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Oscillatory Behaviour Of The Solution Of The Third Order Nonlinear Neutral Delay Difference Equation

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Abstract

In this paper we study oscillatory behaviour of the solution of the third order nonlinear neutral delay difference equation of the form

$$\Delta^2\left(a_n\Delta\left(x_n+p_nx_{n-k}\right)\right)+f\left(n,\sigma(n)\right)=0, n\varepsilon N\left(n_0\right)$$

Key words*: Oscillation, third order, Nonlinear Neutral Delay difference equations*

1. Introduction

We are concerned with the oscillatory behaviour of the solution of the third order nonlinear neutral delay difference equations of the form

$$\Delta^2\left(a_n\Delta\left(x_n+p_nx_{n-k}\right)\right)+f\left(n,\sigma(n)\right)=0, n\varepsilon N\left(n_0\right) \qquad (1.1)$$

Where the following conditions are assumed to hold.

(H1) $\{a_n\}$ is a positive sequence of real numbers for n∈N(n_0) such that $\sum_{n=n_0}^{\infty}\frac{n}{a_n}=\infty$

(H2) $\{p_n\}$ is a real sequence such that $0\le p_n<p<1$ for all n∈N(n_0)

(H3) k is a non negative integer and $\{\sigma(n)\}$ is a sequence of positive integer with $\lim_{x\to\infty}\sigma(n)=\infty$

(H4) $f: N(n_0)\times R\to R$ is continuous and f(n,u) is nondecreasing in u with u f(n,u)> 0 for all u≠ 0 and all n∈N(n_0) and f(n,u)≠ 0 eventually.

By a solution of equation (1.1) we mean real sequence $\{x_n\}$ satisfying (1.1) n={n_0,n_{0+1},n_{0+2},.......} a solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called non oscillatory. The forward difference operator $\Delta x_n=x_{n+1}-x_n$

2. Main Result

In this section we state and prove some lemmas which are useful in establish main result for the sake of convenience we will use of following notations.

$$R(\text{n})=\sum_{s=n_0}^{n-1}\sum_{t=n_0}^{s-1}\frac{t}{a_t}$$

and

$$R(n,N)=\sum_{s=N}^{n-1}\sum_{s=N}^{s-1}\frac{t-1}{a_t}$$

Let $\{x_n\}_{n=n_0}^{\infty}$ be a real sequences we will also associated sequences $\{z_n\}$

$$z_n=x_n+p_{n+k} \qquad n\in N\left(n_0\right) \quad (2.1)$$

Where $\{p_n\}$ and k have been defined above

First we give some relation between the sequence $\{x_n\}$ and $\{z_n\}$

Let $\{x_n\}_{n=n_0}^{\infty}$ be positive sequence, $\{z_n\}$ be sequence by (1.2)

(i) $\lim_{x\to\infty}x_n=\infty$ then $\lim_{x\to\infty}z_n=\infty$

(ii) If $\{z_n\}$ converges to zer then so does $\{x_n\}$

Proof: The proof can be found in [9]

Lemma 2.2

Let $\{x_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of equation (1.1) then there only the following two cases for n large enough

(i) $x_n>0, z_n>0, \Delta z_n>o, a_n\Delta z_n>0, \Delta\left(a_n\Delta z_n\right)>o$

(ii) $x_n>0, z_n>0, \Delta z_n>o, a_n\Delta z_n<0, \Delta\left(a_n\Delta z_n\right)>o$

Lemma 2.3

If $N \geq n_0$ then $\lim_{x\to\infty} \frac{R(n,\mathrm{N})}{R(n)} = 1$

Lemma 2.4

Let $\{x_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of equation (1.1) then there exists an integer $N \in N(n_0)$ and a constant $k_1 > 0$ such that

$\frac{1}{2}\Delta(a_n \Delta z_n) R(n) \leq z_n \leq k_1(R(n)), \mathrm{n} > \mathrm{N}$

Lemma 2.5

Let $\{x_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of equation (1.1) then there exist an integer $n_1 \in N(n_0)$ such that for any integer $N \geq n_1$ we have $z_n \geq \sum_{s=N}^{n-1} R(s,N) f(s,\sigma(n)), n \in N$

The proof of lemmas can be found [7] and [8]

Lemma 2.6

If $\{x_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of equation (1.1) then there exist an integer $n \in N(n_0)$ such that

$\Delta z_n \geq \frac{1}{2}\Delta(a_n \Delta z_n) \Delta R\sigma(n)$ for $n \geq N$ also if $\sigma(n) \leq n$, then

$\Delta z_{\sigma(n)} \geq \frac{1}{2}\Delta(a_n \Delta z_n) \Delta R_{\sigma(n)}$ for $n \geq N$ (2.2)

