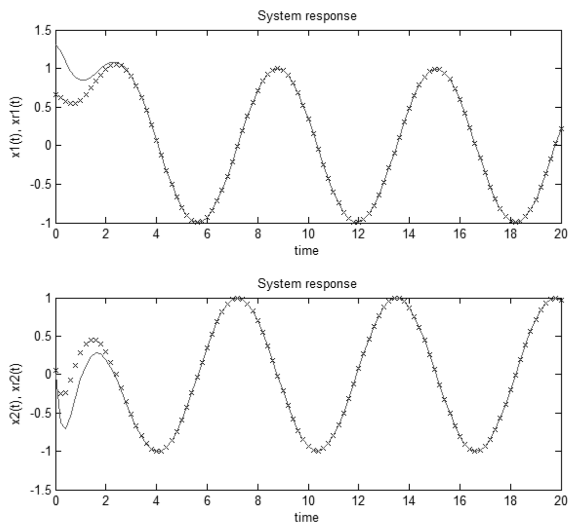

Figure 2. System responses of $x(t)$ (solid line) and $x_r(t)$ (dotted line)

V. CONCLUSION AND FUTURE WORK

The tracking control problem of a polynomial fuzzy model based control system was investigated, and was applied to an inverted pendulum. The inverted pendulum system was made to follow a selected stable reference model. The polynomial feedback gains were determined using SOSTOOLS. To further expand the work, a Simulink model of the inverted pendulum is to be created. The simulation would be used to validate the numerical solutions to the nonlinear equations, and evaluate the performance of the control scheme.

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REFERENCES

[1] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modelling and control," IEEE Trans. Syst., Man, Cybern., vol. SMC-15, no. 1, pp. 116–132, Jan. 1985.

[2] K. Tanaka, H. Yoshida, H. Ohtake, and H. O. Wang, "A sum of squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," IEEE Trans. Fuzzy Syst., vol. 17, no. 4, pp. 911–922, Aug. 2009

[3] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Nonlinear control synthesis by sum-of-squares optimization: A Lyapunov-based approach," in Proc. ASCC, Melbourne, Australia, Feb. 2004, vol. 1, pp. 157–165.

[4] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: A general purpose sum of squares programming solver," in Proc. 41st IEEE Conf. Decision Control, Las Vegas, NV, Dec. 2002, vol. 1, pp. 741–746.

[5] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "SOSTOOLS—Sum of squares optimization toolbox, user's guide," The MathworksInc., Natick,MA, 2002.

[6] H.K.Lam, Hongyi Li, "Output- feedback tracking control for polynomial fuzzy-model-based control systems,"IEEETrans.IE,vol.60, no.12,pp. 5830-5840, Dec 2013.