Proof: From Lemma 2.2 we have for $n \geq n_1 \varepsilon N(n_0)$

$z_n > 0 \quad \Delta z_n > o$ and $\Delta^2(a_n \Delta z_n) < 0$

$$\Delta z_n \geq \sum_{s=n_1}^{n-1} \Delta z_s = \sum_{s=n_1}^{n-1} \frac{1}{a_z} a_z \Delta z_s$$

$$\geq \sum_{s=n_1}^{n-1} \frac{1}{a_s} \sum_{t=n_1}^{s-1} \Delta(a_t \Delta z_t)$$

$$\geq \Delta(a_n \Delta z_n) \sum_{s=n_1}^{n-1} \frac{s-n_1}{a_s}$$

$$\geq \Delta(a_n \Delta z_n) \Delta R(n,n_1) \qquad (2.3)$$

From lemma 2.3 we conclude that there exist an integer $n \geq N$ such that $\Delta R(n,n_1) \geq \frac{1}{2}\Delta R(\mathrm{n})$ for $n \geq N$

Since $\Delta^2(a_n \Delta z_n) < 0$ and $\sigma(n) \leq n$

We have $\Delta z_{\sigma(n)} \geq \frac{1}{2}\Delta(a_n \Delta z_n) \Delta R_{\sigma(n)}$ for $n \geq N$

The proof is complete

Lemma 2.7

If $\{x_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of equation (1.1) then there exist an integer $n \in N(n_0)$ such that $(1-p_n) z_n \leq x_n \leq z_n$ for $n \geq N$

Proof: If $\{x_n\}_{n=n_0}^{\infty}$ is an eventually positive solution of equation (1.1) for $n \geq N$. Then from the definition of z_n we have $z_n > x_n$ for $n \geq N$ from lemma 2.2 we have $z_n > 0$ and $\Delta z_n > o$ for $n \geq N$

$z_n = x_n + p x_{n-k} \qquad x_n = z_n - p_n x_{n-k}$

$x_n \geq z_n - p_n z_{n-k}$

$\geq (1-p_n) z_n$ for $n \geq N$

This completes the proof.

Theorem 2.8

Assume that there exists real sequences $\{q_n\}$ such that $\frac{f(n,u)}{u} \geq M q_n > 0$ for all $u \neq 0, n \geq n_0$

(2.4)

and $\sigma(n) = n - l$ where l is a sequence $\{p_n\}$ such that

$$\limsup_{x\to\infty} \sum_{s=n_0}^{n} \rho_s [(1-p_{z-l}) q_s - \frac{(\Delta\rho_s)^2}{2M \Delta R(s-l) \rho_s^2}] = \infty$$

(2.5)

Then all solutions of equation (1.1) are oscillatory.

Proof: Let $\{x_n\}$ be a nonoscillatory solutions of (1.1) and assume without loss of generality the $\{x_n\}$ is eventually positive. From Lemmas 2.2 and 2.7 we have $z_n > o, z_{n-l} > 0, \Delta z_n > o$ and $\Delta(a_n \Delta z_n) > 0$ for $n \geq N$ and

$x_{n-l} \geq (1-p_n) z_{n-l}$

Define

$$\omega_n = \frac{\rho_n \Delta(a_n \Delta z_n)}{z_{n-l}}, \quad n \ge N$$

Then in view of Lemma 2.6, (2.4) and (2.5) we have

$$\Delta\omega_n \le \frac{\rho_n \Delta^2(a_n \Delta z_n) + \Delta(a_n \Delta z_n)\Delta\rho_n}{z_{n-l}} - \frac{\rho_n \Delta(a_n \Delta z_n)\Delta z_{n-l}}{(z_{n-l})^2}$$

$$\le -Mq_n(1-p_{n-l})\rho_n + \Delta\rho_n \frac{\omega_n}{\rho_n}$$

$$\le -Mq_n(1-p_{n-l})\rho_n + \Delta\rho_n \frac{\omega_n}{\rho_n} - \frac{1}{2\rho_n}\omega^2 \Delta R(n-l)$$

$$\le -Mq_n(1-p_{n-l})\rho_n \frac{(\Delta\rho_n)^2}{2\rho_n \Delta R(n-l)}$$

Summing the last inequality from N to $n \ge N$, we obtain

$$\sum_{s=n_0}^{n} \rho_s[(1-p_{z-l})q_s - \frac{(\Delta\rho_s)^2}{2M\Delta R(s-l)\rho_s^2}] \le \frac{\omega_N}{M}$$

and this contradicts (2.5). Thus the proof is complete.

For the linear equation

$$\Delta^3(x_n + p_n x_{n-\tau}) + q_n x_{n-\sigma} = 0 \qquad (2.6)$$

Where τ and σ are nonnegative integers less than n we obtain from Theorem 2.8 the following corollary

Corollary 2.7

Suppose $q_n \ge 0$ for all $n \ge n_0$ and there exists positive sequences $\{\rho_n\}$ such that

$$\limsup_{x\to\infty} \sum_{s=n_0}^{n} \rho_s[(1-p_{z-l})q_s - \frac{(\Delta\rho_s)^2}{2M\Delta R(s-l)\rho_s^2}] = \infty$$

then all solutions of equation 2.5 are oscillatory.
The proof is complete

Example : Consider the difference equations

$$\Delta^2\left[n(n+1)\Delta\left(x_n + \frac{1}{\sqrt{n-1}}x_{n-1}\right)\right] + nx_{n-1}^{\frac{1}{3}} = 0; n \ge 3 \qquad (2.7)$$

it is easy to see all solutions of the equations(2.7) are oscillatory

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